A Liouville theorem and blow-up behavior for a vector-valued nonlinear heat equation with no gradient structure

Hatem Zaag
Courant Institute and CNRS ENS

Abstract

We prove a Liouville Theorem for the following heat system whose nonlinearity has no gradient structure

$$\partial_t u = \Delta u + v^p, \quad \partial_t v = \Delta v + u^q,$$

where $pq > 1$, $p \geq 1$, $q \geq 1$ and $|p - q|$ small.
We then deduce a localization property and uniform L^∞ estimates of blowing-up solutions of this system.

1 Introduction

In this paper, we are concerned with finite time blow-up for semilinear systems of the heat type

$$\begin{cases} U_t = \Delta U + F(U) \\ U(.,0) = U_0 \end{cases} \tag{1.1}$$

where $U : (x,t) \in \mathbb{R}^N \times [0,T) \to \mathbb{R}^M$, $U_0 : \mathbb{R}^N \to \mathbb{R}^M$, $T > 0$, $(\Delta U)_i = \Delta U_i$, $F \in \mathbf{C}^1(\mathbb{R}^M, \mathbb{R}^M)$ and $N,\ M \in \mathbb{N}$.

The local Cauchy problem for (1.1) can be solved in $L^\infty(\mathbb{R}^N)$. If the maximal solution exists on $[0,T)$ with $T < +\infty$, then the solution blows-up in finite time T in the sense that $\|U(t)\|_{\mathbf{L}^\infty(\mathbb{R}^N)} \to +\infty$ as $t \to T$. In this case, T is called the blow-up time of U. Let us consider a blow-up solution U of (1.1). From the regularizing effect of the heat flow, U is continuous on $\mathbb{R}^N \times (0,T)$ and we can define $a \in \mathbb{R}^N$ to be a blow-up point of U if U is not locally bounded near (a,T) in the sense that $|U(a_n,t_n)| \to +\infty$ for some sequence $(a_n,t_n) \to (a,T)$ as $n \to +\infty$.

Many papers deal with the study of blowing-up solutions of (1.1). However, many of them treat the scalar case ($M = 1$), mainly with positive initial data. Indeed, in this case, the maximum principle applies and allows to obtain many crucial estimates (see for instance Herrero and Velázquez [11], Galaktionov and Vázquez [10], Weissler [16], ..). Unfortunately, in the vector-valued case, the maximum principle does not hold in

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general. However, in the case where there is a potential $G \in \mathbf{C}^2(\mathbb{R}^M, \mathbb{R})$ such that

$$\nabla G = F, \tag{1.2}$$

one can define in some functional space a Lyapunov functional

$$E(U) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla U|^2 dx - \int_{\mathbb{R}^N} G(U) dx,$$

that allows to have some blow-up criteria (see Levine [12], Ball [2],..) or information on the asymptotic blow-up behavior for system (1.1) (see Giga and Kohn [9],...). For instance, let us sketch the main results for the case of the equation

$$U_t = \Delta U + |U|^{p-1}U \text{ with } p > 1 \text{ and } (N-2)p < N+2. \tag{1.3}$$

Under the additional condition

$$(M = 1 \text{ and } U_0 \geq 0) \text{ or } (3N-4)p < 3N+8,$$

Giga and Kohn prove in [8] the existence of some $C > 0$ such that

$$\text{for all } (x,t) \in \mathbb{R}^N \times [0,T), \quad |U(x,t)| \leq C v_0(t)$$

where $v_0(t) = [(p-1)(T-t)]^{-\frac{1}{p-1}}$ is the solution of $v_0' = v_0^p$, $v(T) = +\infty$. The study of the blow-up behavior for solutions of (1.3) is done through the introduction of similarity variables

$$y = \frac{x-a}{\sqrt{T-t}}, \quad s = -\log(T-t), \quad W_a(y,s) = (T-t)^{\frac{1}{p-1}} U(x,t)$$

where a may or not be a blow-up point for U. From (1.3), we see that W_a (or simply W) satisfies the following system : for all $(y,s) \in \mathbb{R}^N \times [-\log T, +\infty)$,

$$W_s = \Delta W - \frac{1}{2} y \cdot \nabla W - \frac{W}{p-1} + |W|^{p-1}W. \tag{1.4}$$

In [15], Merle and Zaag prove the following localization property for $U(x,t)$:

For all $\epsilon > 0$, there exists $C_\epsilon > 0$ such that for all $(x,t) \in \mathbb{R}^N \times [\frac{T}{2}, T)$,

$$|U_t - |U|^{p-1}U| \leq \epsilon |U|^p + C_\epsilon.$$

This identity is a consequence of the following Liouville Theorem for system (1.4) (see [15] and [13]) :

Let W be a solution of (1.4) defined for all $(y,s) \in \mathbb{R}^N \times \mathbb{R}$ such that $W \in \mathbf{L}^\infty(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}^M)$. Then, either $W \equiv 0$ or $W \equiv \kappa\omega_0$ or $W(y,s) = \varphi(s-s_0)\omega_0$ where $s_0 \in \mathbb{R}$, $\omega_0 \in \mathbb{S}^{M-1}$, $\varphi(s) = \kappa(1+e^s)^{-\frac{1}{p-1}}$ and $\kappa = (p-1)^{-\frac{1}{p-1}}$.

From this Theorem and the localization property, one can deduce the asymptotic profile for U near a blow-up point a as t goes to T (see [15], Filippas-Merle [6] and Giga-Kohn [9]).

It is worth noticing that the techniques developed in [15] (see also [13]) for system (1.3) extend naturally to system (1.1) if the nonlinearity F satisfies some conditions, namely (1.2) with a convexity condition on G. Moreover, the techniques of [15] break down if (1.2) no longer holds.

Our aim in this paper is to pass beyond this restriction and to obtain the same type of results in a case where (1.2) does not hold. More precisely, we consider $(u,v) : (x,t) \in \mathbb{R}^N \times [0,T) \to (\mathbb{R}^+)^2$ a solution of the following system

$$\begin{cases} u_t = \Delta u + v^p, & v_t = \Delta v + u^q \\ u(.,0) = u_0, & v(.,0) = v_0, \end{cases} \tag{1.5}$$

blowing-up at time T. where $u,v : (x,t) \in \mathbb{R}^N \times [0,T) \to \mathbb{R}^+$.
In [1], Andreucci, Herrero and Velázquez prove that if

$$pq > 1 \text{ and } \left(q(pN-2) < N+2 \text{ or } p(qN-2) < N+2\right), \tag{1.6}$$

then for all $(x,t) \in \mathbb{R}^N \times [0,T)$,

$$0 \le u(x,t) \le C(T-t)^{-\frac{p+1}{pq-1}} \text{ and } 0 \le v(x,t) \le C(T-t)^{-\frac{q+1}{pq-1}} \tag{1.7}$$

(the same result has been proved by Caristi and Mitidieri [3] in a ball under conditions different from (1.6)).
The study of blow-up solutions for system (1.5) is done through the introduction of the following similarity variables

$$\Phi(y,s) = (T-t)^{\frac{p+1}{pq-1}} u(x,t) \text{ and } \Psi(y,s) = (T-t)^{\frac{q+1}{pq-1}} v(x,t) \tag{1.8}$$

where $$y = \frac{x-a}{\sqrt{T-t}} \text{ and } s = -\log(T-t).$$

From (1.5), Φ and Ψ satisfy the following system

$$
\begin{aligned}
\Phi_s &= \Delta\Phi - \tfrac{1}{2}y.\nabla\Phi + \Psi^p - \left(\tfrac{p+1}{pq-1}\right)\Phi,\\
\Psi_s &= \Delta\Psi - \tfrac{1}{2}y.\nabla\Psi + \Phi^q - \left(\tfrac{q+1}{pq-1}\right)\Psi.
\end{aligned} \tag{1.9}
$$

If (Γ, γ) denotes the only non trivial constant solution of (1.9) defined by

$$
\gamma^p = \Gamma\left(\frac{p+1}{pq-1}\right) \text{ and } \Gamma^q = \gamma\left(\frac{q+1}{pq-1}\right), \tag{1.10}
$$

then it is shown in [1] the following :

Proposition 1.1 (Andreucci-Herrero-Velázquez) *There exists a continuous and positive function ϵ defined in the interval $(1, (N+2)/(N-2))$ if $N \geq 3$ (resp. in $(1, +\infty)$ if $N = 1, 2$) such that if $|p-p_0|+|q-p_0| < \epsilon(p_0)$, for some p_0 satisfying $1 < p_0$ and $(N-2)p_0 < N+2$, then any solution of (1.9) in $\mathbf{L}^\infty(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}^2)$ is either $(\Phi, \Psi) \equiv (0, 0)$ or $(\Phi, \Psi) \equiv (\Gamma, \gamma)$ or satisfies otherwise*

$$
\begin{aligned}
\|\Phi(.,s) - \Gamma\|_{L^2_\rho} + \|\Psi(.,s) - \gamma\|_{L^2_\rho} &\to 0 \quad \textit{as } s \to -\infty,\\
\|\Phi(.,s)\|_{L^2_\rho} + \|\Psi(.,s)\|_{L^2_\rho} &\to 0 \quad \textit{as } s \to +\infty
\end{aligned} \tag{1.11}
$$

where (Γ, γ) is defined in (1.10) and $\mathbf{L}^2_\rho(\mathbb{R}^N)$ is the $\mathbf{L}^2$ space associated with the Gaussian measure $\rho(y) = e^{-\frac{|y|^2}{4}} / (4\pi)^{N/2}$.

Remark : Although condition (1.6) is said to be necessary in the statement of the result of [1], this condition is not used at all in the proof in section 3 of that paper.
This allows the authors to adopt a local (in space) approach in order to describe all the possible blow-up behaviors for (1.5) near a given blow-up point :

Assume that (1.6) holds and that the conclusion of Proposition 1.1 is true. Consider (u, v) a solution of (1.5) which blows-up at time T. Consider $a \in \mathbb{R}$ a blow-up point of (u, v) and define (Φ, Ψ) by (1.8). Then either (Φ, Ψ) goes to (Γ, γ) exponentially fast or there exists $l \in \{1, .., N\}$ such that after an orthogonal change of space coordinates,

(1.12)
$$
\begin{pmatrix} \Phi \\ \Psi \end{pmatrix}(y,s) = \begin{pmatrix} \Gamma \\ \gamma \end{pmatrix} - \frac{C(p,q)}{s}\sum_{k=1}^{l}(2 - y_k^2)\begin{pmatrix} (p+1)\Gamma \\ (q+1)\gamma \end{pmatrix} + o\left(\frac{1}{s}\right)
$$

for some $C(p,q) > 0$, where the convergence takes place in $\mathbf{C}^k_{\mathrm{loc}}(\mathbb{R}^N)$ for any $k \geq 0$.

In the first case, they obtain other profiles, some of them similar to the scalar case of (1.3), and some which are new (see Theorems 3 and 4 in [1] for more details).

Although the profile classification of [1] may seem exhaustive, we should point out that their approach is local and that the convergence speed they got depends on the initial data and on the considered blow-up point. In particular, the uniformity of the convergence of (Φ_a, Ψ_a) to (Γ, γ), with respect to the blow-up point, can not follow from their results. Moreover, it is not likely that their results can provide any stability result (with respect to initial data) of the behavior (1.12) with $l = N$.

