A remark on the energy blow-up behavior for nonlinear heat equations

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1 Introduction

We are concerned with finite time blow-up for the following nonlinear heat equation:

$$\begin{cases} u_t &= \Delta u + |u|^{p-1}u \quad \text{in} \quad \Omega \times [0,T) \\ u &= 0 \quad \text{on} \quad \partial\Omega \times [0,T) \end{cases} \tag{1}$$

with $u(x,0) = u_0(x)$,
where $u : \Omega \times [0,T) \to \mathbb{R}$, Ω is a $C^{2,\alpha}$ convex bounded domain of $\mathbb{R}^N$,
$u_0 \in L^\infty(\Omega)$. We assume that the following condition holds :

$$1 < p,\ (N-2)p < N+2 \text{ and } \left(u_0 \geq 0 \text{ or } p < \frac{3N+8}{3N-4}\right). \tag{2}$$

Therefore, $p+1 > N(p-1)/2$ and the (local in time) Cauchy problem for (1) can be solved in $L^{p+1}(\Omega)$ (see for instance Weissler [19], Theorem 3). If the maximum existence time $T > 0$ is finite, then $u(t)$ is said to blow-up in finite time and in this case

$$\lim_{t\to T} \|u(t)\|_{L^{p+1}(\Omega)} = \lim_{t\to T} \|u(t)\|_{L^\infty(\Omega)} = +\infty \tag{3}$$

(see Corollary 3.2 in [19]). We consider such a blow-up solution $u(t)$ in the following.

From the regularizing effect of the Laplacian, $u(t) \in L^\infty \cap H_0^1(\Omega)$ for all $t \in (0,T)$. We take $\|u\|^2_{H_0^1(\Omega)} = \int_\Omega |\nabla u|^2 dx$. Using the Sobolev embedding and the fact that p is subcritical ($p < \frac{N+2}{N-2}$ if $N \geq 3$), we see that $H_0^1(\Omega) \subset L^{p+1}(\Omega)$. Therefore, (3) implies that

$$\lim_{t\to T} \|u(t)\|_{H_0^1(\Omega)} = +\infty.$$

$a \in \Omega$ is called a blow-up point of u if there exists $(a_n, t_n) \to (a, T)$ such that $|u(a_n, t_n)| \to +\infty$. The set of all blow-up points of $u(t)$ is called the blow-up set and denoted by S. From Giga and Kohn [6] (Theorem 5.3), there are no blow-up points in $\partial\Omega$. Therefore, we see from (3) and the boundedness of Ω that S is not empty.
Many papers are concerned with the Cauchy problem for equation (1) (see for instance [19]) or the problem of finding sufficient blow-up conditions on the initial data (see Ball [2], Levine [10],...). Other papers focus on the description of the blow-up set or the asymptotic behavior of u near blow-up points (Giga and Kohn [8], [7], [6], Herrero and Velázquez [9], [16], [18], [17], Merle and Zaag [13], [14], [15], [12],...). Let us mention for instance the following Liouville Theorem for equation (1) recently proved in [12] and which has many interesting consequences for the study of the blow-up behavior of solutions to (1) (see Fermanian, Merle, Zaag [3], [4], [12]).

Proposition 1 (Merle-Zaag, A Liouville Theorem for equation (1)) *Assume that $1 < p$ and $(N-2)p < N+2$ and consider U a solution of (1) defined for all $(x,t) \in \mathbb{R}^N \times (-\infty, T)$. Assume in addition that $|U(x,t)| \leq C(T-t)^{-\frac{1}{p-1}}$. Then $U \equiv 0$ or there exist $T_0 \geq T$ and $\epsilon \in \{-1, 1\}$ such that $\forall (x,t) \in \mathbb{R}^N \times (-\infty, T)$, $U(x,t) = \epsilon\kappa(T_0 - t)^{-\frac{1}{p-1}}$, where $\kappa = (p-1)^{-\frac{1}{p-1}}$.*

Remark : Note that this result is valid for all subcritical p with no restrictions for $N \geq 2$. For the reader's convenience, a sketch of the proof is given in Appendix A. For more details, see [12], Corollary 1.

