Determination of the curvature of the blow-up set and refined singular behavior for a semilinear heat equation

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Abstract

We consider $u(x,t)$ a solution of $u_t = \Delta u + |u|^{p-1}u$ that blows up at some time $T > 0$, where $u : \mathbb{R}^N \times [0,T) \to \mathbb{R}$, $p > 1$ and $(N-2)p < N+2$. Under a non degeneracy condition, we show that the mere hypothesis that the blow-up set S is continuous and $N-1$ dimensional implies that it is C^2. In particular, we compute the $N-1$ principal curvatures and directions of S. Moreover, a much more refined blow-up behavior is derived for the solution, in terms of the newly exhibited geometric objects. Refined regularity for S and refined singular behavior of u near S are linked through a *new* mechanism of algebraic cancellations that we explain in details.

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1 Introduction

In this paper, we aim at computing the curvature of the blow-up set for the following semilinear heat equation:

$$\begin{aligned} u_t &= \Delta u + |u|^{p-1}u, \\ u(.,0) &= u_0 \in L^\infty(\mathbb{R}^N), \end{aligned} \tag{1}$$

where $u(t) : x \in \mathbb{R}^N \to u(x,t) \in \mathbb{R}$. We assume $p > 1$ and subcritical: if $N \geq 3$, then $1 < p < (N+2)/(N-2)$. A solution $u(t)$ to (1) blows up in finite time if its maximal existence time T is finite. In this case,

$$\lim_{t\to T} \|u(t)\|_{H^1(\mathbb{R}^N)} = \lim_{t\to T} \|u(t)\|_{L^\infty(\mathbb{R}^N)} = +\infty.$$

Let us consider such a solution. T is called the blow-up time of u. A point $a \in \mathbb{R}^N$ is a blow-up point if u is not locally bounded near (a,T). The blow-up set is the set of all blow-up points and will be denoted by S.

Recently, Giga, Matsui and Sasayama [12] extended to all subcritical p the blow-up rate estimate once proved by Giga and Kohn [10] for a smaller range of p or under a positivity condition. Thus, for all subcritical p, it holds that,

$$\forall t \in [0,T), \quad \|u(t)\|_{L^\infty} \leq C(T-t)^{-\frac{1}{p-1}}.$$

Let us remark that with this new result, the previous work of Giga and Kohn [9], [10] and [11], Velázquez (in particular [20] and [21]), and Merle and Zaag (in particular [16], [23] and [24]) extend naturally to cover all the range of subcritical p.

Given $a \in S$, the study of the blow-up behavior of $u(x,t)$ near (a,T) is equivalent to the study of the long-time behavior of $W_a(y,s)$ defined by

$$W_a(y,s) = (T-t)^{\frac{1}{p-1}} u(x,t), \quad y = \frac{x-a}{\sqrt{T-t}}, \quad s = -\log(T-t). \tag{2}$$

Using (1), we see that for all $(y,s) \in \mathbb{R}^N \times [-\log T, \infty)$,

$$\frac{\partial W_a}{\partial s} = \Delta W_a - \frac{1}{2} y.\nabla W_a - \frac{W_a}{p-1} + |W_a|^{p-1} W_a. \tag{3}$$

Giga and Kohn proved in [9], [10] and [11] that

$$W_a(y,s) \to \pm\kappa \equiv (p-1)^{-\frac{1}{p-1}} \text{ as } s \to \infty,$$

in $C(|y| < R)$ for any $R > 0$. In particular, $(T-t)^{\frac{1}{p-1}} u(a,t) \to \pm\kappa$ as $t \to T$. From Filippas and Kohn [7] and Herrero and Velázquez [13], we know that the speed of convergence is either $|\log(T-t)|^{-1}$ (slow) or $(T-t)^\mu$ (fast) for some $\mu > 0$. The dynamical system analysis of solutions of equation (3) in Fermanian, Merle and Zaag [5] easily shows that the slow speed is the only stable, with respect to perturbations of the blow-up point a.

We address in this paper the regularity of the blow-up set S, an issue that has been poorly studied in the literature. Indeed, most contributions focus on single point blow-up, where S is (locally) one isolated point (Weissler [22], Bricmont and Kupiainen [2], Herrero and Velázquez [13] and [20],...). The only pertinent result before [23] is due to Velázquez who showed in [20] that the Hausdorff measure of S is less or equal to $N-1$. In our opinion, authors could not go further in the description of the blow-up set because of the lack of *uniform* estimates of the solution near the blow-up time, with respect to (x,t) and to initial data.

In [14] and [16], we proved with Merle the following Liouville Theorem (or rigidity theorem) for entire solutions of (1):

Consider U a solution of (1) defined for all $(x,t) \in \mathbb{R}^N \times (-\infty, T)$ such that for all $(x,t) \in \mathbb{R}^N \times (-\infty,T)$, $|U(x,t)| \leq C(T-t)^{-\frac{1}{p-1}}$. Then, either $U \equiv 0$ or $U(x,t) = \pm\kappa(T^*-t)^{-\frac{1}{p-1}}$ for some $T^* \geq T$.

This Liouville Theorem allowed us indeed to get uniform estimates in [14], [15], [16], opening thus the door for a new approach in the study of blow-up for equation (1). As a matter of fact, in the case of an isolated blow-up point, we proved with Fermanian and

Merle [6] and [5] the stability of the blow-up profile for equation (1) with respect to initial data; in the case of a non isolated blow-up point, we obtained in [23] and [24] the *first* regularity results for the blow-up set.

Our approach in [23] and [24] (see [25] for a review) is based on two ideas:

- the refinement of the asymptotic behavior of u near (a,T) (or W_a as $s\to\infty$) uniformly with respect to the blow-up point a. The Liouville theorem is crucial in getting uniform estimates.

- the fact that a more refined (uniform) blow-up behavior yields geometrical constraints on the blow-up set S, resulting in more regularity for S.

In this paper, we follow these ideas, and do better! Indeed, we find here a new term in the expansion of u near the blow-up set and exhibit a *new* mechanism of algebraic cancellations which links the refined asymptotic behavior of the solution to the refined regularity for its blow-up set. This mechanism is the heart of our argument and constitutes the main novelty of this paper. As a consequence, we show that the blow-up set is C^2 and that the refined asymptotic behavior of u is expressed in terms of the principal curvatures and directions of its blow-up set.