In this paper, we adopt a global (in space) point of view, and aim at obtaining uniform estimates, with respect to initial data and to the blow-up points, which improve the results of [1]. We would like to adapt the program we did in [15] for (1.3) to the present context. It turns out then that the major difficulty is the proof of a Liouville Theorem for equation (1.9) as we did in [15] for (1.4). Indeed, the non-gradient structure of (1.9) makes the techniques of [15] break down. The key point of our paper is then the proof of the following Liouville Theorem, which strongly improves the classification result of [1], sated in Proposition 1.1 :

THEOREM 1.2 **(A Liouville Theorem for system (1.9))** *There exists a continuous positive function η such that for any $p_0 > 1$ such that $p_0(N-2) < N+2$, for all p, q such that $|p - p_0| + |q - p_0| < \eta(p_0)$, $p \geq 1$ and $q \geq 1$, the following holds :*

Let (Φ, Ψ) be a solution of (1.9) in $\mathbf{L}^\infty(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}^2)$. Then, either $(\Phi, \Psi) = (0,0)$ or $(\Phi, \Psi) = (\Gamma, \gamma)$ or there exists $s_0 \in \mathbb{R}$ such that for all $(y,s) \in \mathbb{R}^N \times \mathbb{R}$, $\Phi(y,s) = \Phi_0(s - s_0)$, $\Psi(y,s) = \Psi_0(s - s_0)$ where $\Phi_0(s) = \Gamma(1+e^s)^{-\frac{p+1}{pq-1}}$ and $\Psi_0(s) = \gamma(1+e^s)^{-\frac{q+1}{pq-1}}$.

Remark : (Φ_0, Ψ_0) is the only solution (up to a time translation) of

$$\begin{cases} \Phi_{0s} = \Psi_0^p - \left(\frac{p+1}{pq-1}\right)\Phi_0, & \Psi_{0s} = \Phi_0^q - \left(\frac{q+1}{pq-1}\right)\Psi_0 \\ (\Phi_0, \Psi_0) \to (\Gamma, \gamma) \text{ as } s \to -\infty \text{ and} & (\Phi_0, \Psi_0) \to (0,0) \text{ as } s \to +\infty \end{cases}$$

This Theorem has an equivalent formulation for solutions of (1.5) :

COROLLARY 1.3 (A Liouville Theorem for system (1.5)) *Assume that p and q satisfy the conditions in Theorem 1.2. Consider (u,v) a solution of*

(1.5) defined on $\mathbb{R}^N \times (-\infty, T)$ *for* $T \in \mathbb{R}$ *such that for all* $(x,t) \in \mathbb{R}^N \times (-\infty, T)$, $0 \le u(x,t) \le C(T-t)^{-\frac{p+1}{pq-1}}$ *and* $0 \le v(x,t) \le C(T-t)^{-\frac{q+1}{pq-1}}$ *for some* $C > 0$. *Then, either* $u \equiv v \equiv 0$ *or for all* $(x,t) \in \mathbb{R}^N \times (-\infty, T)$,

$$u(x,t) = \Gamma(T^* - t)^{-\frac{p+1}{pq-1}} \text{ and } v(x,t) = \gamma(T^* - t)^{-\frac{q+1}{pq-1}}$$

where $T^* \ge T$ *and* (Γ, γ) *is defined in (1.10).*

Theorem 1.2 is the major novelty of our paper. Indeed, once the difficulty of proving this Liouville Theorem is overcome, one can use the same techniques as in [15] to derive for blow-up solutions of (1.5), new results which can not be derived from [1].
The following uniform estimates are the first consequence of Theorem 1.2 :

Theorem 1.4 **(Limits at blow-up of $\mathbf{L}^\infty$ estimates for solutions of (1.5))** *Assume that* p *and* q *satisfy (1.6) and the conditions of Theorem 1.2, and consider* (u_n, v_n) *a sequence of solutions of (1.5) which blow-up at time* T_n *and satisfy*

$$T_n \le T_0 \text{ and } \|u_n(0)\|_{\mathbf{L}^\infty(\mathbb{R}^N)} + \|v_n(0)\|_{\mathbf{L}^\infty(\mathbb{R}^N)} \le C_0 \tag{1.13}$$

for some $T_0 > 0$ *and* $C_0 > 0$. *Then,*
i) $\tau^{\frac{p+1}{pq-1}} \|u_n(T_n - \tau)\|_{\mathbf{L}^\infty(\mathbb{R}^N)} \to \Gamma$,
$\tau^{\frac{(p+1)(q+1)}{pq-1}} \| \left(\frac{u_n(T_n-\tau)}{\Gamma}\right)^{q+1} - \left(\frac{u_n(T_n-\tau)}{\gamma}\right)^{p+1} \|_{L^\infty(\mathbb{R}^N)} \to 0$ *and*
$\tau^{\frac{p+1}{pq-1}+\frac{i}{2}} \|\nabla^i u_n(T_n - \tau)\|_{\mathbf{L}^\infty(\mathbb{R}^N)} \to 0$ $(i = 1, 2)$ *as* τ *goes to* 0, *uniformly in* n.
ii) $\|\Phi_n(s)\|_{\mathbf{L}^\infty(\mathbb{R}^N)} \to \Gamma$, $\| \left(\frac{\Phi_n(s)}{\Gamma}\right)^{q+1} - \left(\frac{\Psi_n(s)}{\gamma}\right)^{p+1} \|_{L^\infty(\mathbb{R}^N)} \to 0$ *and* $\|\nabla^i \Phi_n(s)\|_{\mathbf{L}^\infty(\mathbb{R}^N)} \to 0$ *(for* $i = 1, 2$*) as* $s \to +\infty$, *uniformly in* n *(*(Φ_n, Ψ_n) *is defined from* (u_n, v_n) *and* T_n *by (1.8)).*
The same holds for v_n *and* Ψ_n *with obvious changes.*

Remark : The notation $\nabla^2 u$ stands for the second differential of u.
The following localization result is the second consequence of the Theorem 1.3 :

Theorem 1.5 **(A localization property for blow-up solutions of (1.5))** *Under the assumptions of Theorem 1.4, consider* $C_0 > 0$ *and* $T_0 > 0$. *Then, for all* $\epsilon > 0$, *there exists* $C(C_0, T_0, \epsilon) > 0$ *such that for all solution* (u, v) *of (1.5) which blows-up at time* T *and satisfies*

$$T \le T_0 \text{ and } \|u(0)\|_{\mathbf{C}^2(\mathbb{R}^N)} + \|v(0)\|_{\mathbf{C}^2(\mathbb{R}^N)} \le C_0,$$

we have for all $(x,t) \in \mathbb{R}^N \times [0,T)$,

$$\begin{aligned} &|u_t - v^p| \le \epsilon v^p + C, \quad |v_t - u^q| \le \epsilon u^q + C \ \textit{and} \\ &\left| \left(\tfrac{u}{\Gamma}\right)^{q+1} - \left(\tfrac{v}{\gamma}\right)^{p+1} \right| \le \epsilon u^{q+1} + C. \end{aligned} \tag{1.14}$$

As a striking consequence of Theorem 1.5, we have the following Corollary which asserts that the coupled system (1.5) is in some sense (at least in the singular region where u and v are large enough) equivalent to two uncoupled ordinary differential equations.

Corollary 1.6 (Uniform ODE comparison until blow-up) *Under the assumptions of Theorem 1.4, we have for all* $\epsilon > 0$ *and* $(x,t) \in \mathbb{R}^N \times [0,T)$

$$\left| \partial_t u - \gamma^p \left(\frac{u}{\Gamma}\right)^{\frac{p(q+1)}{p+1}} \right| \le \epsilon u^{\frac{p(q+1)}{p+1}} + C(\epsilon, C_0, T_0).$$

v *satisfies of course an analogous estimate.*

An immediate and important consequence of this Corollary is the following.

Corollary 1.7 *Under the assumptions of Theorem 1.4, consider* (u,v) *a solution of (1.5) blowing-up at time* T.
i) **(Continuity in $\bar{\mathbb{R}}$ near a blow-up point)** *For all blow-up point* $a \in \mathbb{R}^N$, $u(x,t) \to +\infty$ *and* $v(x,t) \to +\infty$ *as* $(x,t) \to (a,T)$.
ii) **(No oscillation in time)** *There exists* $\delta > 0$ *such that for all blow-up point* $a \in \mathbb{R}^N$, $\forall (x,t) \in B(a,\delta) \times [T-\delta, T)$, $\frac{\partial u}{\partial t}(x,t) > 0$ *and* $\frac{\partial v}{\partial t}(x,t) > 0$.

Remark : i) is to be compared with the definition of a blow-up point, where one requires that $u(a_n,t_n) \to +\infty$ and $v(a_n,t_n) \to +\infty$, just for one sequence (a_n,t_n) going to (a,T).

We also have the following uniform convergence estimate and a blow-up exclusion criterion, localized at the point.

Proposition 1.8 (Uniform convergence at blow-up points) *Under the assumptions of Theorem 1.4, consider* (u,v) *a solution of (1.5) blowing-up at time* T*, and denote by* S *the set of all blow-up points of* (u,v). *Then,*
i) $\sup_{a \in S} \left| (T-t)^{\frac{p+1}{pq-1}} u(a,t) - \Gamma \right| + \left| (T-t)^{\frac{q+1}{pq-1}} u(a,t) - \gamma \right| \to 0$ *as* $t \to T$.
ii) For all $\epsilon > 0$, $\exists \delta(\epsilon) > 0$ *such that if for some* $x_0 \in \mathbb{R}^N$ *and* $t_0 \in [T - \delta(\epsilon), T)$, $u(x_0,t_0) < (\Gamma - \epsilon)(T-t_0)^{-\frac{p+1}{pq-1}}$ *or* $v(x_0,t_0) < (\gamma-\epsilon)(T-t_0)^{-\frac{q+1}{pq-1}}$, *then* x_0 *is not a blow-up point of* (u,v).

Furthermore, we suspect that the techniques of [14] can be adapted to refine the results of Theorem 1.4 until the first order and obtain (under the same hypotheses) :

There exists $C(C_0, T_0) > 0$ *such that* $\forall s \geq -\log T$, $\|\Phi(s)\|_{L^\infty} \leq \Gamma + Cs^{-1}$, $\|\nabla^i \Phi(s)\|_{L^\infty} \leq Cs^{-\frac{i}{2}}$ *(for* $i = 1$ *or* 2*) and* $\| \left(\frac{\Phi_n(s)}{\Gamma}\right)^{q+1} - \left(\frac{\Psi_n(s)}{\gamma}\right)^{p+1} \|_{L^\infty(\mathbb{R}^N)} \leq Cs^{-1}$ *(the same for* ψ*).*

The proof of this fact should be much more technical than [14]. With these uniform estimates, one can do as in [14] and give a new proof of (1.12) and of the existence of a blow-up profile, with a convergence speed independent from the considered blow-up point. Therefore, we suspect one can adapt the techniques of [5] to show that the profile given by (1.12) with $l = N$ is stable with respect to perturbations in initial data.

As we mentioned before, the novelty of our paper is the Liouville Theorem. Deriving consequences for blow-up solutions (Theorems 1.4 and 1.5, Corollaries 1.6 and 1.7 and Proposition 1.8) is done in the same way as in [15]. Therefore, we focus on the proof of Theorem 1.2, and also on the proofs of Theorems 1.4 and 1.5, because the non-homogeneousness of the nonlinearity in (1.5) makes this case more delicate. For the other results, we just sketch the proofs and refer to [15] for details.