In this paper, we crucially use the Liouville Theorem to study how the Lyapunov functional

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p+1}\int_\Omega |u|^{p+1} dx \tag{4}$$

associated with (1) behaves under the nonlinear heat flow.
It has been shown by Giga in [5] that under the positivity condition

$$u_0 \geq 0,$$

we have

$$E(u(t)) \to -\infty \text{ as } t \to T. \tag{5}$$

Let us remark that Giga's proof relies on another Liouville Theorem related to equation (1) :

Assume $p > 1$ and $p(N-2) < N+2$. Then, there is no nonnegative solution for the problem

$$\begin{cases} \Delta u + u^p = 0 \ \text{in } \mathbb{R}^N, \\ u(0) > 0. \end{cases}$$

In this paper, we use the new Liouville Theorem stated in Proposition 1 and ideas from [5] to extend the validity of the limit (5) to the more general case (2).

Theorem 2 (Limit of the energy at blow-up) *Assume (2). Then, $E(u(t))$ goes to $-\infty$ as t goes to T.*

In Bahri [1], the study of critical points of E is related to the study of those of a functional J associated with E and defined for all $v \in \Sigma$, the unit sphere of $H_0^1(\Omega)$ by

$$J(v) = \sup_{\lambda > 0} E(\lambda v).$$

In other words, $J(v)$ is the supremum of E in the direction of v. Note that J is positive by (4). The following is shown in [1] (Proposition 1) :

(i) $J \in C^2(\Sigma, \mathbb{R})$;

(ii) For all $v \in \Sigma$, we have $E'(\lambda(v)v) = \lambda(v)^{-1}J'(v)$, where $\lambda(v)$ is the unique positive solution of $J(v) = E(\lambda v)$;

(iii) There is a one-to-one correspondence between the critical points of J and E by means of the transformation

$$\omega \in \Sigma \to \lambda(\omega)\omega = \omega_1, \quad J(\omega) = E(\omega_1).$$

With this correspondence, Bahri reduces to the study of some topological properties of level sets of J. He shows in particular that the level sets of J have contractibility properties one into another. More precisely (see Lemma 1 in [1]), if we define

$$J_a = \{v \in \Sigma \mid J(v) \geq a\},$$

then

for all $a > 0$, there exists $\mu(a) \geq a$ such that $J(\mu(a))$ is contractible in $J(a)$.

Remark : If $B \subset A$, then B is said to be contractible in A if there is a continuous mapping $\theta(t,.) : [0,1] \times B \to A$ such that for all $x \in B$, $\theta(0,x) = x$ and $\theta(0,x) = x_0 \in A$.

Our second concern in this paper is to understand the effect on J of the nonlinear heat flow of equation (1) (composed with the projection over Σ).

In other words, we want to understand the behavior of $J\left(\frac{u(t)}{\|u(t)\|_{H_0^1(\Omega)}}\right)$ as $t \to T$. We claim the following :

Theorem 3 (Blow-up limit of the directional supremum of the energy) *The Rayleigh quotient for the solution* $\|u(t)\|_{H_0^1}/\|u(t)\|_{L^{p+1}}$ *goes to* $+\infty$ *as* $t \to T$ *and so does*

$$J\left(\frac{u(t)}{\|u(t)\|_{H_0^1(\Omega)}}\right) = \sup_{\lambda>0} E(\lambda u(t)) = \frac{p-1}{2(p+1)}\left(\frac{\|u(t)\|_{H_0^1}}{\|u(t)\|_{L^{p+1}}}\right)^{\frac{2(p+1)}{p-1}}. \qquad (6)$$

Roughly speaking, one consequence of this Theorem is that the nonlinear heat flow of equation (1) (composed with the projection over Σ) maps any element of a given level set J_a into J_b, for any $b > a$ (Note that this mapping raises the level set of J, in the contrary of the contractibility result of [1] which lowers the value of J).
Another consequence of Theorem 3 is that $E(u(t))$ can not tend to $-\infty$ "radially". More precisely,

Corollary 4 *We can not have* $u(.,t) \sim \lambda(t)\varphi$ *in* $H_0^1(\Omega)$ *as* $t \to T$.

Indeed, if is was the case, then $J\left(\frac{u(t)}{\|u(t)\|_{H_0^1(\Omega)}}\right) \sim J\left(\frac{\varphi}{\|\varphi\|_{H_0^1(\Omega)}}\right)$ as $t \to T$ since J is continuous. This contradicts Theorem 3.