Before stating our results, we would like to mention that the regularity of the blow-up set is an issue that arises also for semilinear wave equations like

$$u_{tt} = \Delta u + |u|^{p-1}u. \tag{4}$$

Due to the finite speed of propagation, a blow-up solution of this equation has a blow-up time $T(x)$ for each $x\in\mathbb{R}^N$, and its blow-up set is the graph $\{t=T(x)\}$, a subset of $\mathbb{R}^N\times\mathbb{R}$. By definition, the blow-up set is $1-$ Lipschitz (see Alinhac [1]). In [4] and [3], Caffarelli and Friedman showed under restrictions on the dimension, the power and initial data the C^1 regularity of the blow-up set. In [17], [19] and [18], we find with Merle the blow-up rate near the blow-up set for (4) when $1<p\le 1+4/(N-1)$, which is the first step towards the regularity of the blow-up set with no restrictions.

Let us now go back to the heat equation (1) and present our results. In [23], we showed that if the blow-up set is continuous (in some precise sense), then it is a C^1 manifold. In [24], we showed that if in addition, the blow-up set is of codimension 1, then it is $C^{1,\alpha}$ for any $\alpha\in(0,\frac12)$.

In order to stress more the novelty of our present contribution, we assume that we already know that locally near $\hat a$, S is a C^1 hypersurface, and that $(T-t)^{\frac{1}{p-1}}u(\hat a,t)\to\kappa$ with the (slow) speed $|\log(T-t)|^{-1}$, which is a reasonable hypothesis since this speed can be shown to be a stable behavior with the techniques of [5]. In [23] and [24], we have shown then that for some $t_0<T$ and $\delta>0$, for all $K_0>0$ and $t\in[t_0,T)$, we have:

For all $x\in B(\hat a,\delta)$ such that $\operatorname{dist}(x,S)\le K_0\sqrt{(T-t)|\log(T-t)|}$,

$$\left|(T-t)^{\frac{1}{p-1}}u(x,t)-f_1\left(\frac{\operatorname{dist}(x,S)}{\sqrt{(T-t)|\log(T-t)|}}\right)\right|\le C_0'(K_0)\frac{\log|\log(T-t)|}{|\log(T-t)|} \tag{5}$$

where

$$f_1(z) = \left(p - 1 + \frac{(p-1)^2}{4p} z_1^2\right)^{-\frac{1}{p-1}} \tag{6}$$

and

$$(T-t)^{\frac{1}{p-1}} \left| u(x,t) - \tilde{u}_{\sigma(P_S(x))} (\mathrm{dist}(x,S), t) \right| \le C(T-t)^{\frac{1}{2}} |\log(T-t)|^{\frac{3}{2}+C_0} \tag{7}$$

where $P_S(x)$ is the projection of x over S, $\sigma \in C(S \cap B(\hat{a}, \delta), \mathbb{R})$,

$$\tilde{u}_\sigma(x_1, t) = e^{-\frac{\sigma}{p-1}} \tilde{u}(e^{-\frac{\sigma}{2}} x_1, T - e^{-\sigma}(T-t)) \tag{8}$$

and $\tilde{u}(x_1, t)$ is a positive symmetric one dimensional solution of (1) that has the same profile f_1, and which decays on $(0, \infty)$ and blows up at time T only at the origin. Note that

$$\text{in the self-similar variables (2), } \tilde{u}_\sigma \text{ becomes } \tilde{w}(.,. + \sigma) \tag{9}$$

where $\tilde{w}(y_1, s)$ is defined by

$$\tilde{w}(y_1, s) = (T-t)^{\frac{1}{p-1}} \tilde{u}(x_1, t), \quad y_1 = \frac{x_1}{\sqrt{T-t}}, \quad s = -\log(T-t) \tag{10}$$

is a solution of (3).

In (7), the N-dimensional solution $u(x,t)$ appears as a superposition of one dimensional solutions of (1), $\tilde{u}_\sigma$, functions of the normal variable $\mathrm{dist}(x,S)$. Note that $\tilde{u}_\sigma$ for any $\sigma > 0$ are all dilations of the same one dimensional solution of (1), $\tilde{u}$, and that they all blow up at time T only at the origin. In other words, all the blow-up modalities in N dimensions are already contained in the one dimensional case. Estimate (7) may appear less interesting than (5) because the profile is explicit in (5), unlike (7). However, having a non explicit profile in (7) is the price to pay in order to get a smaller error term, which yields better regularity. In this paper, we are able to get to even smaller error terms and prove the C^2 regularity of the blow-up set. More precisely,

Theorem 1 (C^2 regularity of the blow-up set) *Assume $N \ge 2$ and consider u a solution of (1) that blows up at time T on a set S which is locally near some $\hat{a}$, a C^1 hypersurface. Assume in addition that $(T-t)^{\frac{1}{p-1}} u(\hat{a}, t) \to \kappa$ as $t \to T$ with the (slow) speed $|\log(T-t)|^{-1}$. Then, S is a C^2 hypersurface, locally near $\hat{a}$.*

Remark: From the sign invariance of the equation, the same conclusion holds if $(T-t)^{\frac{1}{p-1}} u(\hat{a}, t) \to -\kappa$.
This regularity result follows from the refinement of the blow-up behavior of $u(x,t)$ near (a, T) (or $W_a(y,s)$ as $s \to \infty$), uniformly with respect to the blow-up point a in a neighborhood of $\hat{a}$. More precisely, when a is a blow-up point, we link in the following proposition the blow-up behavior of u near (a, T) to the geometrical description of the blow-up set at a (normal vector, tangent space, principal curvatures and directions):

Proposition 2 (Link between the refined uniform blow-up behavior of the solution and the principal curvatures of its blow-up set) *Under the hypotheses of Theorem 1, there exist $\delta > 0$ and $s_0 \ge -\log T$ such that for all $a \in S \cap B(\hat{a}, \delta)$:*