The paper is organized as follows : In section 2 we prove Theorem 1.2. In section 3, we prove applications of the Liouville Theorem (Theorems 1.4 and 1.5, Corollary 1.7 and Proposition 1.8. Note that Corollary 1.6 easily follows from Theorem 1.5). Let us mention that in Appendix B, we give a local lower bound on the blow-up rate for (1.5), in the same spirit as in [9].

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2 A Liouville Theorem for equation (1.9)

In this section, we prove Theorem 1.2. Let us first remark that the nonlinearity

$$\begin{pmatrix} \Phi \\ \Psi \end{pmatrix} \to \begin{pmatrix} \Psi^p \\ \Phi^q \end{pmatrix} \text{ in system (1.9)}$$

has no gradient structure, so that the method of [15] can not apply. More precisely, what breaks down first in the method of [15] is the proof of the existence of limits as $s \to \pm\infty$ for solutions of (1.9) defined in all $\mathbb{R}^{N+1}$, since this proof relies strongly on the existence of a Lyapunov functional for the system. The existence of limits has been proved by Andreucci, Herrero and Velázquez in [1] through a perturbation argument around the problem for the particular value $(p,q) = (p_0, p_0)$, which reduces in fact to a scalar equation (see Proposition 1.1). Theorem 1.2 will be proved if we completely characterize the case (1.11) of Proposition 1.1 and show the existence of $s_0 \in \mathbb{R}$ such that $\forall (y,s) \in \mathbb{R}^N \times \mathbb{R}$,

$$\Phi(y,s) = \Gamma(1+e^{s-s_0})^{-\frac{p+1}{pq-1}} \text{ and } \Psi(y,s) = \gamma(1+e^{s-s_0})^{-\frac{q+1}{pq-1}} \tag{2.1}$$

where Γ and γ are given by (1.10). For this, we will use ideas from [15] and [13] based on a blow-up criterion for system (1.9). Let us point out that the blow-up criterion of [15] breaks down here since we have no gradient structure, and the finite time blow-up criterion of [13] does not hold since we are not in the scalar case. Nevertheless, we have the following infinite time blow-up criterion, which is crucial for our argument.

Proposition 2.1 **(An infinite time blow-up criterion for system (1.9))** *Assume $p \geq 1$ and $q \geq 1$. Let (Φ, Ψ) be a solution of system (1.9) defined for all $(y,s) \in \mathbb{R}^N \times [s_0, +\infty)$ for some $s_0 \in \mathbb{R}$ such that $z(s_0) > 0$ where*

$$z(s) = q\gamma \int \Phi(y,s)\rho(y)dy + p\Gamma \int \Psi(y,s)\rho(y)dy - (p+q)\gamma\Gamma. \tag{2.2}$$

Then, $z(s) \to +\infty$ as $s \to +\infty$.

Proof : Roughly speaking, the conclusion follows on one hand from the fact that $z(s)$ is an eigenfunction with eigenvalue 1 of the system (1.9) linearized in the space $\mathbf{L}^2_\rho(\mathbb{R}^N)$ around the constant solution (Γ, γ), and on the other hand from the convexity of the nonlinearity, since both p and q are greater than 1. We will not linearize (1.9) here, but we will directly derive a differential inequality satisfied by $z(s)$.

From system (1.9) we write the following equation for z :

$$\begin{aligned} z'(s) &= -\tfrac{q(p+1)}{pq-1}\gamma \int \Phi(y,s)\rho(y)dy - \tfrac{p(q+1)}{pq-1}\Gamma \int \Psi(y,s)\rho(y)dy \\ &\quad + q\gamma \int \Psi(y,s)^p\rho(y)dy + p\Gamma \int \Phi(y,s)^q\rho(y)dy. \end{aligned} \tag{2.3}$$

From convexity ($p \geq 1$ and $q \geq 1$), we write

$$\Psi^p \geq \gamma^p + p\gamma^{p-1}(\Psi - \gamma) \text{ and } \Phi^q \geq \Gamma^q + q\Gamma^{q-1}(\Phi - \Gamma). \tag{2.4}$$

Plugging this in (2.3), we get (use $\int_{\mathbb{R}^N} \rho(y)dy = 1$)

$$\begin{aligned} z'(s) \ &\geq \left(\int \Phi(y,s)\rho(y)dy - \Gamma\right)\left[-\tfrac{q(p+1)}{pq-1}\gamma + pq\Gamma^q\right] \\ &+ \left(\int \Psi(y,s)\rho(y)dy - \gamma\right)\left[-\tfrac{p(q+1)}{pq-1}\Gamma + pq\gamma^p\right] \\ &- \tfrac{q(p+1)}{pq-1}\gamma\Gamma - \tfrac{p(q+1)}{pq-1}\Gamma\gamma + q\gamma^{p+1} + p\Gamma^{q+1}. \end{aligned}$$

Using (1.10) and (2.2), we end-up with

$$\forall s \geq s_0, \ z'(s) \geq z(s).$$

Since $z(s_0) > 0$, this concludes the proof of Proposition 2.1. ∎

We now consider a solution (Φ, Ψ) of (1.9) satisfying case (1.11) of Proposition 1.1, and proceed in 3 parts to completely characterize it and then finish the proof of Theorem 1.2.

- In Part I, following ideas from [13], we linearize system (1.9) around (Γ, γ) and do a kind of center manifold theory as $s \to -\infty$ to show that (Φ, Ψ) behaves at most in three different ways.
- In Part II, we show that one of these three ways actually corresponds to the case (2.1).
- In Part III, we rule out the two remaining cases using an argument based on the invariance of the system (1.9) under the following geometric transformation :

$$a \in \mathbb{R}^N \to \left[\begin{pmatrix} \Phi_a \\ \Psi_a \end{pmatrix} : (y,s) \to \begin{pmatrix} \Phi(y + ae^{s/2}, s) \\ \Psi(y + ae^{s/2}, s) \end{pmatrix}\right] \tag{2.5}$$

and the infinite time blow-up criterion of Proposition 2.1.

Let us note that with proposition 1.1, our strategy becomes quite similar to the one of [13] and [15], except for the delicate point of the blow-up criterion which is not the same. Moreover, the equations we obtain here are similar to those of [13] and [15]. Therefore, we refer to these papers for most of the proofs of the Propositions we write below.

Part I : First order asymptotic expansion of (Φ, Ψ) as $s \to -\infty$

Let us first linearize system (1.9) around (Γ,γ) as $s\to-\infty$. If we introduce

$$(\varphi,\psi)=(\Phi-\Gamma,\Psi-\gamma), \tag{2.6}$$

then we see from (1.9) that it satisfies the following system

$$\frac{\partial}{\partial s}\begin{pmatrix}\varphi\\ \psi\end{pmatrix}=(\mathbb{L}_0 Id+M)\begin{pmatrix}\varphi\\ \psi\end{pmatrix}+F\begin{pmatrix}\varphi\\ \psi\end{pmatrix} \tag{2.7}$$

(2.8)

where

$$\mathbb{L}_0=\Delta-\frac{1}{2}y.\nabla,$$

Id is the identity of $\mathbb{R}^2$,

$$M=\begin{pmatrix}-\frac{p+1}{pq-1} & p\gamma^{p-1}\\ q\Gamma^{q-1} & -\frac{q+1}{pq-1}\end{pmatrix} \tag{2.9}$$

and

$$F\begin{pmatrix}\varphi\\ \psi\end{pmatrix}=\begin{pmatrix}(\gamma+\psi)^p-\gamma^p-p\gamma^{p-1}\psi\\ (\Gamma+\varphi)^q-\Gamma^q-q\Gamma^{q-1}\varphi\end{pmatrix}$$

satisfies $|F(\psi,\varphi)|\le C(|\varphi|^2+|\psi|^2)$ since $\|(\varphi,\psi)\|_{L^\infty(\mathbb{R}^N\times\mathbb{R})}<+\infty$.

One can easily compute that M has two eigenvalues $\lambda_1=1$ and $\lambda_2=-\frac{(p+1)(q+1)}{pq-1}$ with as eigenvectors respectively

$$e_1=\begin{pmatrix}(p+1)\Gamma\\ (q+1)\gamma\end{pmatrix}\text{ and } e_2=\begin{pmatrix}p\Gamma\\ -q\gamma\end{pmatrix}. \tag{2.10}$$

If we perform the following change of functions

$$\begin{pmatrix}\varphi\\ \psi\end{pmatrix}=ge_1+he_2, \tag{2.11}$$

then the linear part of system (2.7) uncouples and (g,h) satisfies

$$\begin{aligned}\frac{\partial g}{\partial s}&=(\mathbb{L}_0+1)g+F_1(g,h)\\ \frac{\partial h}{\partial s}&=(\mathbb{L}_0-\frac{(p+1)(q+1)}{pq-1})h+F_2(g,h)\end{aligned} \tag{2.12}$$

for some $F_i : \mathbb{R}^2 \to \mathbb{R}$, $i = 1, 2$ satisfying

$$|F_i(g,h)| \le C(|g|^2 + |h|^2). \tag{2.13}$$

Moreover, from (2.11), (2.6) and (1.11), we have

$$\|g(s)\|_{L^2_\rho} \to 0 \text{ and } \|h(s)\|_{L^2_\rho} \to 0 \text{ as } s \to -\infty. \tag{2.14}$$

In all this part, we shall study system (2.12) which is equivalent to (1.9). The dynamics of (2.12) are mainly determined by its linear part. Let us study it in the following.

$\mathbb{L}_0$ is a self-adjoint operator on $\mathbb{D}(\mathbb{L}_0) \subset L^2_\rho(\mathbb{R}^N, \mathbb{R})$. Its spectrum consists of eigenvalues :

$$\text{spec } \mathbb{L}_0 = \{-\frac{m}{2} \mid m \in \mathbb{N}\}.$$

The eigenspace of $\lambda = -\frac{m}{2}$ is finite dimensional and is spanned by

$$H_\alpha(y) = h_{\alpha_1}(y_1)...h_{\alpha_N}(y_N) \tag{2.15}$$

where $\alpha = (\alpha_1, ..., \alpha_N) \in \mathbb{N}^N$ satisfies $|\alpha| = \alpha_1 + ... + \alpha_N = m$ and

$$h_n(\xi) = \sum_{k=0}^{[\frac{n}{2}]} \frac{n!}{k!(n-2k)!}(-1)^k \xi^{n-2k}$$

are dilatations of Hermite polynomials. The family $(H_\alpha)_{\alpha \in \mathbb{N}^N}$ spans all the space
$L^2_\rho(\mathbb{R}^N, \mathbb{R})$, and we can write for every $v \in L^2_\rho(\mathbb{R}^N, \mathbb{R})$

$$v(y) = \sum_{m \in \mathbb{N}} v_m(y) \text{ where } v_m(y) = \sum_{|\alpha| = m} v_\alpha H_\alpha(y) \tag{2.16}$$

is the $\mathbf{L}^2_\rho$ projection of v on the eigenspace of $\mathbb{L}_0$ associated with $\lambda = -\frac{m}{2}$ and $v_\alpha \in \mathbb{R}$.