In [1] (Proposition 2), it is shown that J satisfies the following property :

$\forall (u_n)$; $u_n \in \Sigma$; u_n *goes weakly to zero in* $H_0^1(\Omega) \Leftrightarrow J(u_n) \to +\infty$.

Therefore, Theorem 3 is equivalent to the following :

Proposition 5 $\frac{u(t)}{\|u(t)\|_{H_0^1(\Omega)}}$ *goes to* 0 *as* $t \to T$*, weakly in* $H_0^1(\Omega)$.

The paper is organized as follows. In Section 2, we use the Liouville Theorem of [12] and prove Theorem 2. In Section 3, we use results from [7] and some consequences of the Liouville Theorem to prove Proposition 5 and Theorem 3.

2 Energy blow-up behavior

We prove Theorem 2 in this section. We proceed in two Parts. We recall some results from [7] and [12] for blow-up solutions of (1) in the first Part. Then, the proof of Theorem 2 is presented in the second Part.

Part 1 : L^∞ estimates for Blow-up solutions of (1)

The following uniform L^∞ bound for blow-up solutions of (1) is proved in [12] (Theorem 2).

Proposition 2.1 (Giga-Kohn, A uniform L^∞ bound on $u(t)$ at blow-up) *There exists $C_0 > 0$ such that*

$$\forall t \in [0,T), \;\; \|u(t)\|_{L^\infty} \le C_0 (T-t)^{-\frac{1}{p-1}}. \tag{7}$$

In the following Proposition, we derive the existence of a blow-up profile for $u(t)$.

Proposition 2.2 (Existence of the blow-up profile) *There exists $u^*(x)$ defined on $\Omega \backslash S$ such that*
$u^* \in L^\infty_{loc}(\Omega \backslash S)$,
$u(t) \to u^*$ uniformly on each compact set of $\Omega \backslash S$ as $t \to T$.

Proof : See Merle [11] for example. ■

In [12] (Proposition 4), Merle and Zaag generalize a result by Velázquez (see [18], [17] and [16]), and prove the following result on the size of the blow-up set S.

Proposition 2.3 (Size of the blow-up set) *S is compact and the $(N-1)-$Hausdorff measure of S is finite.*

Remark : Since $u_0 \in L^\infty(\Omega)$, $u(t) \in L^\infty \cap H^1_0(\Omega)$ for all $t > 0$, from the regularizing effect of the Laplacian. Therefore, Proposition 4 of [12] applies.

Part 2 : Proof of Theorem 2

Our proof relies strongly on the Liouville Theorem presented in Proposition 1. We proceed by contradiction. Since $E(u(t))$ is decreasing in time, it goes to some finite $A \in \mathbb{R}$ as $t \to T$. Therefore, multiplying (1) by $\frac{\partial u}{\partial t}$ and integrating over $\Omega \times [0,T)$, we get

$$\int_0^T dt \int_\Omega dx \left| \frac{\partial u}{\partial t}(x,t) \right|^2 = E(u_0) - A \equiv B < +\infty. \tag{8}$$

In a first Step, we will use a compactness procedure to derive a solution of (1) which satisfies the hypotheses of the Liouville Theorem (Proposition 1). In a second Step, we apply Proposition 1 on one hand and use (8) with scaling arguments on the other hand to get a contradiction.

Step 1 : A compactness procedure

Let us consider $a \in \Omega$ a blow-up point of $u(t)$ and any sequence $t_k \to T$ as $k \to +\infty$.
From the uniform blow-up bound of Proposition 2.1 and Giga and Kohn [6], we know that

$$u(a, t_k) \sim \epsilon\kappa(T - t_k)^{-\frac{1}{p-1}} \text{ as } k \to +\infty \tag{9}$$

where $\epsilon \in \{-1, 1\}$. We can assume $\epsilon = 1$ from the sign invariance of (1).
For each $k \in \mathbb{N}$, we define for all $\xi \in (\Omega - a)(T - t_k)^{-\frac{1}{2}}$ and $\tau \in (-\frac{t_k}{T-t_k}, 1)$

$$v_k(\xi, \tau) = (T - t_k)^{\frac{1}{p-1}} u(a + \xi\sqrt{T - t_k}, t_k + \tau(T - t_k)). \tag{10}$$

From (1), (7) and (9), we see that v_k satisfies for all $\xi \in (\Omega - a)(T - t_k)^{-\frac{1}{2}}$ and $\tau \in (-\frac{t_k}{T-t_k}, 1)$

$$\frac{\partial v_k}{\partial \tau} = \Delta v_k + |v_k|^{p-1} v_k,\ \ |v_k(\xi, \tau)| \le C_0(1 - \tau)^{-\frac{1}{p-1}} \text{ and } v_k(0, 0) \to \kappa$$

as $k \to +\infty$.