(i) For all $s \geq s_0$,

$$\left\| W_a(M(a)y, s) - \tilde{w}(y_1, s+\sigma(a)) - \frac{\kappa e^{-\frac{s}{2}}}{4ps} y_1 \sum_{j=2}^{N} l_j(a)(y_j^2 - 2) \right\|_{L^2_\rho} \leq C e^{-\frac{s}{2}} s^{-1-\nu}$$

where $\nu \in (0, \frac{1}{2})$, W_a is defined in (2), $\tilde{w}$ is the one dimensional solution defined in (10), $\sigma(a) \in \mathbb{R}$ is continuous, $(l_j(a))_{2\leq j\leq N}$ and $(M(a)e_j)_{2\leq j\leq N}$ are respectively the principal curvatures and the unitary principal vectors of the blow-up set at a, and $M(a)e_1$ is a unitary normal vector. Note that $l_j(a)$ for all $2 \leq j \leq N$ and $M(a)e_1$ are continuous in terms of a. The convergence takes place in L^2_ρ, the L^2 space with respect to the weight

$$\rho(y) = e^{-|y|^2/4} / (4\pi)^{N/2} \tag{11}$$

as well as in $W^{2,\infty}(|y| < R)$ for any $R > 0$.

(ii) For all $R_0 > 0$, $t \in [T - s^{-s_0}, T)$ and $x \in B(\hat{a}, \frac{\delta}{2})$ such that $\operatorname{dist}(x, S) < R_0\sqrt{T-t}$,

$$(T-t)^{\frac{1}{p-1}} \left| u(x,t) - \left\{ \tilde{u}_{\sigma(P_S(x))}\left(d(x,S), t\right) - \frac{\kappa \tilde{d}(x,S)}{2p|\log(T-t)|} m(P_S(x)) \right\} \right|$$
$$\leq C(R_0) \frac{(T-t)^{\frac{1}{2}}}{|\log(T-t)|^{1+\nu}}$$

where $P_S(x)$ is the projection of x on S, $\sigma : S \to \mathbb{R}$ is continuous, $m : S \to \mathbb{R}$ is the mean curvature and $\tilde{d}(x,S)$ is the signed distance to the blow-up set.

(iii) It holds that for all $j \geq 2$,

$$l_j(a) = \lim_{t\to T} \frac{p}{4\kappa} |\log(T-t)|^{-1} (T-t)^{\frac{1}{p-1}-\frac{1}{2}}$$
$$\times \int_{\mathbb{R}^N} u(x,t)(x-a).M(a)e_1 \left(((x-a).M(a)e_j)^2 - 2 \right) \frac{e^{-\frac{|x-a|^2}{4(T-t)}}}{(4\pi)^{N/2}} dx.$$

Remark: Note that in (i) of the previous proposition, the direction y_1 is along the normal vector $M(a)e_1$, while the directions y_j (along the principal vector $M(a)e_j$) for $j \geq 2$ lay in the tangent space to the blow-up set at a. Unfortunately, we are unable to decide whether the principal vectors $M(a)e_j$ for $2 \leq j \leq N$ are continuous or not.

Remark: Estimate (ii) in Proposition 2 improves (7) proved in [24]. It is better because it has a smaller error term and a further term in the expansion involving a geometric feature of the blow-up set, the mean curvature. However, the price to pay is to reduce the convergence domain in space from $\operatorname{dist}(x,S) < \sqrt{(T-t)|\log(T-t)|}$ to $\operatorname{dist}(x,S) < \sqrt{T-t}$.

Remark: Unlike what may be understood from (iii) of Proposition 2, it is not necessary to know a principal direction in order to compute the corresponding principal curvature. Indeed, if we just know a normal vector and any orthogonal basis of the tangent space, then we can compute the second fundamental form in this basis, which gives by diagonalizing the principal curvatures and directions at once. See Proposition 4.1.

In Theorem 1, we derive C^2 regularity assuming C^1 regularity. Since we have derived in [23] and [24] $C^{1,\alpha}$ regularity assuming just continuity (in addition to a non-degeneracy property), we actually have a stronger version of Theorem 1 which derives C^2 regularity just assuming continuity. Stating this new version requires additional technical notation.

Let us consider a non isolated blow-up point $\hat{a}$ where for all $K_0 > 0$,

$$\sup_{|z|\le K_0}\left|(T-t)^{\frac{1}{p-1}}u\left(\hat{a}+Q(\hat{a})z\sqrt{(T-t)|\log(T-t)|},t\right)-f_1(z)\right|\to 0 \text{ as } t\to T \qquad (12)$$

where $Q(\hat{a})$ is an orthonormal $N\times N$ matrix and f_1 is defined in (6). We may take $Q(\hat{a})=\mathrm{Id}$. According to Theorem 2 in [20], for all $\epsilon>0$, there is $\delta(\epsilon)>0$ such that

$$S\cap B(\hat{a},\delta)\subset\Omega_{\hat{a},\pi,\epsilon}\equiv\{x\mid |P_\pi(x-\hat{a})|\ge(1-\epsilon)|x-\hat{a}|\},$$

where P_π is the orthogonal projection over π, the subspace spanned by e_2, ..., e_N. Note that $\Omega_{\hat{a},\pi,\epsilon}$ is a cone with vertex $\hat{a}$ that shrinks to $\hat{a}+\pi$ as $\epsilon\to 0$. In fact, $\hat{a}+\pi$ is the candidate for the tangent plane to S at $\hat{a}$. We assume there is $a\in C((-1,1)^{N-1},\mathbb{R}^N)$ such that $a(0)=\hat{a}$ and $\mathrm{Im}\,a\subset S$ where $\mathrm{Im}\,a$ is at least $(N-1)$-dimensional in the sense that

$$\begin{array}{l}\forall b\in\mathrm{Im}\,a,\text{ there are }(N-1)\text{ independent vectors }v_1,...,v_{N-1}\text{ in }\mathbb{R}^N\text{ and}\\ a_1,..,a_{N-1}\text{ functions in }C^1([0,1],\mathrm{Im}\,a)\text{ such that }a_i(0)=b\text{ and }a_i'(0)=v_i.\end{array} \qquad (13)$$