If we consider equation (2.12) under this spectral information for $\mathbb{L}_0$, we see that the linear operator for the second equation is negative, whereas the spectrum of the linear operator $\mathbb{L}_0 + 1$ of the first equation contains a positive part (1 and $\frac{1}{2}$), a null and a negative one. If we define

$$g_+(y,s) = \sum_{m=0}^{1} g_m(y,s), \; g_{null} = g_2 \text{ and } g_- = \sum_{m \ge 3} g_m H_\alpha, \tag{2.17}$$

then one can follow ideas from center manifold theory and use (2.12), (2.13), (2.14), and perform in a straightforward way the same type of estimates as in the scalar case in [13] (Proposition 3.5) to show the following :

Proposition 2.2 **(Finite dimensional reduction of the problem as $s \to -\infty$)** *As $s \to -\infty$,*

$$\text{(2.18)} \quad \textit{either} \quad \|h(s)\|_{L^2_\rho} + \|g_-(s)\|_{L^2_\rho} + \|g_+(s)\|_{L^2_\rho} = o\left(\|g_{null}(s)\|_{L^2_\rho}\right)$$

$$\text{(2.19)} \quad \textit{or} \quad \|h(s)\|_{L^2_\rho} + \|g_-(s)\|_{L^2_\rho} + \|g_{null}(s)\|_{L^2_\rho} = o\left(\|g_+(s)\|_{L^2_\rho}\right).$$

Remark : Center manifold theory can not apply for the nonlinear term $F_i(g,h)$ is not quadratic with respect to the $L^2_\rho(\mathbb{R}^N)$ or $H^1_\rho(\mathbb{R}^N)$ norm (see [4] page 834-835 for more details).

Proof of Proposition 2.2 : The steps 1 and 2 in Appendix A in [13] can be adapted in a straightforward way to handle this vectorial case. ∎

In the following Proposition, we use Proposition 2.2 to reduce the study of (2.12) to a finite dimensional ODE problem and find first order expansions for (g,h) as $s \to -\infty$. More precisely, we show that when (2.19) occurs in Proposition 2.2, then either g_0 or g_1 dominates the other. Therefore, we reduce to the study of the projection of g on the eigenspace of $(\mathbb{L}_0 + 1)$ spanned by 1, 1/2 or 0, which is finite dimensional.

Proposition 2.3 **(First order expansion for (g,h) as $s \to -\infty$)** *As $s \to -\infty$, one of the following cases occurs :*
i) $\|h(s)\|_{L^2_\rho} + \|g_-(s)\|_{L^2_\rho} + \|g_{null}(s)\|_{L^2_\rho} + |g_1(s)| = o\left(g_0(s)\right)$ *and* $g_0(s) = C_0 e^s + O\left(e^{2(1-\epsilon)s}\right)$ *as $s \to -\infty$, for some $C_0 \in \mathbb{R}$ and for all $\epsilon > 0$.*
ii) $\|h(s)\|_{L^2_\rho} + \|g_-(s)\|_{L^2_\rho} + \|g_{null}(s)\|_{L^2_\rho} + |g_0(s)| = o\left(g_1(s)\right)$, $g_1(s) \sim C_1 e^{s/2}$ *and* $g_0(s) = o\left(se^s\right)$ *as $s \to -\infty$, for some $C_1 \in \mathbb{R}^N \backslash \{0\}$.*
iii) $\|h(s)\|_{L^2_\rho} + \|g_-(s)\|_{L^2_\rho} + \|g_+(s)\|_{L^2_\rho} = o\left(\|g_{null}(s)\|_{\mathbf{L}^2_\rho}\right)$ *and there exists $l \in \{1,...,N\}$ and Q an orthonormal $N \times N$ matrix such that* $g_{null}(Qy,s) = \frac{2pq+p+q}{8(p+1)(q+1)pqs}\left(2l - \sum_{i=1}^{l} y_i^2\right) + O\left(\frac{1}{s^{1+\delta}}\right)$ *in L^2_ρ,* $g_1(s) = O\left(\frac{1}{s^2}\right)$ *and* $g_0(s) = O\left(\frac{1}{s^2}\right)$ *as $s \to -\infty$, for some $\delta > 0$.*

Proof : From Proposition 2.2, we see that g dominates h in the L^2_ρ norm. Therefore, we concentrate on the study of the equation satisfied by g. From (1.9), (2.6) and (2.11), g satisfies the equation : $\forall (y,s)\in\mathbb{R}^N\times\mathbb{R}$

$$\frac{\partial g}{\partial s}=(\mathbb{L}_0+1)\,g+F_1(g,h)$$

where $|F_1(g,h)-\frac{(p+1)(q+1)pq}{2pq+p+q}g^2|\le C\left(\left(g^2+h^2\right)^{1/2}|h|+C\left(g^2+h^2\right)^{3/2}\right)$. Let us remark that this case is very similar to the scalar case studied in [13] where the linearized equation of (1.9) around $\kappa=(p-1)^{-\frac{1}{p-1}}$ is : $\forall (y,s)\in\mathbb{R}^N\times\mathbb{R}$,

$$\frac{\partial v}{\partial s}=(\mathbb{L}_0+1)\,v+F(v)\text{ with }|F(v)-\frac{p}{2\kappa}v^2|\le C|v|^3.$$

We claim that since $\|h(s)\|_{L^2_\rho}=o\left(\|g(s)\|_{L^2_\rho}\right)$ as $s\to-\infty$ in our case, the asymptotic study of v in [13] holds with obvious changes for the proof of Proposition 2.3. See in [13] Step 3 in the proof of Proposition 3.5 (Appendix A) and the proof of Proposition 3.10 (Appendix C). ∎

Part II : The relevant case : Characterization of the L^∞_{loc} connection between (Γ,γ) and $(0,0)$ in (1.9)

In this Part, we prove the following Proposition :

Proposition 2.4 **(Case i) of Proposition 2.3 : the relevant case)** *Assume that case i) of Proposition 2.3 holds. Then, there exists $s_0\in\mathbb{R}$ such that $\forall (y,s)\in\mathbb{R}^N\times\mathbb{R}$, $\Phi(y,s)=\Phi_0(s-s_0)$ and $\Psi(y,s)=\Psi_0(s-s_0)$ where $\Phi_0(s)=\Gamma(1+e^s)^{-\frac{p+1}{pq-1}}$ and $\Psi_0(s)=\gamma(1+e^s)^{-\frac{q+1}{pq-1}}$.*

Proof : Through the transformations (2.11) and (2.6) and the definitions (2.16) and (2.17), i) of Proposition 2.3 reads

$$\begin{aligned}&\|\Phi(y,s)-\{\Gamma+(p+1)\Gamma C_0e^s\}\,H_0(y)\|_{L^2_\rho}=o(e^s)\\&\|\Psi(y,s)-\{\gamma+(q+1)\gamma C_0e^s\}\,H_0(y)\|_{L^2_\rho}=o(e^s)\end{aligned}\tag{2.20}$$

as $s\to-\infty$ with $H_0(y)=1$ (see (2.15)). Let us remark that we already have a solution (Φ^*,Ψ^*) of (1.9) defined in $\mathbb{R}^N\times(-\infty,s_*]$ for some $s_*\in\mathbb{R}$ and which satisfies the same expansion :

(2.21)- if $C_0=0$, just take (Γ,γ),

(2.22)- if $C_0<0$, take $(\Phi_0,\Psi_0)(s-s_0)$ where $s_0=\log(-C_0(pq-1))$

(2.23)- if $C_0>0$, take $(\Phi_b,\Psi_b)(s-s_0)$ where $s_0=\log(C_0(pq-1))$ and $(\Phi_b,\Psi_b)(s)=\left(\Gamma(1-e^s)^{-\frac{p+1}{pq-1}},\gamma(1-e^s)^{-\frac{q+1}{pq-1}}\right)$

is a solution of (1.9) which blows-up at $s = 0$ but is bounded for all $s \leq -1$. Since the expansion of (Φ, Ψ) is supported by H_0 which is the one dimensional eigenspace of $\mathbb{L}_0 + 1$, one expects from a dimension argument that (Φ^*, Ψ^*) is the only solution satisfying (2.20).

In the following, we will prove that $(\Phi, \Psi) \equiv (\Phi^*, \Psi^*)$ on $\mathbb{R}^N \times (-\infty, s_*]$. For this, we will linearize (1.9) around (Φ^*, Ψ^*) and not around (Γ, γ) as we did in Part I. Note that since

$$\forall s \leq s^*, \ \ |(\Phi^*, \Psi^*) - (\Gamma, \gamma)| \leq Ce^s,$$

one expects to have the same equations as in Part I, up to a perturbation of size e^s.

Let us introduce for all $(y, s) \in \mathbb{R}^N \times (-\infty, s^*]$

$$(\varphi_1, \psi_1) = (\Phi - \Phi^*, \Psi - \Psi^*). \tag{2.24}$$

From (1.9) and (2.20), we see that

$$\|\varphi_1(s)\|_{L^2_\rho} + \|\psi_1(s)\|_{L^2_\rho} = o(e^s) \text{ as } s \to -\infty, \tag{2.25}$$

$$\frac{\partial}{\partial s}\begin{pmatrix} \varphi_1 \\ \psi_1 \end{pmatrix} = (\mathbb{L}_0 Id + M + L(s))\begin{pmatrix} \varphi_1 \\ \psi_1 \end{pmatrix} + F_1\begin{pmatrix} \varphi_1 \\ \psi_1 \end{pmatrix} \tag{2.26}$$

where M is given in (2.9) and $L(s)$ is a 2×2 matrix satisfying $|L_{i,j}(s)| \leq Ce^s$ and $|F_1(\varphi_1, \psi_1)| \leq C(\varphi_1^2 + \psi_1^2)$.

Using the same argument as in the proof of Proposition 3.7 in [13], we prove that (2.25) and (2.26) imply that $(\varphi_1, \psi_1) \equiv (0, 0)$ for all $s \leq s_*$. Therefore, $\forall (y, s) \in \mathbb{R}^N \times (-\infty, s^*)$

$$(\Phi(y, s), \Psi(y, s)) = (\Phi^*(s), \Psi^*(s)). \tag{2.27}$$

From the uniqueness of the Cauchy problem for equation (1.9) and since (Φ, Ψ) is defined for all $(y, s) \in \mathbb{R}^N \times \mathbb{R}$, (Φ^*, Ψ^*) is also defined for all $(y, s) \in \mathbb{R}^N \times \mathbb{R}$, and (2.27) holds for all $(y, s) \in \mathbb{R}^N \times \mathbb{R}$. Therefore, case (2.23) can not hold. Moreover, (2.21) is ruled out by (1.11). Thus, only case (2.22) holds and there exists $s_0 \in \mathbb{R}$ such that $\forall s \in \mathbb{R}$,

$$(\Phi(y, s), \Psi(y, s)) = (\Phi^*(s), \Psi^*(s)) = (\Phi_0, \Psi_0)(s - s_0).$$

This finishes the proof of Proposition 2.4. ∎

Part III : Cases ii) and iii) of Proposition 2.3

In this Part, we assume that case ii) or iii) of Proposition 2.3 holds and use the invariance of system (1.9) under the geometric transformation (2.5) to show the existence of some $a_0 \in \mathbb{R}^N$ and $s_0 \in \mathbb{R}$ such that $(\Phi_{a_0}, \Psi_{a_0})(s_0)$ satisfies the infinite blow-up criterion of Proposition 2.1. More precisely, we have the following Proposition :

PROPOSITION 2.5 **(The irrelevant cases ii) and iii) of Proposition 2.3)** *Assume that case ii) or case iii) of Proposition 2.3 holds, then,*
a- In case ii) :
$\int g(y+ae^{s/2},s)\rho(y)dy = a.C_1 e^s + o(|a|e^s) + O(se^s)$ *as* $(ae^{\frac{s}{2}}, s) \to (0,-\infty)$.
b- In case iii) :
$\int g(y+ae^{s/2},s)\rho(y)dy = \frac{2pq+p+q}{16(p+1)(q+1)pq|s|}\sum_{i=1}^{l}\int (z_i^2-2)(Qae^{s/2}.z)^2\rho(z)dz + O\left(s^{-2}\right) + O\left(|a|^2 e^s |s|^{-1-\delta}\right) + O\left(|a|^3 e^{3s/2} s^{-1}\right)$ *as* $(ae^{s/2}, s) \to (0,-\infty)$.