Since $a \notin \partial\Omega$ and $t_k \to T$ as $k \to +\infty$, v_k is defined (at least) for all $(\xi, \tau) \in D_n$, for all $n \in \mathbb{N}^*$ and $k \ge k_0(n)$, where $D_n = \bar{B}(0, n) \times [-n, 1 - \frac{1}{n}]$. Moreover, it satisfies $\|v_k\|_{L^\infty(D_n)} \le C_0 n^{\frac{1}{p-1}}$. Using parabolic regularity for equation (1) in $D_{n+1} \supset\supset D_n$, we obtain $\|v_k\|_{C^{2,1}_\alpha(D_n)} \le C(n)$, for all $n \in \mathbb{N}^*$ and $k \ge k_0(n + 1)$, where

$$\|h\|_{C^{2,1}_\alpha(D)} = \|h\|_{C_\alpha(D)} + \|\nabla h\|_{C_\alpha(D)} + \|\nabla^2 h\|_{C_\alpha(D)} + \|\partial_\tau h\|_{C_\alpha(D)}, \tag{11}$$

$$\|h\|_{C_\alpha(D)} = \|h\|_{L^\infty(D)} + \sup_{(\xi,\tau),(\xi',\tau')\in D} \frac{|h(\xi, \tau) - h(\xi', \tau')|}{(|\xi - \xi'|^2 + |\tau - \tau'|)^{\alpha/2}}$$

and $\alpha \in (0, 1)$. Using the compactness of the embedding of $C_\alpha(D_n)$ into $C(D_n)$, we find $v(\xi, \tau)$ a solution of (1) defined for all $(\xi, \tau) \in \mathbb{R}^N \times (-\infty, 1)$ and satisfying $v_k \to v$ in $C^{2,1}_{loc}(\mathbb{R}^N \times (-\infty, 1))$ (up to a subsequence), $\forall(\xi, \tau) \in \mathbb{R}^N \times (-\infty, 1)$,

$$\frac{\partial v}{\partial \tau} = \Delta v + |v|^{p-1} v,\ \ |v(\xi, \tau)| \le C_0(1 - \tau)^{-\frac{1}{p-1}} \text{ and } v(0, 0) = \kappa. \tag{12}$$

Step 2 : Conclusion of the proof of Theorem 2

From the Liouville Theorem of Proposition 1, (12) yields

$$\forall(\xi, \tau) \in \mathbb{R}^N \times (-\infty, 1),\ \ v(\xi, \tau) = \kappa(1 - \tau)^{-\frac{1}{p-1}}. \tag{13}$$

From the convergence of v_k, we have for all $R > 0$,
$\int_{-R}^0 d\tau \int_{B(0,R)} d\xi \left|\frac{\partial v}{\partial \tau}(\xi,\tau)\right|^2 = \lim_{k\to+\infty} \int_{-R}^0 d\tau \int_{B(0,R)} d\xi \left|\frac{\partial v_k}{\partial \tau}(\xi,\tau)\right|^2.$

From (10), (8) and scaling argument, we easily compute
$\int_{-R}^0 d\tau \int_{B(0,R)} d\xi \left|\frac{\partial v_k}{\partial \tau}(\xi,\tau)\right|^2$
$= (T-t_k)^\beta \int_{t_k-R(T-t_k)}^{t_k} dt \int_{B(a,R\sqrt{T-t_k})} dx \left|\frac{\partial u}{\partial t}(x,t)\right|^2$
$\leq (T-t_k)^\beta \int_0^T dt \int_\Omega dx \left|\frac{\partial u}{\partial t}(x,t)\right|^2 \leq B(T-t_k)^\beta$ where

$$\beta = \frac{p+1}{p-1} - \frac{N}{2} > 0$$

since p is subcritical.