This hypothesis means that b is actually non isolated in $(N-1)$ independent directions. We also assume that $\hat{a}$ is not an endpoint in $\mathrm{Im}\,a$ in the sense that

$$\begin{array}{l}\forall\epsilon>0,\text{ the projection of }a((-\epsilon,\epsilon)^{N-1})\text{ on the plane }\hat{a}+\pi\\ \text{contains an open ball with center }\hat{a}.\end{array} \qquad (14)$$

We claim the following:

Theorem 3 (C^2 regularity of the blow-up set) *Assume that $(T-t)^{\frac{1}{p-1}}u(\hat{a},t)\to\kappa$ with the speed $|\log(T-t)|^{-1}$. Consider $a\in C((-1,1)^{N-1},\mathbb{R}^N)$ such that $\hat{a}=a(0)\in\mathrm{Im}\,a\subset S$ and $\mathrm{Im}\,a$ is at least $(N-1)$-dimensional in the sense (13). If $\hat{a}$ is not an endpoint (in the sense (14)), then, the blow-up set is a C^2 hypersurface, locally near $\hat{a}$. The conclusion of Proposition 2 holds in this case too.*

Remark: If $(T-t)^{\frac{1}{p-1}}u(\hat{a},t)\to-\kappa$, then the same conclusion holds with a sign change.

Our paper is organized in 3 sections:
- In Section 2, after introducing suitable local charts for the blow-up set and making a dynamical system formulation for equation (3), we explain a mechanism of a geometric constraint which links the blow-up behavior of the solution to the regularity of its blow-up set.
- In Section 3, we use this mechanism to get algebraic cancellations and find the refined blow-up behavior of the solution.
- In Section 4, we use again the geometric constraint mechanism and this refined behavior to derive the C^2 regularity of the blow-up set (Theorem 1). We then conclude the proof of Proposition 2. Finally, we briefly show how Theorem 3 follows from Theorem 1 and [23].

Throughout the paper, we work under the hypotheses of Theorem 1, which are also the hypotheses of Proposition 2. At the very end of the paper, we assume the weaker hypotheses stated in Theorem 3 and prove it using Theorem 1 and [23].

2 Setting of the problem and strategy of the proof

We find in this section a sharp estimate of the blow-up behavior, uniformly with respect to the blow-up set. To this end, we introduce a crucial geometric constraint mechanism on the blow-up set, which constitutes the heart of our argument and the main novelty of the paper.

2.1 Local $C^{1,\alpha}$ charts of the blow-up set

Under the hypotheses of Theorem 1, we know from Velázquez [21] and [20] (see also Filippas and Liu [8] and Filippas and Kohn [7]) that the local behavior of u near the blow-up point $\hat{a}$ is given by (12). Therefore, we can apply Proposition 3 of [24] and derive the existence of $\delta > 0$ such that $S_\delta \equiv S \cap B(\hat{a}, 2\delta)$ is a $C^{1,\alpha}$ manifold for any $\alpha \in (0, \frac{1}{2})$. In the following, we fix such an α. For convenience, we will use a local chart at every point $a \in S_\delta$ in the form of a graph $\{(\varphi_a(\xi_2, .., \xi_N), \xi_2, .., \xi_N)\}$ of a $C^{1,\alpha}([-\eta_a, \eta_a]^{N-1})$ function φ_a in a (direct) orthonormal basis $(a, n(a), \tau_2(a), ..., \tau_N(a))$ for some $\eta_a > 0$ where $n(a)$ and $\tau_i(a)$ are of norm 1 and respectively normal and tangent to S_δ. It is possible to take $n(a)$ and $\tau_i(a)$ of class C^α in terms of a. By construction, we have for all $i \geq 2$,

$$\varphi_a(0) = \frac{\partial \varphi_a}{\partial \xi_i}(0) = 0. \tag{15}$$

In other words, S_δ is locally near a given by the set

$$\{a + \varphi_a(\xi)n(a) + \sum_{i=2}^{N} \xi_i \tau_i(a) \mid |\xi| \leq \eta_a\} \text{ where } \xi = (\xi_2, ..., \xi_N). \tag{16}$$

Theorem 1 will be proved if we prove that for all $a \in S_\delta$, φ_a is C^2 at $\xi = 0$ and if we compute $\dfrac{\partial^2 \varphi_a}{\partial \xi_i \partial \xi_j}(0)$ for all $i,\ j \geq 2$. In dimension 2, there is only one number to compute, $\varphi_a''(0)$, which is precisely the curvature.

Let $Q(a)$ be the (direct) orthonormal matrix whose columns are $n(a)$ and $\tau_i(a)$, $i \geq 2$. Hence,

$$n(a) = Q(a)e_1 \text{ and } \tau_i(a) = Q(a)e_i \text{ and } Q(a) \text{ is of class } C^\alpha. \tag{17}$$

If w_a is defined by

$$w_a(y, s) = (T-t)^{\frac{1}{p-1}} u(x, t), \quad y = Q(a)^T \left(\frac{x-a}{\sqrt{T-t}} \right), \quad s = -\log(T-t), \tag{18}$$

then we see from (2) that $w_a(y, s) = W_a(Q(a)y, s)$ and that w_a is also a solution of (3) for all $(y, s) \in \mathbb{R}^N \times [-\log T, \infty)$. We have proved in [24] (Proposition 2.1) the following stronger version of (7) in self-similar variables[1]: for all $a \in S_\delta$ and $s \geq -\log T$,

$$\|w_a(y, s) - \tilde{w}(y_1, s + \sigma(a))\|_{L^2_\rho} \leq C e^{-\frac{s}{2}} s^{C_0}, \tag{19}$$

[1] In [23] and [24], y was equal to $\frac{x-a}{\sqrt{T-t}}$ in the definition analogous to (18). Therefore, estimates (19) and (20) come with $w_a(Q(a)y, s)$ instead of $w_a(y, s)$ in the cited papers.

where $\tilde{w}$ is defined in (10) and $\sigma : S_\delta \to \mathbb{R}$ is a continuous function. Note that any other value of $\sigma \neq \sigma(a)$ in (19) gives an error of order s^{-2} (see [24]), which is bad. The choice of $\sigma(a)$ gives the smallest error. It was made possible in [24] by means of modulation theory.