PROOF : Using Proposition 2.3, the proof of Lemma 2.6 of [15] holds here with no adaptations. ∎

This Proposition allows us to conclude. Indeed,

- if case ii) of Proposition 2.3 holds, then we fix s_0 negative enough and $a_0 = \frac{e^{-s_0/2}}{|s_0|}\frac{C_1}{|C_1|}$ to get

$$\int g(y+a_0 e^{s_0/2}, s_0)\rho(y)dy \geq \frac{1}{2}e^{s_0} a_0.C_1 = \frac{e^{s_0/2}}{2|s_0|}|C_1| > 0.$$

This implies through (2.11), (2.6) and (2.5) that

$$q\gamma\int_{\mathbb{R}^N}\Phi_{a_0}(y,s_0)\rho(y)dy + p\Gamma\int \Psi_{a_0}(y,s_0)\rho(y)dy > (p+q)\gamma\Gamma$$

where (Φ_{a_0}, Ψ_{a_0}) is also a solution of (1.9) defined from (Φ,Ψ) through the geometrical transformation (2.5). From Proposition 2.1, (Φ_{a_0}, Ψ_{a_0}) blows-up in infinite time. This contradicts the fact that $\|(\Phi_{a_0}, \Psi_{a_0})\|_{\mathbf{L}^\infty(\mathbb{R}^N\times\mathbb{R})} = \|(\Phi,\Psi)\|_{\mathbf{L}^\infty(\mathbb{R}^N\times\mathbb{R})} < +\infty$. Thus, case ii) of Proposition 2.3 does not hold.

- if case iii) of Proposition 2.3 holds, then we fix s_0 negative enough and $a_0 = \frac{e^{-s_0/2}}{|s_0|^{1/4}}Q^{-1}\epsilon_1$ where $\epsilon_1 = (1,0,...,0)$ so that we get

$$\int g(y+a_0 e^{s_0/2}, s_0)\rho(y)dy \geq \frac{1}{2}\frac{2pq+p+q}{16|s|(p+1)(q+1)pq} \times 8 > 0.$$

This leads to a contradiction by the same argument as before. Therefore, case iii) of Proposition 2.3 can not hold.

This concludes the characterization of case (1.11) in Proposition 1.1 and concludes the proof of Theorem 1.2.

3 Uniform estimates and uniform comparison with an ODE of blow-up solutions of (1.5)

We derive in this section applications of the Liouville Theorem for blow-up solutions of (1.5). Basically, the techniques are the same as we did in [15] for (1.3). However, the fact that the nonlinearity in (1.5) is non homogeneous makes the proof more complicated technically, at least for Theorems 1.4 and 1.5. Therefore, we give the details of Theorems 1.4 and 1.5. On the contrary, we just sketch the proofs of Corollary 1.7 and Proposition 1.8. Let us first recall an upper bound on the blow-up rate for (1.5) from [1].

3.1 An upper bound on the blow-up rate for (1.5)

If one carefully reads the proof of Theorem 1 in [1], then he sees that the result is actually stronger than stated there. We state it in the following Proposition :

PROPOSITION 3.1 (Uniform $\mathbf{L}^\infty$ bound for t near T from [1]) *Assume (1.6) and consider (u,v) a solution of (1.5) which blows-up at time T. Then, for all $\delta \in (0,1)$, there exists $C_1(\delta) > 0$ such that for all $(x,t) \in \mathbb{R}^N \times [\frac{\delta}{1+\delta}T, T)$,*

$$(T-t)^{\frac{p+1}{pq-1}}\|u(t)\|_{\mathbf{L}^\infty} + (T-t)^{\frac{q+1}{pq-1}}\|v(t)\|_{\mathbf{L}^\infty} \le C_1.$$

Using the continuity of solutions to (1.5) with respect to initial data in L^∞, we then claim the following :

PROPOSITION 3.2 (Uniform L^∞ estimate) *Assume (1.6) and consider $C_0 > 0$ and $T_0 > 0$. Then, there exists $C(C_0, T_0) > 0$ such that for all solution (u,v) of (1.5) which blows-up at time T and satisfies*

$$T \le T_0 \text{ and } \|u(0)\|_{\mathbf{L}^\infty(\mathbb{R}^N)} + \|v(0)\|_{\mathbf{L}^\infty(\mathbb{R}^N)} \le C_0,$$

we have $\forall t \in [0,T)$,

$$\|u(t)\|_{\mathbf{L}^\infty(\mathbb{R}^N)} \le C(T-t)^{-\frac{p+1}{pq-1}} \text{ and } \|v(t)\|_{\mathbf{L}^\infty(\mathbb{R}^N)} \le C(T-t)^{-\frac{q+1}{pq-1}}.$$

Proof : The argument is similar to the one used in [15] to derive Theorem 2 from Proposition 3.4. ∎

3.2 Limits at blow-up of $\mathbf{L}^\infty$ estimates for solutions of (1.5)

We prove Proposition 1.4 here. Let us first note that $T_n \geq t_0(C_0) > 0$. This follows from the following Lemma.

Lemma 3.3 (Boundedness of the solution in $\mathbf{L}^\infty$) *Consider (u,v) a solution of (1.5) satisfying $\|u(0)\|_{L^\infty} + \|v(0)\|_{L^\infty} \leq C_0$. Then, there exists $t_0(C_0) > 0$ such that for all $t \in [0, t_0]$,* $\max(\|u(t)\|_{L^\infty}, \|v(t)\|_{L^\infty}) \leq 2C_0$.

Proof : Omitted. See Lemma 3.1 in [15] for a similar argument. ∎

We prove here that

$$\tau^{\frac{p+1}{pq-1}}\|u_n(T_n-\tau)\|_{\mathbf{L}^\infty} \to \Gamma \text{ as } \tau \to 0 \text{ uniformly in } n. \tag{3.1}$$

The proof of the other estimates follows in the same way (see Theorem 1.1 in [13] for a similar case). We proceed by contradiction and assume that for some $\epsilon_0 > 0$, there exists $\tau_n \to 0$ as $n \to +\infty$ and (u_n, v_n) a blow-up solution of (1.5) satisfying (1.13) and either $\tau_n^{\frac{p+1}{pq-1}}\|u_n(T_n-\tau_n)\|_{\mathbf{L}^\infty} \geq \Gamma+\epsilon_0$ or $\tau_n^{\frac{p+1}{pq-1}}\|u_n(T_n-\tau_n)\|_{\mathbf{L}^\infty} \leq \Gamma-\epsilon_0$. Let us mention the following classical lower bound on the blow-up rate for equation (1.5).

Proposition 3.4 (A lower bound on the blow-up rate) *Assume that $pq > 1$ and consider a solution (u,v) of (1.5) which blows-up at time T. Then, for all $t \in [0,T)$,*

$$\textit{either } \|u(t)\|_{\mathbf{L}^\infty} \geq \Gamma(T-t)^{-\frac{p+1}{pq-1}} \textit{ or } \|v(t)\|_{\mathbf{L}^\infty} \geq \gamma(T-t)^{-\frac{q+1}{pq-1}}.$$

Proof : See Appendix A. ∎

Up to extracting a subsequence and from Proposition 3.4, we can assume that

$$\text{either} \qquad \forall n \in \mathbb{N},\quad \tau_n^{\frac{p+1}{pq-1}}\|u_n(T_n-\tau_n)\|_{\mathbf{L}^\infty} \geq \Gamma+\epsilon_0, \tag{3.2}$$

or

(3.3)

$$\forall n \in \mathbb{N},\quad \tau_n^{\frac{p+1}{pq-1}}\|u_n(T_n-\tau_n)\|_{\mathbf{L}^\infty} \leq \Gamma-\epsilon_0 \text{ and } \tau_n^{\frac{q+1}{pq-1}}\|v_n(T_n-\tau_n)\|_{\mathbf{L}^\infty} \geq \gamma$$

By Proposition 3.2, we have for all $n \in \mathbb{N}$ and $t \in [0, T_n)$,

$$\|u_n(t)\|_{\mathbf{L}^\infty}(T_n - t)^{\frac{p+1}{pq-1}} + \|v_n(t)\|_{\mathbf{L}^\infty}(T_n - t)^{\frac{q+1}{pq-1}} \le C^*(C_0, T_0). \tag{3.4}$$

Let us define

$$\begin{aligned} U_n(\xi, \tau) &= \tau_n^{\frac{p+1}{pq-1}} u_n\left(\xi\sqrt{\tau_n}, T_n - \tau_n + \tau\tau_n\right) \\ V_n(\xi, \tau) &= \tau_n^{\frac{q+1}{pq-1}} v_n\left(\xi\sqrt{\tau_n}, T_n - \tau_n + \tau\tau_n\right) \end{aligned} \tag{3.5}$$

which is still a solution of (1.5) defined for all $(\xi, \tau) \in \mathbb{R}^N \times [\frac{\tau_n - T_n}{\tau_n}, 1)$. By (3.4), we have for all $n \in \mathbb{N}$ and $(\xi, \tau) \in \mathbb{R}^N \times [-\frac{\tau_n - T_n}{\tau_n}, 1)$,

$$U_n(\xi, \tau) \le C^*(1 - \tau)^{-\frac{p+1}{pq-1}} \text{ and } V_n(\xi, \tau) \le C^*(1 - \tau)^{-\frac{q+1}{pq-1}}. \tag{3.6}$$

Since $T_n \ge t_0(C_0)$ and $\tau_n \to 0$, we have $\frac{\tau_n - T_n}{\tau_n} \to -\infty$ as $n \to +\infty$. We claim the following parabolic regularity result :

Lemma 3.5 (Parabolic regularity for system (1.5)) *Assume (u, v) is a solution of (1.5) defined for all $(\xi, \tau) \in D = B(0, \eta) \times [0, t^*]$ and satisfying $\|u\|_{\mathbf{L}^\infty(D)} + \|v\|_{\mathbf{L}^\infty(D)} \le M$. Consider $t_1 \in (0, t^*)$, then, there exists $\alpha \in (0, 1)$ and $K(t_1, \eta, M)$ such that*

$$\|u\|_{\mathbf{C}^{2,1}(D')} + |\nabla^2 u|_{\alpha, D'} + |\partial_t u|_{\alpha, D'} \le K$$

where $D' = B(0, \frac{\eta}{2}) \times [t_1, t^*]$, $\|u\|_{\mathbf{C}^{2,1}} = \|u\|_{\mathbf{L}^\infty} + \|\nabla u\|_{\mathbf{L}^\infty} + \|\nabla^2 u\|_{\mathbf{L}^\infty} + \|\partial_t u\|_{\mathbf{L}^\infty}$ and

$$|a|_{\alpha, D'} = \sup_{(\xi,\tau),(\xi,\tau') \in D'} \frac{|a(\xi, \tau) - a(\xi', \tau')|}{\left(|\xi - \xi'| + |\tau - \tau'|^{1/2}\right)^\alpha}. \tag{3.7}$$

The same holds for v.