Therefore, $\int_{-R}^0 d\tau \int_{B(0,R)} d\xi \left|\frac{\partial v_k}{\partial \tau}(\xi,\tau)\right|^2 \to 0$ as $k \to +\infty$ and so

$$\forall R > 0, \quad \int_{-R}^0 d\tau \int_{B(0,R)} d\xi \left|\frac{\partial v}{\partial \tau}(\xi,\tau)\right|^2 = 0. \tag{14}$$

A contradiction follows from (13) and (14), and Theorem 2 is proved.

3 Blow-up behavior of the directional maximum of the energy

We prove Proposition 5 and Theorem 3 in this section. As stated in the introduction, Theorem 3 is a direct consequence of Proposition 5, thanks to a result of [1] (Proposition 2). Since this fact can be proved in a simple and short way, we present a proof of it in the following.

Proposition 5 implies Theorem 3 :

Since p is subcritical, we have $p+1 < 2^* = \frac{2N}{N-2}$ whenever $N \geq 3$. Hence, $H_0^1(\Omega)$ is compactly embedded in $L^{p+1}(\Omega)$. Therefore, assuming Proposition 5, we get

$$\frac{\|u(t)\|_{L^{p+1}}}{\|\nabla u(t)\|_{L^2}} \to 0 \text{ as } t \to T. \tag{15}$$

The expression of the Rayleigh quotient given in (6) can be easily checked from (4). Thus, (15) yields Theorem 3.

Now, we use information on the blow-up set S from section 2 to prove Proposition 5.

Proof of Proposition 5 :

It is enough to show that for all $\varphi \in C^\infty(\Omega)$ with supp $\varphi \subset\subset \Omega$ and for all $\epsilon > 0$, there exists $t_0(\epsilon) < T$ such that for all $t \in [t_0(\epsilon), T)$, we have:

$$\left| \frac{\int_\Omega \nabla u(x,t).\nabla\varphi(x)dx}{\left(\int_\Omega |\nabla u(x,t)|^2 dx\right)^{1/2}} \right| \le \epsilon \left(1 + \|\nabla\varphi\|_{L^\infty(\Omega)}\right).$$

From Proposition 2.3, we know that S is compact in Ω and that its Lebesgue measure $|S| = 0$. Therefore, we may consider the following open set

$$V_\epsilon = \{x \in \Omega \mid d(x,S) < \delta_\epsilon\}$$

where δ_ϵ is small enough so that

$$|V_\epsilon| \le \epsilon^2.$$

We then write

$$\frac{\int_\Omega \nabla u(x,t).\nabla\varphi(x)dx}{\left(\int_\Omega |\nabla u(x,t)|^2 dx\right)^{1/2}} = I + II$$

where

$$I = \frac{\int_{V_\epsilon} \nabla u(x,t).\nabla\varphi(x)dx}{\left(\int_\Omega |\nabla u(x,t)|^2 dx\right)^{1/2}} \text{ and } II = \frac{\int_{\Omega\setminus V_\epsilon} \nabla u(x,t).\nabla\varphi(x)dx}{\left(\int_\Omega |\nabla u(x,t)|^2 dx\right)^{1/2}}.$$

By Cauchy Schwartz inequality, we have for all $t \in [0,T)$

$$\begin{aligned} |I| &= \left| \frac{\int_{V_\epsilon} \nabla u(x,t).\nabla\varphi(x)dx}{\left(\int_\Omega |\nabla u(x,t)|^2 dx\right)^{1/2}} \right| \le \frac{\left(\int_{V_\epsilon} |\nabla u|^2 dx\right)^{1/2}}{\left(\int_\Omega |\nabla u|^2 dx\right)^{1/2}} \|\nabla\varphi\|_{L^\infty(\Omega)} |V_\epsilon|^{1/2} \\ &\le \epsilon \|\nabla\varphi\|_{L^\infty(\Omega)}. \end{aligned} \tag{16}$$

According to Giga and Kohn, no blow-up occurs near he boundary $\partial\Omega$ (see [6], Theorem 5.3). Therefore, using Proposition 2.2 and parabolic regularity, we find $M(\epsilon) > 0$ such that

$$\forall x \in \Omega\setminus V_\epsilon,\ \forall t \in [\frac{T}{2}, T),\ |u(x,t)| + |\nabla u(x,t)| \le M(\epsilon).$$