Recall that we already know from [23] (in particular in Proposition 3.1) that $w_a(y,s)$ and $\tilde{w}(y_1,s)$ have the same profile $f_1\left(\frac{y_1}{\sqrt{s}}\right)$ (6). In particular [2],

$$\sup_{a\in S_\delta,\ s\geq -\log T}\left\|w_a(y,s)-\left\{\kappa+\frac{\kappa}{2ps}\left(1-\frac{y_1^2}{2}\right)\right\}\right\|_{W^{2,\infty}(|y|<2)} \leq C\frac{\log s}{s^2}, \tag{20}$$

$$\sup_{s\geq -\log T}\left\|\tilde{w}(y_1,s)-\left\{\kappa+\frac{\kappa}{2ps}\left(1-\frac{y_1^2}{2}\right)\right\}\right\|_{W^{2,\infty}(|y_1|<2)} \leq C\frac{\log s}{s^2} \tag{21}$$

(the convergence takes place in L^2_ρ in [23] and the $W^{2,\infty}_{loc}$ estimate follows by parabolic regularity).

2.2 A dynamical system formulation

We need to refine estimate (19) on $w_a(y,s)-\tilde{w}(y_1,s+\sigma(a))$, the difference between two solutions of equation (3) which have the same profile f_1 (6).

The formulation is the same as we did with Fermanian in [6] for the difference of two solutions with the *radial* profile

$$f_N(z)=\left(p-1+\frac{(p-1)^2}{4p}|z|^2\right)^{-\frac{1}{p-1}}. \tag{22}$$

Therefore, we follow in extent the strategy of [6] and emphasize the novelties. However, some technical details -most of them are straightforward and long- are omitted. The reader can find them in [6]. Consider an arbitrary $a \in S_\delta$. If we define

$$g_a(y,s)=w_a(y,s)-\tilde{w}(y_1,s+\sigma(a)), \tag{23}$$

then we see from (3) that for all $(y,s)\in\mathbb{R}^N\times[-\log T+\sigma_0,\infty)$ where $\sigma_0=\max_{a\in B(\hat{a},\delta)\cap S}|\sigma(a)|$,

$$\partial_s g_a(y,s)=(\mathcal{L}+\alpha_a)\,g_a, \tag{24}$$

where $\mathcal{L}=\Delta-\frac{1}{2}y.\nabla+1$ and

$$\alpha_a(y,s)=\frac{|\ w_a\ |^{p-1}\ w_a(y,s)-\tilde{w}^p(y_1,s+\sigma(a))}{w_a-\tilde{w}}-\frac{p}{p-1} \tag{25}$$

if $w_a(y,s)\neq\tilde{w}(y_1,s+\sigma)$, and in general,

$$\alpha_a(y,s)=p\ |\ \bar{w}_a(y,s)\ |^{p-1}-\frac{p}{p-1} \tag{26}$$

for some $\bar{w}_a(y,s)\in\big(w_a(y,s),\tilde{w}(y_1,s+\sigma(a))\big)$.

[2]See previous footnote.

According to (19) which was proved in [24] and (23), $g_a \to 0$ in L^2_ρ as $s \to \infty$. More precisely, for all $s \geq -\log T + \sigma_0$,

$$\|g_a(s)\|_{L^2_\rho} \leq C e^{-\frac{s}{2}} s^{C_0}. \tag{27}$$

In this paper, we refine this estimate and find an equivalent of g_a in L^2_ρ. In order to do so, we need first to understand the dynamics of equation (24) satisfied by g_a. In practice, we need to know more about $\mathcal{L}$ and α_a.

- *The operator* $\mathcal{L}$: Operator $\mathcal{L}$ is self-adjoint on $\mathcal{D}(\mathcal{L}) \subset L^2_\rho(\mathbb{R}^N)$ where ρ is defined in (11). The spectrum of $\mathcal{L}$ consists of eigenvalues

$$\operatorname{spec}\mathcal{L} = \{1 - \frac{m}{2},\ \ m \in \mathbb{N}\}.$$

Note that except two positive eigenvalues (1 and $\frac{1}{2}$) and a null eigenvalue, all the spectrum is negative. The eigenfunctions of $\mathcal{L}$ are

$$h_\beta(y) = h_{\beta_1}(y_1)...h_{\beta_N}(y_N), \tag{28}$$

where $\beta = (\beta_1, ..., \beta_N) \in \mathbb{N}^N$ and for each $m \in \mathbb{N}$, h_m is the rescaled Hermite polynomial

$$h_m(\xi) = \sum_{j=0}^{[m/2]} \frac{m!}{j!(m-2j)!}(-1)^j \xi^{m-2j}. \quad \text{We note } \ k_m = \frac{h_m}{\|h_m\|^2_{L^2_{\rho_1}(\mathbb{R})}}, \tag{29}$$

where $L^2_{\rho_1}(\mathbb{R})$ is the L^2 space with the measure

$$\rho_1(\zeta) = \frac{e^{-\frac{\zeta^2}{4}}}{\sqrt{4\pi}} \text{ that satisfies } \rho(y) = \prod_{i=1}^{N} \rho_1(y_i). \tag{30}$$

The polynomials h_m and h_β satisfy the following

$$\begin{aligned} &\mathcal{L}h_\beta = (1 - \tfrac{|\beta|}{2})h_\beta, \qquad \int_{\mathbb{R}} h_m(\zeta)h_j(\zeta)\rho_1(\zeta)d\zeta = 2^m m!\delta_{m,j}, \\ &h'_m(\zeta) = m h_{m-1}(\zeta), \quad \frac{\partial h_\beta}{\partial y_i}(y) = \beta_i h_{\beta - e_i}(y) = \beta_i h_{\beta_i - 1}(y_i) \prod_{j=1,\, j\neq i}^{j=N} h_{\beta_j}(y_j) \end{aligned} \tag{31}$$

where e_i is the i-th vector of the canonical basis of $\mathbb{R}^N$ and with the convention that $h_m \equiv 0$ if $m \leq -1$.