Proof : This is a consequence of a result by Friedman (Theorem 3 p. 406 in [7]). See Lemma 2.8 in [14] for a proof (Although the proof of [14] is given for a scalar heat equation, it extends naturally to the case of the system (1.5)). ∎

From this Lemma and (3.6), we get compactness of (U_n, V_n) in every compact set of $\mathbb{R}^N \times (-\infty, 1)$ and find (U, V) a solution of (1.5) defined for all $(\xi, \tau) \in \mathbb{R}^N \times (-\infty, 1)$ and satisfying

$$U(\xi, \tau) \le C^*(1 - \tau)^{-\frac{p+1}{pq-1}} \text{ and } V(\xi, \tau) \le C^*(1 - \tau)^{-\frac{q+1}{pq-1}}$$

such that (up to a subsequence) $(U_n, V_n) \to (U, V)$ uniformly on compact sets. Moreover, from (3.2) and (3.3), we have either $\|U(0)\|_{\mathbf{L}^\infty} \geq \Gamma + \epsilon_0$, or $\|U(0)\|_{\mathbf{L}^\infty} \leq \Gamma - \epsilon_0$ and $\|V(0)\|_{\mathbf{L}^\infty} \geq \gamma$. By the Liouville Theorem of Corollary 1.3, (U, V) does not depend on ξ and there is a contradiction in both cases. Therefore, (3.1) holds.

The proof of the estimates on $\left(\frac{u_n}{\Gamma}\right)^{q+1} - \left(\frac{v_n}{\gamma}\right)^{p+1}$, v_n and the derivatives of u_n and v_n follow by the same compactness argument which yields in all cases a contradiction by Corollary 1.3, because all the solutions given by this corollary satisfy $\nabla U \equiv \nabla V \equiv 0$, $\nabla^2 U \equiv \nabla^2 V \equiv 0$ and $\left(\frac{U}{\Gamma}\right)^{q+1} - \left(\frac{V}{\gamma}\right)^{p+1} \equiv 0$. This concludes the proof of Proposition 1.4.

3.3 Uniform ODE comparison of the solutions of (1.5)

We prove Theorem 1.5 here and omit the proof of Corollary 1.6 since it is obvious if one assumes Theorem 1.5.

PROOF OF THEOREM 1.5 : We proceed by contradiction and consider some $\epsilon_0 > 0$ and a sequence (u_n, v_n) of solutions of (1.5) such that (u_n, v_n) blows-up at time T_n and satisfies

$$T_n \leq T_0, \ \ \|u_n(0)\|_{\mathbf{C}^2(\mathbb{R}^N)} + \|v_n(0)\|_{\mathbf{C}^2(\mathbb{R}^N)} \leq C_0 \text{ and} \tag{3.8}$$

$$\begin{array}{ll} \text{either} & |\Delta u_n(x_n, t_n)| \geq \epsilon_0 v_n^p + n \text{ or } |\Delta v_n(x_n, t_n)| \geq \epsilon_0 u_n^q + n \\ \text{or} & \left| \left(\frac{u_n(x_n,t_n)}{\Gamma}\right)^{q+1} - \left(\frac{v_n(x_n,t_n)}{\gamma}\right)^{p+1} \right| \geq \epsilon_0 u_n(x_n, t_n)^{q+1} + n \end{array}$$

for some $(x_n, t_n) \in \mathbb{R}^N \times [0, T_n)$. From translation invariance of (1.5), we assume $x_n = 0$. Since the roles of (u, p) and (v, q) are symmetric, we can assume, up to extracting a subsequence, that

$$\text{either} \quad \forall n \in \mathbb{N}, \ \ |\Delta u_n(0, t_n)| \geq \epsilon_0 v_n(0, t_n)^p + n \tag{3.9}$$

$$\text{or} \ \forall n \in \mathbb{N}, \ \ \left| \left(\frac{u_n(0,t_n)}{\Gamma}\right)^{q+1} - \left(\frac{v_n(0,t_n)}{\gamma}\right)^{p+1} \right| \geq \epsilon_0 u_n(0, t_n)^{q+1} + n. \tag{3.10}$$

We proceed in three steps.

- In Step 1, we show that $(T_n - t_n)^{\frac{p+1}{pq-1}} u_n(0, t_n) + (T_n - t_n)^{\frac{q+1}{pq-1}} v_n(0, t_n) \to 0$.
- In Step 2, we use this to show that up to a scaling of the type

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} \to \begin{pmatrix} u_{n,\lambda_n,\sigma_n} \\ v_{n,\lambda_n,\sigma_n} \end{pmatrix} (\xi, \tau) = \begin{pmatrix} \lambda_n^{\frac{2(p+1)}{pq-1}} u_n(\lambda_n \xi, \lambda_n^2 \tau + \sigma_n) \\ \lambda_n^{\frac{2(q+1)}{pq-1}} v_n(\lambda_n \xi, \lambda_n^2 \tau + \sigma_n) \end{pmatrix} \tag{3.11}$$

which keeps (1.5) invariant, (u_n, v_n) is flat around $(0, t_n)$.
- In Step 3, we find a contradiction with (3.9) and (3.10).

Step 1 : $(T_n - t_n)^{\frac{p+1}{pq-1}} u_n(0, t_n) + (T_n - t_n)^{\frac{q+1}{pq-1}} v_n(0, t_n) \to 0.$
We prove the following Lemma in this step :

LEMMA 3.6 *For all $n \in \mathbb{N}$,*
i) $\forall t \in [0, T_n)$, $(T_n - t)^{\frac{p+1}{pq-1}} \|u_n(t)\|_{\mathbf{L}^\infty(\mathbb{R}^N)} + (T_n - t)^{\frac{q+1}{pq-1}} \|v_n(t)\|_{\mathbf{L}^\infty(\mathbb{R}^N)} \le C^*(C_0, T_0)$.
ii) For $i = 1, 2$, $\forall t \in [0, T_n)$,

$$(T_n - t)^{\frac{p+1}{pq-1} + \frac{i}{2}} \|\nabla^i u_n(t)\|_{\mathbf{L}^\infty} + (T_n - t)^{\frac{q+1}{pq-1} + \frac{i}{2}} \|\nabla^i v_n(t)\|_{\mathbf{L}^\infty} \le C_1(C_0, T_0).$$

iii) $T_n - t_n \to 0$. More precisely, $T_n - t_n \le \left(\frac{C_1}{n}\right)^{\frac{pq-1}{p(q+1)}}$ if (3.9) holds, and $T_n - t_n \le \left(\frac{C_2}{n}\right)^{\frac{pq-1}{(p+1)(q+1)}}$ for some $C_2 > 0$ if (3.10) holds.
iv) $(T_n - t_n)^{\frac{p+1}{pq-1}} u_n(0, t_n) + (T_n - t_n)^{\frac{q+1}{pq-1}} v_n(0, t_n) \to 0$ as $n \to +\infty$.

PROOF : i) is a consequence of (3.8) and the uniform estimates of Proposition 3.2.
ii) Consider (Φ_n, Ψ_n) the solution of (1.9) defined by (1.8) from u_n, v_n and T_n. From i), we have $\|\Phi_n\|_{\mathbf{L}^\infty} + \|\Psi_n\|_{\mathbf{L}^\infty} \le C^*$. Just use (3.8) and parabolic regularity to get ii).

iii) - Case (3.9) : From (3.9) and ii) we have
$n \le |\Delta u_n(0, t_n)| \le C_1 (T_n - t_n)^{-\frac{p(q+1)}{pq-1}}$.
- Case (3.10) : From (3.10) and i), we have
$n \le \left(\frac{u_n(0,t_n)}{\Gamma}\right)^{q+1} + \left(\frac{v_n(0,t_n)}{\gamma}\right)^{p+1}$
$\le \left(\left(\frac{C^*}{\Gamma}\right)^{q+1} + \left(\frac{C^*}{\gamma}\right)^{p+1}\right)(T_n - t_n)^{-\frac{(p+1)(q+1)}{pq-1}}$. In both cases, this yields iii).
iv) Let us define for all $(\xi, \tau) \in \mathbb{R}^N \times [-\frac{t_n}{T_n - t_n}, 1)$,

$$\begin{aligned} U_n(\xi, \tau) &= (T_n - t_n)^{\frac{p+1}{pq-1}} u_n\left(\xi\sqrt{T_n - t_n}, t_n + \tau(T_n - t_n)\right) \\ V_n(\xi, \tau) &= (T_n - t_n)^{\frac{q+1}{pq-1}} v_n\left(\xi\sqrt{T_n - t_n}, t_n + \tau(T_n - t_n)\right). \end{aligned} \tag{3.12}$$

(U_n, V_n) is still a solution of (1.5). From i), we have for all $n \in \mathbb{N}$ and $(\xi, \tau) \in \mathbb{R}^N \times [-\frac{t_n}{T_n - t_n}, 1)$,

$$0 \le U_n(\xi, \tau) \le C^*(1 - \tau)^{-\frac{p+1}{pq-1}} \text{ and } 0 \le V_n(\xi, \tau) \le C^*(1 - \tau)^{-\frac{q+1}{pq-1}}.$$

Note that $-\frac{t_n}{T_n-t_n} \to -\infty$ since $t_n \geq t_0(C_0)$ (Lemma 3.3) and iii) holds. Using Lemma 3.5 and a compactness procedure, we obtain (U,V) a solution of (1.5) defined for all $(\xi,\tau) \in \mathbb{R}^N \times (-\infty, 1)$ and satisfying

$$(3.13) \quad 0 \leq U(\xi,\tau) \leq C^*(1-\tau)^{-\frac{p+1}{pq-1}} \text{ and } 0 \leq V(\xi,\tau) \leq C^*(1-\tau)^{-\frac{q+1}{pq-1}}$$

such that $(U_n, V_n) \to (U,V)$ uniformly on compact sets of $\mathbb{R}^N \times (-\infty,1)$ (up to a subsequence). By (3.9) and (3.10), we have either $\epsilon_0 V(0,0)^p \leq |\Delta U(0,0)|$ or
$\epsilon_0 U(0,0)^{q+1} \leq \left| \left(\frac{U(0,0)}{\Gamma}\right)^{q+1} - \left(\frac{V(0,0)}{\gamma}\right)^{p+1} \right|$. From Corollary 1.3, we have

$$\text{either } U \equiv V \equiv 0 \text{ or } (U,V)(\tau) = \left(\Gamma(T-\tau)^{-\frac{p+1}{pq-1}}, \gamma(T-\tau)^{-\frac{q+1}{pq-1}}\right)$$

for some $T \geq 1$. Therefore, $U \equiv V \equiv 0$ and $U_n(0,0) + V_n(0,0) \to 0$ as $n \to +\infty$, which concludes the proof of Lemma 3.6. ∎

Step 2 : Flatness of (u_n, v_n) around $(0, t_n)$

In this step, we find a (3.11) type scaling of (u_n, v_n) and show that (u_n, v_n) is flat in this scale.
With Lemma 3.6, we are able to introduce for all $\delta > 0$ and n large,

(3.14)
$$t'_{\delta,n} = \inf\left\{ t \in [s_n, t_n] \mid (T_n - t)^{\frac{p+1}{pq-1}} u_n(0,t) + (T_n - t)^{\frac{q+1}{pq-1}} v_n(0,t) \leq \delta \right\}$$

$$(3.15) \qquad \text{where } s_n = T_n - n^{-\frac{pq-1}{2(p+1)(q+1)}}.$$

$$(3.16) \qquad \text{It is clear that } T_n - t'_{\delta,n} \to 0 \text{ as } n \to +\infty$$

$$(3.17) \qquad \text{and } (T_n - t'_{\delta,n})^{\frac{p+1}{pq-1}} u_n(0, t'_{\delta,n}) + (T_n - t'_{\delta,n})^{\frac{q+1}{pq-1}} v_n(0, t'_{\delta,n}) \leq \delta.$$

Similarly as we did in (3.12) and what follows, we introduce

$$(3.18) \qquad \begin{aligned} \hat{U}_n(\xi,\tau) &= (T_n - t'_n)^{\frac{p+1}{pq-1}} u_n\left(\xi\sqrt{T_n - t'_n}, t'_n + \tau(T_n - t'_n)\right) \\ \hat{V}_n(\xi,\tau) &= (T_n - t'_n)^{\frac{q+1}{pq-1}} v_n\left(\xi\sqrt{T_n - t'_n}, t'_n + \tau(T_n - t'_n)\right) \end{aligned}$$

which is a solution of (1.5) on $\mathbb{R}^N \times [-\frac{t'_n}{T_n - t'_n}, 1)$ (we omit the δ dependence in the notation for simplicity), and introduce $(\hat{U}, \hat{V})$ a solution of (1.5) satisfying (3.13) such that $(\hat{U}_n, \hat{V}_n) \to (\hat{U}, \hat{V})$ uniformly on compact sets of $\mathbb{R}^N \times (-\infty, 1)$ (up to a subsequence). From the Liouville Theorem of Corollary 1.3, we know that $(\hat{U}, \hat{V})(\xi, \tau) = (\hat{U}, \hat{V})(\tau)$. The following Proposition shows that $(\hat{U}_n, \hat{V}_n)$ is flat in $D = B(0,1) \times [0,1)$ and has compactness properties which allows us to show that this convergence is actually uniform in D.