We then write for all $t \ge \frac{T}{2}$,

$$|II| = \left| \frac{\int_{\Omega\setminus V_\epsilon} \nabla u(x,t).\nabla\varphi(x)dx}{\left(\int_\Omega |\nabla u(x,t)|^2 dx\right)^{1/2}} \right| \le \frac{M(\epsilon)\|\nabla\varphi\|_{L^\infty}|\Omega|}{\left(\int_\Omega |\nabla u|^2 dx\right)^{1/2}}.$$

Since $\int_\Omega |\nabla u(x,t)|^2 dx \to +\infty$, we may take $t \geq t_1(\epsilon)$ large enough so that

$$|II| \leq \epsilon. \tag{17}$$

Combining (16) and (17) yields: $\forall t \geq t_0(\epsilon) \equiv \max\left(t_1(\epsilon), \frac{T}{2}\right)$,

$$\left|\frac{\int_\Omega \nabla u(x,t).\nabla \varphi dx}{\int_\Omega |\nabla u(x,t)|^2 dx}\right| \leq \epsilon \left(\|\nabla \varphi\|_{L^\infty} + 1\right).$$

This concludes the proof of Proposition 1 and the proof of Theorem 1 also. ■

A Sketch of the proof of the Liouville Theorem

We give in this appendix a sketch of the proof of Proposition 1. For more details, one can find a complete proof in [12].

Let U be a solution of (1) defined for all $(x,t) \in \mathbb{R}^N \times (-\infty, T)$ and satisfying $|U(x,t)| \leq C(T-t)^{-\frac{1}{p-1}}$. If $w(y,s)$ is defined by the following self-similar change of variables

$$y = \frac{x}{\sqrt{T-t}},\ s = -\log(T-t),\ w(y,s) = (T-t)^{\frac{1}{p-1}} u(x,t), \tag{18}$$

then w satisfies the following equation for all $(y,s) \in \mathbb{R}^N \times \mathbb{R}$:

$$\partial_s w = \Delta w - \frac{1}{2} y.\nabla w - \frac{w}{p-1} + |w|^{p-1} w \tag{19}$$

and $\|w\|_{L^\infty(\mathbb{R}^N \times \mathbb{R})} \leq C$. Let us introduce the following Lyapunov functional associated with equation (19)

$$\mathcal{E}(w) = \int_{\mathbb{R}^N} \left(\frac{1}{2}|\nabla w|^2 + \frac{1}{2(p-1)}|w|^2 - \frac{1}{p+1}|w|^{p+1}\right) \rho(y) dy$$

where $\rho(y) = e^{-|y|^2/4}/(4\pi)^{N/2}$.

With the change of variables (18), Proposition 1 is equivalent to the following :

Proposition A.1 *Assume that $1 < p$ and $(N-2)p < N+2$. Consider w a solution of (19) defined for all $(y,s) \in \mathbb{R}^N \times \mathbb{R}$ and satisfying $\|w\|_{L^\infty(\mathbb{R}^N \times \mathbb{R})} \leq C$. Then either $w \equiv 0$ or $w \equiv \epsilon\kappa$ or for all $(y,s) \in \mathbb{R}^N \times \mathbb{R}$, $w(y,s) = \epsilon\varphi(s - s_0)$ where $\kappa = (p-1)^{-\frac{1}{p-1}}$, $\epsilon \in \{-1, 1\}$ and $\varphi(s) = \kappa(1+e^s)^{-\frac{1}{p-1}}$ is a solution of*

$$\varphi' = -\frac{\varphi}{p-1} + \varphi^p,\ \varphi(-\infty) = \kappa,\ \varphi(+\infty) = 0. \tag{20}$$

Therefore, we reduce to the proof of Proposition A.1.

We proceed in 3 Parts :

- In Part I, we use the monotonicity of $s \mapsto \mathcal{E}(w(s))$ to show that $w(.,s)$ has limits $w_{\pm\infty}$ as $s \to \pm\infty$ (in $L^2_\rho(\mathbb{R}^N)$ and $C^k_{\text{loc}}(\mathbb{R}^N)$) which are stationary solutions of (19). From [8], we know that either $w_{\pm\infty} \equiv 0$ or $w_{\pm\infty} \equiv \epsilon\kappa$ where $\epsilon = \pm 1$. We focus then on the non trivial case $(w_{-\infty}, w_{+\infty}) = (\kappa, 0)$.

- In Part II, we linearize (19) around the constant solution κ as $s \to -\infty$ and show that w behaves in 3 possible ways.