- *The function* α_a: We claim the following:

Lemma 2.1 (Estimates on α) *For all $a \in S_\delta$, $y \in \mathbb{R}^N$ and $s \geq -\log T + \sigma_0$,*

$$\alpha_a(y,s) \leq \frac{C}{s}, \ \ |\alpha_a(y,s)| \leq \frac{C}{s}(1+|y|^2) \textit{and } |\alpha_a(y,s) + \frac{1}{4s}h_2(y_1)| \leq \frac{C}{s^{3/2}}(1+|y|^3). \tag{32}$$

Proof: See Lemmas 2.5 in [24] and [6]. ■

From the first estimate, we see that α_a doesn't shift much the spectrum of the linear operator in the positive direction. Therefore, it is convenient to project equation (24) on the eigenfunctions of $\mathcal{L}$ in order to understand its dynamics.

- *Decomposition of g_a with respect to the spectrum of $\mathcal{L}$*: Since the family $\{h_\beta(y)\ |\beta \in \mathbb{N}\}$ spans all the space $L^2_\rho(\mathbb{R}^N)$, let us introduce the orthogonal projection of $g_a(.,s)$ on h_β:

$$g_{a,\beta}(s) = \int_{\mathbb{R}^N} k_\beta(y) g_a(y,s)\rho(y)dy \text{ where } k_\beta(y) = \| h_\beta \|^{-2}_{L^2_\rho} h_\beta(y). \tag{33}$$

If P_n is the orthogonal projector of L^2_ρ over the eigenspace of $\mathcal{L}$ corresponding to $1-\frac{n}{2}$ and $R_k g_a$ the sum of all $P_n g_a$ for $n \geq k$, then

$$\begin{cases} P_n g_a(y,s) = \sum_{|\beta|=n} g_{a,\beta}(s) h_\beta(y), \\ g_a(y,s) = \sum_{n\in\mathbb{N}} P_n g_a = \sum_{\beta\in\mathbb{N}^N} g_{a,\beta}(s)h_\beta(y) = \sum_{|\beta|\leq k} g_{a,\beta}(s) h_\beta(y) + R_{k+1} g_a(y,s) \\ \|g_a(s)\|^2_{L^2_\rho} \equiv I_a(s)^2 = \sum_{n\in\mathbb{N}} l_{a,n}(s)^2 = \sum_{n\leq k} l_{a,n}(s)^2 + r_{a,k+1}(s)^2. \end{cases} \tag{34}$$

where $l_{a,n}(s) \equiv \|P_n g_a\|_{L^2_\rho}$ and $r_{a,k}(s) \equiv \|R_k g_a\|_{L^2_\rho}$. In the following, we project equation (24) on the different modes:

Lemma 2.2 (Projection of (24) on the different modes) *There exist $s_1 \geq -\log T$ and $s_* > 0$ such that for all $a \in S_\delta$, $s \geq s_1$, $n \in \mathbb{N}$ and $\beta \in \mathbb{N}^N$, we have the following:*
(i) $|l'_{a,n} + (\frac{n}{2}-1) l_{a,n}| \leq C(n)\frac{I_a(s)}{s}$,
(ii) $I'_a(s) \leq \left(1 - \frac{n+1}{2} + \frac{C_0}{s}\right) I_a(s) + \sum_{k=0}^{n} \frac{1}{2}(n+1-k) l_{a,k}(s)$.
(iii) $\left| g'_{a,\beta}(s) + \left(-1 + \frac{|\beta|}{2} + \frac{\beta_1}{s}\right) g_{a,\beta}(s)\right| \leq C(\beta)\frac{I_a(s)}{s}$.
(iv) $\left| g'_{a,\beta}(s) + \left(-1 + \frac{|\beta|}{2} + \frac{\beta_1}{s}\right) g_{a,\beta}(s)\right| \leq C(\beta)\left(\frac{l_{a,|\beta|-2}(s) + l_{a,|\beta|+2}(s)}{s} + \frac{I_a(s)}{s^{\frac{3}{2}}}\right)$.
(v) $r'_{a,n} \leq (1-\frac{n}{2}) r_{a,n} + \frac{C}{s} I_a(s-s_*)$.

Proof: The calculations are based on (24) and the definition of α_a (25). They are straightforward. For (i) and (ii), see the proof of Lemma 2.7 in [6], where the same equation is treated (with a function α derived from the radial profile (22) instead of f_1 (6)). Note that (iii) follows immediately from (iv). For (iv), see Appendix V.1 in [24] for a similar calculation. Thus, we only prove (v) in Appendix A. ■

2.3 Strategy of the proof: a geometric constraint linked to the asymptotic behavior

Using the equations of Lemma 2.2, we will refine in the space L^2_ρ the estimate (27) on g_a and get to the first significant term as $s \to \infty$, uniformly in $a \in S_\delta$. This term depends continuously on $a \in S_\delta$. Using parabolic regularity and the definition (23) of g_a, this yields an expansion for w_a in $W^{2,\infty}(|y| < 2)$, uniformly in $a \in S_\delta$. Using the definition

(18) of w_a, we see that in the $u(x,t)$ formulation, the domain of validity of this expansion is $B(a, 2\sqrt{T-t})$, for each $a \in S_\delta$. These domains overlap, leading to as many expansions for the *same thing*, namely u and its derivatives at a given point (x,t), as there are points in $S_\delta \cap B(x, 2\sqrt{T-t})$. Of course, the leading terms of these expansions must be the same. This is how a geometric constraint (here, the overlapping and later, the regularity) is related to algebraic relations in the coefficients of the asymptotic behavior of w_a.