PROPOSITION 3.7 **(Boundedness and flatness of (U_n, V_n) near the origin)**
i) There exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$, if $n \geq n_0(\delta)$, then for all $(\xi, \tau) \in \bar{B}(0,4) \times [0,1)$, $\hat{U}_n(\xi, \tau) \leq M_1(p,q)\delta(1-\tau)^{-\frac{p+1}{pq-1}}$ and $\hat{V}_n(\xi, \tau) \leq M_2(p,q)\delta(1-\tau)^{-\frac{q+1}{pq-1}}$.
ii) There exists $\delta_1 > 0$ such that for all $\delta \in (0, \delta_1]$, if $n \geq n_1(\delta)$, then for all $(\xi, \tau) \in \bar{B}(0,2) \times [-1,1)$, $\hat{U}_n(\xi,\tau) + \hat{V}_n(\xi,\tau) \leq C_3(C_0, T_0)$.
iii) For all $\delta < \delta_1$ and $n \geq n(\delta)$, $|\hat{U}_n|_{\alpha,D} + |\hat{V}_n|_{\alpha,D} + |\Delta \hat{U}_n|_{\alpha,D} + |\Delta \hat{V}_n|_{\alpha,D} \leq C_4(C_0, T_0)$ where $D = B(0,1) \times [0,1)$, $\alpha \in (0,1)$ and $|.|_{\alpha,D}$ is defined in (3.7).

PROOF : From (3.18), (3.17), i) of Lemma 3.6, (3.16) and Proposition 1.4, we have the following facts

$$\hat{U}_n(0,0) + \hat{V}_n(0,0) \leq \delta, \tag{3.19}$$

$\forall \tau \in [-\frac{t'_n}{T_n - t'_n}, 1)$,

$$(1-\tau)^{\frac{p+1}{pq-1}} \|\hat{U}_n(\tau)\|_{\mathbf{L}^\infty} + (1-\tau)^{\frac{q+1}{pq-1}} \|\hat{U}_n(\tau)\|_{\mathbf{L}^\infty} \leq C^*(C_0, T_0), \tag{3.20}$$

$$\sup_{\tau \in [0,1)} \left((1-\tau)^{\frac{p+1}{pq-1} + \frac{i}{2}} \|\nabla^i \hat{U}_n(\tau)\|_{\mathbf{L}^\infty} + (1-\tau)^{\frac{q+1}{pq-1} + \frac{i}{2}} \|\nabla^i \hat{V}_n(\tau)\|_{\mathbf{L}^\infty} \right) \to 0. \tag{3.21}$$

i) For all $\xi \in \bar{B}(0,4)$, $|\hat{U}_n(\xi, 0) - \hat{U}_n(0,0)| \leq 4\|\nabla \hat{U}_n(0)\|_{\mathbf{L}^\infty} \to 0$ as $n \to +\infty$ by (3.21). The same holds for $\hat{V}_n$. If n is large enough, then this implies by (3.19)

$$\forall \xi \in \bar{B}(0,4), \ \ \hat{U}_n(\xi, 0) \leq 2\delta \text{ and } \hat{V}_n(\xi, 0) \leq 2\delta. \tag{3.22}$$

From (3.21), we have for n large enough and $\tau \in [0,1)$,
$(1-\tau)^{\frac{p(q+1)}{pq-1}} \|\Delta \hat{U}_n(\tau)\|_{\mathbf{L}^\infty} + (1-\tau)^{\frac{q(p+1)}{pq-1}} \|\Delta \hat{V}_n(\tau)\|_{\mathbf{L}^\infty} \leq \delta^2$.
Since $(\hat{U}_n, \hat{V}_n)$ is a solution of (1.5), (3.22) and this imply that for all $(\xi,\tau) \in \bar{B}(0,4) \times [0,1)$,

$$
\begin{aligned}
&|\partial_\tau \hat{U}_n - \hat{V}_n^p| \leq \delta^2 (1-\tau)^{-\frac{p(q+1)}{pq-1}}, \ |\partial_\tau \hat{V}_n - \hat{U}_n^q| \leq \delta^2 (1-\tau)^{-\frac{q(p+1)}{pq-1}} \\
&\hat{U}_n(\xi,0) \leq 2\delta \text{ and } \hat{V}_n(\xi,0) \leq 2\delta.
\end{aligned} \tag{3.23}
$$

The conclusion of i) then follows by straightforward a priori estimates.

ii) If $\tau \in [-1,0]$, then (3.20) gives the estimate, since $-\frac{t'_n}{T_n - t'_n} \to -\infty$. If $\tau \in [0,1)$, then the estimate follows directly from i) and the following Proposition which is an adaptation of Theorem 2.1 in [9] to the case of equation (1.5) where the nonlinearity can be non homogeneous.

Proposition 3.8 (A local lower bound on the blow-up rate)
Assume $p \geq 1$, $q \geq 1$ and $pq > 1$. Consider (u,v) a solution of (1.5). Assume that for all $(x,t) \in B(x_0,A) \times [T-\eta,T)$,

$$
u(x,t)(T-t)^{\frac{p+1}{pq-1}} + v(x,t)(T-t)^{\frac{q+1}{pq-1}} \leq \delta \tag{3.24}
$$

for some $x_0 \in \mathbb{R}^N$, $A > 0$, $T > 0$, $\eta \in [0,T)$ and $\delta > 0$. We claim that if $\delta \leq \delta_0(A,\eta,N,p,q)$, then for all $(x,t) \in B(x_0,A/2) \times [T-\eta,T)$,

$$
u(x,t) + v(x,t) \leq C(A,\eta,N,p,q)\delta.
$$

In particular, (u,v) does not blow-up at time T at the point x_0.

Proof : See Appendix B. ∎

iii) Just use ii) and apply Lemma 3.5. This concludes the proof of Proposition 3.7. ∎

Step 3 : Conclusion of the proof : contradiction with (3.9) or (3.10)

Now, we fix $\delta = \min(\delta_1, (\Gamma+\gamma)/2)$ where δ_1 is defined in Proposition 3.7, and still write t'_n for $t'_{\delta,n}$. From iii) of Proposition 3.7, we see that $\hat{U}_n$ and $\Delta \hat{U}_n$ are uniformly continuous in D and therefore $\hat{U}_n(\xi,\tau) \to \hat{U}(\tau)$

and $\Delta\hat{U}_n(\xi,\tau) \to \Delta\hat{U}(\tau) = 0$ as $n \to +\infty$, uniformly for $(\xi,\tau) \in D$ (the same holds for $\hat{V}_n$). In particular, for $\tau_n = \frac{t_n - t'_n}{T_n - t'_n}$, we have

(3.25)
$$|\hat{U}_n(0,\tau_n) - \hat{U}(\tau_n)| + |\hat{V}_n(0,\tau_n) - \hat{V}(\tau_n)| + |\Delta\hat{U}_n(0,\tau_n)| + |\Delta\hat{V}_n(0,\tau_n)|$$

$\to 0$ as $n \to +\infty$. We consider two cases in the following.

Case 1 : $t'_n > s_n$. It follows then from (3.14) that $(T_n - t'_n)^{\frac{p+1}{pq-1}} u_n(0,t'_n) + (T_n - t'_n)^{\frac{q+1}{pq-1}} v_n(0,t'_n) = \delta$ which gives by (3.18) as $n \to +\infty$ $\hat{U}(0) + \hat{V}(0) = \delta > 0$. Corollary 1.3 then implies that

(3.26)
$$\forall \tau \in (-\infty, 1), \quad \hat{U}(\tau) = \Gamma(T^* - \tau)^{-\frac{p+1}{pq-1}} \text{ and } \hat{V}(\tau) = \gamma(T^* - \tau)^{-\frac{q+1}{pq-1}}$$

for some $T^* > 1$.
- If (3.9) holds, then (3.18) gives $|\Delta\hat{U}_n(0,\tau_n)| \geq \epsilon_0 \hat{V}(0,\tau_n)^p$. As $n \to +\infty$, (3.25) yields $0 \geq \epsilon_0 \min\limits_{\tau\in[0,1)} \hat{V}(\tau)^p$. Contradiction with (3.26).
- If (3.10) holds, then (3.18) gives $\left| \left(\frac{\hat{U}_n(0,\tau_n)}{\Gamma}\right)^{q+1} - \left(\frac{\hat{V}_n(0,\tau_n)}{\gamma}\right)^{p+1} \right| \geq$ $\epsilon_0 \hat{U}_n(0,\tau_n)^{q+1}$. As $n \to +\infty$, (3.25) yields $0 \geq \epsilon_0 \min\limits_{\tau\in[0,1)} \hat{U}(\tau)^{q+1}$. Contradiction with (3.26).

Case 2 : $t'_n = s_n$ **defined in (3.15)**.
- If (3.9) holds, then (3.25) and (3.18) give $\Delta u_n(0,t_n) = o\left((T_n - s_n)^{-\frac{p(q+1)}{pq-1}}\right) = O\left(n^{\frac{p}{2(p+1)}}\right) = o(n)$ by (3.15). This contradicts (3.9) for n large.
- If (3.10) holds, then we have from (3.18), (3.25) and (3.26)
$$\left| \left(\frac{u_n(0,t_n)}{\Gamma}\right)^{q+1} - \left(\frac{v_n(0,t_n)}{\gamma}\right)^{p+1} \right| =$$
$$\left| \left(\frac{\hat{U}_n(0,\tau_n)}{\Gamma}\right)^{q+1} - \left(\frac{\hat{V}_n(0,\tau_n)}{\gamma}\right)^{p+1} \right| (T_n - s_n)^{-\frac{(p+1)(q+1)}{pq-1}}$$
$= O\left((T_n - s_n)^{-\frac{(p+1)(q+1)}{pq-1}}\right) = O(\sqrt{n})$ by (3.15). This contradicts (3.10) for n large.

This concludes the proof of Theorem 1.5. ∎

3.4 Application of the ODE comparison to blow-up solutions of (1.5)

We sketch the proofs of Corollary 1.7 and Proposition 1.8 here.