- In Part III, we show that one of these 3 ways corresponds to the case $w(y,s) = \varphi(s-s_0)$ where φ is defined in (20). In the two other cases, we show that w satisfies a finite-time blow-up criterion for (19), which contradicts the fact that w is defined for all $(y,s) \in \mathbb{R}^N \times \mathbb{R}$ and $\|w\|_{L^\infty(\mathbb{R}^N\times\mathbb{R})} \le C < +\infty$. Thus, we rule out these two cases.

Part I : Existence of limits for w as $s \to \pm\infty$

We have the following :

Lemma A.2 *As $s \to +\infty$, $w(.,s) \to w_{+\infty}$ in $H^1_\rho(\mathbb{R}^N)$ and $C^k_{\text{loc}}(\mathbb{R}^N)$ for all $k \in \mathbb{N}$, where either $w_{+\infty} = 0$ or $w_{+\infty} = \epsilon\kappa$ with $\epsilon = \pm 1$. An analogous statement holds for the limit as $s \to -\infty$.*

Sketch of the proof : For a complete proof, see Proposition 2.2 in [12] and Step 1 in section 3 in [14].
Since $\|w\|_{L^\infty(\mathbb{R}^N\times\mathbb{R})} \le C$, parabolic regularity applied to equation (19) implies that for all $R > 0$, $\|w\|_{C^{2,1}_\alpha(B(0,R)\times[-R,R])} \le M(R)$ where $\|a\|_{C^{2,1}_\alpha(D)}$ is defined in (11). Using the compactness of the embedding of $C_\alpha(D)$ in $C(D)$ and considering subsequences $w_j(y,s) = w(y, s+s_j)$ where $s_j \to +\infty$, the following identity

$$\forall s_1, s_2 \in \mathbb{R}, \quad \int_{s_1}^{s_2} \int_{\mathbb{R}^N} |\partial_s w(y,s)|^2 \rho(y) dy ds = \mathcal{E}(w(s_1)) - \mathcal{E}(w(s_2)) \qquad (21)$$

allows us to find $w_{+\infty}(y)$, a stationary solution of (19) such that $w(.,s) \to w_{+\infty}$ as $s \to +\infty$ in $C^2_{\text{loc}}(\mathbb{R}^N)$. The conclusion follows from the following result by Giga and Kohn in [8] :

Claim A.3 (Giga-Kohn) *If $p > 1$ and $(N-2)p < N+2$, then the only stationary solutions of (19) are 0, κ and $-\kappa$.*

Letting $s_2 \to +\infty$ and $s_1 \to -\infty$ in (21), we obtain

$$\mathcal{E}(w_{-\infty}) - \mathcal{E}(w_{+\infty}) = \int_{-\infty}^{+\infty} \int_{\mathbb{R}^N} |\partial_s w(y,s)|^2 \rho(y) dy ds \ge 0.$$

Therefore, two cases arise :
- Case 1 : $\mathcal{E}(w_{-\infty}) - \mathcal{E}(w_{+\infty}) = 0$. Therefore, $\partial_s w \equiv 0$ and w is a stationary solution of (19). Claim A.3 implies then that $w \equiv 0$, κ or $-\kappa$. This corresponds to the first cases expected in Proposition A.1.
- Case 2 : $\mathcal{E}(w_{-\infty}) - \mathcal{E}(w_{+\infty}) > 0$. Since $\mathcal{E}(\kappa) = \mathcal{E}(-\kappa) > 0 = \mathcal{E}(0)$, this implies that $w_{+\infty} \equiv 0$ and $w_{-\infty} \equiv \kappa$ or $-\kappa$. From sign invariance of (19), we reduce to the case

$$(w_{-\infty}, w_{+\infty}) = (\kappa, 0).$$

Part II : Linear behavior of w near κ

We introduce $v = w - \kappa$. From (19), v satisfies the following equation

$$\partial_s v = \mathcal{L}v + f(v) \tag{22}$$

where $|f(v)| \leq C|v|^2$ and $\mathcal{L} = \Delta - \frac{1}{2}y.\nabla + 1$ is a self-adjoint operator on $\mathcal{D}(\mathcal{L}) \subset L^2_\rho(\mathbb{R}^N)$ whose spectrum consists of eigenvalues $\{1 - \frac{m}{2} \mid m \in \mathbb{N}\}$. Therefore, we can expand v on the eigenspaces of $\mathcal{L}$. Since $\|v\|_{L^\infty(\mathbb{R}^N\times\mathbb{R})} \leq C$, we use hard analysis where the key point is the control of the quadratic term in (22), and prove that one of the modes 1, $\frac{1}{2}$ or 0 dominates the others as $s \to -\infty$. More precisely, we have the following :