To illustrate this mechanism, we fix some $a \in S_\delta$, $|y| < 2$ and $s \geq -\log T$. In the $u(x,t)$ formulation, the point (y,s) from the domain of w_a (18) becomes the point (x,t) where

$$x = a + e^{-\frac{s}{2}} Q(a) y = a + e^{-\frac{s}{2}} \left(y_1 n(a) + \sum_{j=2}^{N} y_j \tau_j(a) \right), \tag{35}$$

$$t = T - e^{-s} \text{ and } w_a(y,s) = (T-t)^{\frac{1}{p-1}} u(x,t)$$

(see (17) and (18)). Now, take an arbitrary $b \in S_\delta$ and use again (18) (but in the other way) to write

$$u(x,t) = (T-t)^{-\frac{1}{p-1}} w_b(Y,s) \text{ where } Y = Q(b)^T \left(\frac{x-b}{\sqrt{T-t}} \right).$$

Therefore, we have

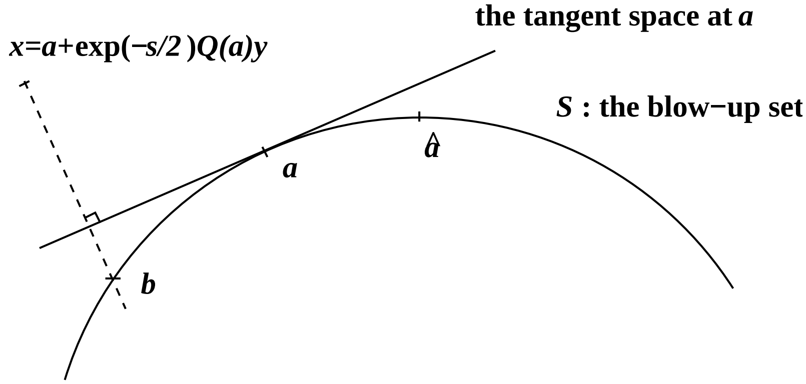
$$w_a(y,s) = w_b(Y,s) \text{ where } Y = Q(b)^T \left(Q(a) y + e^{\frac{s}{2}} (a-b) \right). \tag{36}$$

Since $n(a)$ and $\tau_i(a)$ for $i \geq 2$ are the column vectors of $Q(a)$ (see (17)), we can differentiate (36) with respect to y_i, $i \geq 2$, and write two formulations for the tangential derivative of u with respect to $\tau_i(a)$ at the point (x,t) defined in (35):

$$(T-t)^{\frac{1}{p-1}+\frac{1}{2}} \frac{\partial u}{\partial \tau_i(a)}(x,t) = \frac{\partial w_a}{\partial y_i}(y,s) = \tau_i(a).n(b) \frac{\partial w_b}{\partial y_1}(Y,s) + \sum_{k=2}^{N} \tau_i(a).\tau_k(b) \frac{\partial w_b}{\partial y_k}(Y,s). \tag{37}$$

Now, if we use the equations of Lemma 2.2 to refine the smallness condition (27) on g_a and use parabolic regularity, we obtain an expansion on $\frac{\partial w_a}{\partial y_k}$ and $\frac{\partial w_b}{\partial y_k}$ as $s \to \infty$, uniform for all points in S_δ. Thus, we will obtain two expansions for the same object (the tangential derivative of u) expressed in (37). These expansions have naturally to agree (up to error terms), leading to constraints on their coefficients and then to more regularity for the blow-up set.

Given $a \in S_\delta$, $|y| < 2$ and $s \geq -\log T$, b is still free in S_δ for the moment. However, there is one choice that makes the comparison of the two expansions particularly simple: we fix b as the (non orthogonal!) projection on the blow-up set of $x = a + e^{-\frac{s}{2}} Q(a) y$ defined in (35), in the orthogonal direction to the tangent space to the blow-up set at a.



Thus, in the local chart (16), b has the same components on the tangent space spanned by $\{\tau_j(a),\ j \geq 2\}$ as x and has the following particularly simple expression:

$$b = b(a, y, s) = a + \varphi_a(e^{-\frac{s}{2}}\tilde{y})n(a) + e^{-\frac{s}{2}} \sum_{j=2}^{N} y_j \tau_i(a) \text{ where } \tilde{y} = (y_2, ..., y_N). \tag{38}$$

We claim the following:

Proposition 2.3 (A geometric constraint on the expansion of w_a)
There exists $s_2 \geq -\log T$ such that for all $a \in S_\delta$, $|y| \leq 1$, $s \geq s_2$ and $i = 2, ..., N$, it holds that

$$\left|\frac{\partial w_a}{\partial y_i}(y, s) - \left\{\frac{\partial w_b}{\partial y_i}(y_1, 0, ..., 0, s) + \frac{\partial \varphi_a}{\partial \xi_i}(e^{-\frac{s}{2}}\tilde{y})y_1\frac{\kappa}{2ps}\right\}\right| \tag{39}$$
$$\leq C\left|\frac{\partial \varphi_a}{\partial \xi_i}(e^{-\frac{s}{2}}\tilde{y})\right|\left[|y_1|\frac{\log s}{s^2} + \frac{e^{-\frac{\alpha}{2}s}}{s}\right] + Ce^{-\frac{1+\alpha}{2}s}s^{C_0},$$

where b and $\tilde{y}$ are defined in (38).

Proof: We transform here some terms appearing in (37) in order to get (39).

(a) **Term** $\tau_i(a).n(b)$. Since φ_a is $C^{1,\alpha}$ and $\varphi_a(0) = \nabla\varphi_a(0) = 0$, we have for all $|\xi| < \eta_a$,

$$|\varphi(\xi)| \leq C|\xi|^{1+\alpha} \text{ and } |\nabla\varphi(\xi)| \leq C|\xi|^\alpha. \tag{40}$$

Using the local coordinates (38) and (40), we have

$$n(b) = \left(n(a) - \sum_{j=2}^{N} \frac{\partial \varphi_a}{\partial \xi_j}(e^{-\frac{s}{2}}\tilde{y})\tau_j(a)\right) / \sqrt{1 + |\nabla\varphi_a|(e^{-\frac{s}{2}}\tilde{y})^2} \tag{41}$$
$$\text{and } \left|\tau_i(a).n(b) + \frac{\partial \varphi_a}{\partial \xi_i}(e^{-\frac{s}{2}}\tilde{y})\right| \leq \left|\frac{\partial \varphi_a}{\partial \xi_i}(e^{-\frac{s}{2}}\tilde{y})\right| e^{-\alpha s}$$

where $\tilde{y}$ is defined in (38).