PROOF OF COROLLARY 1.7 : Corollary 1.7 can be derived from Corollary 1.3 in the same way we did for Corollary 2 in [15]. ∎

PROOF OF PROPOSITION 1.8 : ii) is a direct consequence of i). Let us prove i).
i) We proceed by contradiction. Let us assume that for some $\epsilon_0 > 0$, there is a sequence $t_n \to T$ and $a_n \in S$ such that (for example)

$$(T-t_n)^{\frac{p+1}{pq-1}} u(a_n,t_n) \to \Gamma - \epsilon_0 \text{ as } n \to +\infty. \tag{3.27}$$

If we define for all $\tau \in [0,1)$,

$$U_n(\tau) = (T-t_n)^{\frac{p+1}{pq-1}} u(a_n, t_n + \tau(T-t_n)), \tag{3.28}$$

then we have from Corollary 1.7 and (3.27)

$$\forall \epsilon > 0,\ \ \forall \tau \in [0,1),\ \ \left| U_n' - \gamma^p \left(\frac{U_n}{\Gamma}\right)^{\frac{p(q+1)}{p+1}} \right| \le \epsilon {U_n}^{\frac{p(q+1)}{p+1}} + C_\epsilon (T-t_n)^{\frac{p(q+1)}{pq-1}}$$

and $U_n(0) \to \Gamma - \epsilon_0$ as $n \to +\infty$. Since
$U^*(\tau) = \left[(\Gamma - \epsilon_0)^{\frac{1-pq}{p+1}} - \tau \Gamma^{\frac{1-pq}{p+1}} \right]^{-\frac{p+1}{pq-1}}$ is bounded for all $\tau \in [0,1]$ and solves

$$U^{*\prime} = \gamma^p \left(\frac{U^*}{\Gamma}\right)^{\frac{p(q+1)}{p+1}} \text{ with } U^*(0) = \Gamma - \epsilon_0,$$

we deduce that $U_n(\tau) \to U^*(\tau)$ as $n \to +\infty$ uniformly for $\tau \in [0,1)$. As n is large, we obtain $\limsup_{\tau \to 1} U_n(\tau) \le 2U^*(1)$ which gives by (3.28)

$$\limsup_{t \to T} u(a_n, t) \le 2U^*(1)(T-\tau_n)^{-\frac{p+1}{pq-1}},$$

and this contradicts the fact that a_n is a blow-up point of (u,v) (see i) of Corollary 1.7). ∎

Appendix A

A lower bound on the blow-up rate for equation (1.5)

We prove Proposition 3.4 in this appendix. We proceed by contradiction and assume that for some $t_0 \in [0,T)$, we have $\|u(t_0)\|_{\mathbf{L}^\infty} < \Gamma(T-t_0)^{-\frac{p+1}{pq-1}}$ and $\|v(t_0)\|_{\mathbf{L}^\infty} < \gamma(T-t_0)^{-\frac{q+1}{pq-1}}$. We then fix some $T_0(t_0) > T$ such that

$$\|u(t_0)\|_{\mathbf{L}^\infty} < \Gamma(T_0-t_0)^{-\frac{p+1}{pq-1}} \text{ and } \|v(t_0)\|_{\mathbf{L}^\infty} < \gamma(T_0-t_0)^{-\frac{q+1}{pq-1}}. \tag{A.1}$$

We claim that for all $t \in [t_0,T)$,

$$\|u(t)\|_{\mathbf{L}^\infty} \le \Gamma(T_0-t)^{-\frac{p+1}{pq-1}} \text{ and } \|v(t)\|_{\mathbf{L}^\infty} \le \gamma(T_0-t)^{-\frac{q+1}{pq-1}}, \tag{A.2}$$

which yields a contradiction with the fact that (u,v) blows-up at time $T<T_0$ and concludes the proof. It remains for us to prove (A.2).
From (A.1), (A.2) is true at $t=t_0$. Let us assume by contradiction that (A.2) is true for all $t \in [t_0,t^*]$ and (for example)

$$\|u(t^*)\|_{\mathbf{L}^\infty} = \Gamma(T_0-t^*)^{-\frac{p+1}{pq-1}}. \tag{A.3}$$

Then, from (1.5), we have $u(t^*) = S(t^*-t_0)u(t_0) + \int_{t_0}^{t^*} S(t^*-t)v(t)^p dt$ where $S(t)$ is the heat kernel. Therefore,
$\|u(t^*)\|_{\mathbf{L}^\infty} \le \|u(t_0)\|_{\mathbf{L}^\infty} + \int_{t_0}^{t^*} \|v(t)\|^p_{\mathbf{L}^\infty} dt$
$< \Gamma(T_0-t_0)^{-\frac{p+1}{pq-1}} + \gamma^p \int_{t_0}^{t^*} (T_0-t)^{-\frac{p(q+1)}{pq-1}} dt$ (by (A.1) and (A.2))
$= \Gamma(T_0-t^*)^{-\frac{p+1}{pq-1}}$ (by (1.10)). This contradicts (A.3) and concludes the proof of (A.2) and the proof of Proposition 3.4.

Appendix B

A local lower bound on the blow-up rate for (1.5) We show in this appendix how to adapt the proof of Theorem 2.1 in [9] to get Proposition 3.8. We first remark that Proposition 3.8 is valid also if u and v are vector-valued and satisfy the following parabolic inequalities

$$|u_t - \Delta u| \le K(1+|v|^p) \text{ and } |v_t - \Delta v| \le K(1+|u|^q).$$

Let us note that introducing

$$U(\xi,\tau) = \eta^{\frac{p+1}{pq-1}} u\left(x_0+\xi\sqrt{\eta}, T-\eta+\tau\eta\right)$$
$$V(\xi,\tau) = \eta^{\frac{q+1}{pq-1}} u\left(x_0+\xi\sqrt{\eta}, T-\eta+\tau\eta\right)$$

we can assume $x_0 = 0$, $\eta = 1$ and $T = 1$.
The hypothesis (3.24) can be rewritten as

(B.1)
$$\forall (x,t) \in B(0,A) \times [0,1), \quad u(x,t)(1-t)^{\mu \frac{p+1}{pq-1}} + v(x,t)(1-t)^{\mu \frac{q+1}{pq-1}} \le \delta$$

where $\mu = 1$.
If $p = q$, then [9] directly gives the result. Unfortunately, the proof of Giga and Kohn is strongly attached to the homogeneity of the nonlinearity. In our sketch of the proof, we consider the case $p \neq q$ (actually $p > q$) and proceed by a priori estimates as in [9] in order to decrease inductively the parameter μ in (B.1) to zero, which gives the desired result.
Assuming (B.1), we localize equation (1.5) in order to show that (B.1) holds with a parameter $\mu < 1$ (up to shrinking the space domain). Let us introduce a cut-off function χ such that $\chi(x) = 0$ if $|x| \ge A$ and $\chi(x) = 1$ if $|x| \le \lambda A$, and note

$$C_0(A,\lambda) = \max\left(\|\Delta\chi\|_{\mathbf{L}^\infty}, \|\nabla\chi\|_{\mathbf{L}^\infty}\right) \tag{B.2}$$

We then define

$$f(x,t) = \chi(x)u(x,t) \text{ and } g(x,t) = \chi(x)v(x,t). \tag{B.3}$$

From (1.5) and (B.1), we see that f satisfies the following equation :

$$\begin{array}{l} f(x,t) = \\ S(t)f(0) + \int_0^t S(t-s)\left(g(s)v(s)^{p-1} + u(s)\Delta\chi - 2\nabla.\left(u(s)\nabla\chi\right)\right)ds \end{array}$$

where $S(t)$ is the heat semi-group. Therefore,

$$\begin{aligned} \|f(t)\|_{\mathbf{L}^\infty} \quad &\le \|f(0)\|_{\mathbf{L}^\infty} + \int_0^t \Big(\|g(s)\|_{\mathbf{L}^\infty}\|v(s)\|_{\mathbf{L}^\infty}^{p-1} + \|u(s)\|_{\mathbf{L}^\infty}\|\Delta\chi\|_{\mathbf{L}^\infty} \\ &+ \frac{2K'(N)}{\sqrt{t-s}}\|u(s)\|_{\mathbf{L}^\infty}\|\nabla\chi\|_{\mathbf{L}^\infty}\Big)\, ds. \end{aligned}$$

Using (B.1), (B.3), (B.2), introducing for all $\nu \in \mathbb{R}$ and $t \in [0,1)$,

$$I(\nu,t) = \int_0^t (t-s)^{-\frac{1}{2}}(1-s)^{-\nu} ds,$$

and proceeding similarly for b, we obtain for all $t \in [0,1)$,

(B.4)
$$\begin{aligned} a(t) &\le \delta + \delta^{p-1}\int_0^t b(s)(1-s)^{-\mu\frac{(q+1)(p-1)}{pq-1}} ds + K(N)C_0\delta I\left(\mu\frac{p+1}{pq-1}, t\right) \\ b(t) &\le \delta + \delta^{q-1}\int_0^t a(s)(1-s)^{-\mu\frac{(p+1)(q-1)}{pq-1}} ds + K(N)C_0\delta I\left(\mu\frac{q+1}{pq-1}, t\right). \end{aligned}$$

By straightforward (but long!) a priori estimates, we claim the following :

Claim B.1 *There exists* δ_1 *such that if* $\delta \le \delta_1$*, then for all* $t \in [0,1)$*,*

$$\|f(t)\|_{\mathbf{L}^\infty} \le M_1 \delta (1-t)^{-\alpha} \text{ and } \|g(t)\|_{\mathbf{L}^\infty} \le M_2 \delta (1-t)^{-\beta} \tag{B.5}$$

where

(B.6)
$$\alpha = \frac{p+1}{pq-1} - \frac{1}{2}\min\left(\frac{1}{2}, \frac{q+1}{pq-1}\right), \quad \beta = \frac{q+1}{pq-1} - \frac{1}{2}\min\left(\frac{1}{2}, \frac{q+1}{pq-1}\right)$$

and M_1*,* M_2 *and* δ_0 *are fixed in terms of* (A, λ, N, p, q)*.*

Therefore, up to changing A by λA and shrinking δ, (B.1) holds with $\mu = \max\left(\frac{pq-1}{p+1}\alpha, \frac{pq-1}{q+1}\beta\right) \in (0,1)$.

We then start again the localization process from (B.1), assuming now that $\mu < 1$. Let

$$\mu_0 = \min\left(\frac{pq-1}{2(p+1)}, \frac{pq-1}{2(q+1)}, \frac{pq-1}{(p-1)(q+1)}, \frac{pq-1}{(q-1)(p+1)}\right)$$

(if $q = 1$, ignore the last number). Since $p > q$, we have $(p-1)(q+1) > pq-1$, therefore, $\mu_0 < 1$. Using (B.4), one can make straightforward (though long!) a priori estimates to show that, up to replacing A by λA and shrinking δ, (B.1) holds with a smaller parameter $\mu' = 0$ if $\mu < \mu_0$, which terminates the proof. If $\mu \ge \mu_0$, one can show that (B.1) holds with $\mu' = \mu - \min\left(\frac{(1-\mu)(pq-1)}{p-q}, \frac{\mu_0}{2}\right) < \mu$. In this case, one has just to start once more the whole process, and it is easy to see that in a finite number of steps, one shows that (B.1) holds with $\mu = 0$, which finishes the proof.

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Current address :

Courant Institute, NYU, 251 Mercer Street, NY 10012, New York, USA.

`e-mail : Hatem.Zaag@ens.fr`

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