Lemma A.4 *As $s \to -\infty$, one of the following cases occur :*
*i) (**mode** $\lambda = 1$) : $\|w(y,s) - \{\kappa - C_0 e^s\}\|_{H^1_\rho(\mathbb{R}^N)} = o(e^s)$ where $C_0 > 0$.*
*ii) (**mode** $\lambda = \frac{1}{2}$) : $\|w(y,s) - \{\kappa + e^{\frac{s}{2}} C_1.y\}\|_{H^1_\rho(\mathbb{R}^N)} = o(e^{\frac{s}{2}})$ where $C_1 \in \mathbb{R}^N \backslash \{0\}$.*
*iii) (**mode** $\lambda = 0$) : $\|w(Qy,s) - \{\kappa + \frac{\kappa}{2ps}\left(l - \frac{1}{2}\sum_{i=1}^{l} y_i^2\right)\}\|_{H^1_\rho(\mathbb{R}^N)} = o(\frac{1}{s})$ where Q is an orthonormal $N \times N$ matrix and $l \in \{1, ..., N\}$.*

Proof : See Proposition 2.4 in [12] and Propositions 3.5, 3.6, 3.9 and 3.10 in [14]. ■

Part III : Conclusion of the proof

Case 1 : mode $\lambda = 1$ dominates, the relevant case

We remark that we already know a solution of (19) which behaves like w as $s \to -\infty$: it is $\varphi(s - s_0)$ where φ satisfies (20) and $s_0 = -\log\left(\frac{C_0(p-1)}{\kappa}\right)$. Therefore, $\|w(y,s) - \varphi(s - s_0)\|_{H^1_\rho(\mathbb{R}^N)} = o(e^s)$ as $s \to -\infty$. Let us prove that in fact

$$w(y,s) = \varphi(s - s_0), \text{ for all } (y,s) \in \mathbb{R}^N \times \mathbb{R}. \tag{23}$$

For this, we introduce $V(y,s) = w(y,s) - \varphi(s - s_0)$ which satisfies $\|V(y,s)\|_{H^1_\rho(\mathbb{R}^N)} = o(e^s)$ and show that $V \equiv 0$. See Proposition 2.5 in [12] for more details. Therefore, (23) holds and this gives the last case expected in Proposition A.1.

Case 2 and 3 : mode $\lambda = \frac{1}{2}$ or 0 dominates, irrelevant cases

Here we use the invariance of (19) under the following geometric transformation

$$(a_0, s_0) \in \mathbb{R}^N \times \mathbb{R} \mapsto \left(w_{a_0,s_0} : (y,s) \mapsto w(y + a_0 e^{\frac{s}{2}}, s + s_0)\right)$$

and the following blow-up criterion for equation (19) :

Lemma A.5 (A blow-up criterion for equation (19)) *Consider W a solution of (19) satisfying $I(W(0)) > 0$ where*

$$I(v) = -2\mathcal{E}(v) + \frac{p-1}{p+1}\left(\int_{\mathbb{R}^N} |v(y)|^2 \rho(y) dy\right)^{\frac{p+1}{2}}.$$

Then, W blows-up in finite time $S > 0$.

Proof : See Proposition 2.1 in [12]. ■

Using the asymptotic expansions of Lemma A.4, we find $(a_0, s_0) \in \mathbb{R}^N \times \mathbb{R}$ such that $I\left(w_{a_0,s_0}\right) > 0$. Therefore, w_{a_0,s_0} blows-up in finite time $S > 0$. This contradicts the fact that w_{a_0,s_0} is defined for all $(y,s) \in \mathbb{R}^N \times \mathbb{R}$ and satisfies $\|w_{a_0,s_0}\|_{L^\infty(\mathbb{R}^N \times \mathbb{R})} = \|w\|_{L^\infty(\mathbb{R}^N \times \mathbb{R})} \le C < +\infty$. Thus, cases 2 and 3 of Lemma A.4 actually do not hold. For more details, see [12], Section 2, Part II, Step 2.

This concludes the sketch of the proof of Propositions A.1 and 1.

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