(b) **Term** $\tau_i(a).\tau_k(b)$. Using (38) and (40), we see that $|a - b| \leq |\varphi_a(\tilde{y}e^{-\frac{s}{2}})| + |e^{-\frac{s}{2}}\tilde{y}| \leq Ce^{-\frac{s}{2}}$. Since $n(a)$ and $\tau_j(a)$ are C^α, it holds then that $|n(a) - n(b)| + |\tau_j(a) - \tau_j(b)| \leq C|a - b|^\alpha \leq Ce^{-\frac{\alpha}{2}s}$. Therefore,

$$\begin{aligned} &|n(a).n(b) - 1| \leq Ce^{-\frac{\alpha}{2}s}, && |n(a).\tau_j(b)| \leq Ce^{-\frac{\alpha}{2}s}, \\ &|\tau_i(a).\tau_j(b) - \delta_{i,j}| \leq Ce^{-\frac{\alpha}{2}s}, && |\tau_j(a).n(b)| \leq Ce^{-\frac{\alpha}{2}s}. \end{aligned} \tag{42}$$

(c) **The point** $Y(a,y,s)$. Using the expressions (17), (36) and (38) of $\tau_i(a)$, Y and b, we write $Y_j = Y.e_j = \left(y_1 - e^{\frac{s}{2}}\varphi_a(e^{-\frac{s}{2}}\tilde y)\right) n(a).Q(b)e_j$. Using (17), (40) and (42), we get

$$|Y_1 - y_1| \le Ce^{-\frac{\alpha}{2}s} \text{ and } \forall j\ge 2,\ \ |Y_j| \le Ce^{-\frac{\alpha}{2}s}. \tag{43}$$

(d) **Term** $\frac{\partial w_b}{\partial y_1}(Y,s)$. Using parabolic regularity, we get by differentiation from (27) and (23) the following estimate:

$$\left|\frac{\partial w_b}{\partial y_1}(Y,s) - \frac{\partial \tilde w}{\partial y_1}\left(Y_1, s+\sigma(b)\right)\right| + \sum_{k=2}^{N}\left|\frac{\partial w_b}{\partial y_k}(Y,s)\right| + \sup_{|z|<2,(i,j)\neq(1,1)}\left|\frac{\partial^2 w_b}{\partial y_i\partial y_j}(z,s)\right| \tag{44}$$
$$\le Ce^{-\frac{s}{2}}s^{C_0}.$$

From (21), we have $\left|\frac{\partial^2\tilde w}{\partial y_1^2}(z,s') + \frac{\kappa}{2ps'}\right| \le C\frac{\log s'}{s'^2}$, for all $s' \ge -\log T$ and $|z|<2$. Since $\frac{\partial \tilde w}{\partial y_1}(0,s') \equiv 0$ ($\tilde u$ and $\tilde w$ are symmetric), we interpolate to get

$$\left|\frac{\partial \tilde w}{\partial y_1}\left(Y_1, s+\sigma(b)\right) + Y_1\frac{\kappa}{2ps}\right| \le C|Y_1|\frac{\log s}{s^2}. \tag{45}$$

Using (44), (45) and (43), we get

$$\left|\frac{\partial w_b}{\partial y_1}(Y,s) + y_1\frac{\kappa}{2ps}\right| \le C|y_1|\frac{\log s}{s^2} + C\frac{e^{-\frac{\alpha}{2}s}}{s}. \tag{46}$$

(e) **Term** $\frac{\partial w_b}{\partial y_i}(Y,s)$ where $i\ge 2$. Using (43) and the estimate of the second derivative in (44), we get

$$\left|\frac{\partial w_b}{\partial y_i}(Y,s) - \frac{\partial w_b}{\partial y_i}(y_1,0,...,0,s)\right| \le Ce^{-\frac{(1+\alpha)}{2}s}s^{C_0}. \tag{47}$$

We have just found expansions for all the terms involved in (37): (41) for $\tau_i(a).n(b)$, (46) for $\frac{\partial w_b}{\partial y_1}(Y,s)$, (42) for $\tau_i(a).\tau_k(a)$, (44) for $\frac{\partial w_b}{\partial y_k}(Y,s)$ when $k\neq i$ and (47) for $\frac{\partial w_b}{\partial y_k}(Y,s)$ when $k=i$. Using this with (37) yields (39) and concludes the proof of Proposition 2.3.■

3 Refined blow-up behavior

In this section, we use the equations of Lemma 2.2 and the mechanism of geometric constraint in order to refine estimate (27) and find an equivalent of g_a in L^2_ρ, continuous in terms of the blow-up point a. The continuous matrix of the coefficients of that equivalent will be shown to be equal (up to some constant factors) to the second fundamental form of the blow-up set, which gives the C^2 regularity (section 4). In the following proposition, we find an equivalent for g_a:

Proposition 3.1 (Term of order $\frac{e^{-\frac{s}{2}}}{s}$ in the expansion of w) *There exist $\mu \in (0, \frac{1}{2})$ and continuous functions $a \to \lambda_\beta(a)$ for all $\beta \in \mathbb{N}^N$ with $|\beta| = 3$ and $\beta_1 = 1$ such that for all $a \in S_\delta$ and $s \geq s_1$,*

$$\left\| g_a(y,s) - \frac{e^{-\frac{s}{2}}}{s} \sum_{|\beta|=3,\ \beta_1=1} \lambda_\beta(a) h_\beta(y) \right\|_{L^2_\rho} \leq C e^{-\frac{s}{2}} s^{-1-\mu}$$

where s_1 is defined in Lemma 2.2.

This section is devoted to the proof of this proposition. In regard of estimate (27) and the decomposition (34), the modes of the eigenvalue $-\frac{1}{2}$, namely $g_{a,\beta}$ for $|\beta| = 3$ are resonant. We will call high frequencies the modes $g_{a,\beta}$ for $|\beta| \leq 2$ and low frequencies the modes $g_{a,\beta}$ for $|\beta| \geq 4$, or simply $R_4 g_a$.

Plugging the rough estimate (27) into the equations on the different modes in Proposition 2.2, we will find a first expansion (terms of order $e^{-\frac{s}{2}}$). Using the geometric constraint mechanism of Proposition 2.3, we show that all terms of order $e^{-\frac{s}{2}}$ are zero. We then reiterate the process and use again Proposition 2.2 in order to get the terms of order $\frac{e^{-\frac{s}{2}}}{s}$ and conclude the proof of Proposition 3.1.

3.1 Vanishing of the term of order $e^{-\frac{s}{2}}$

Let us recall from (27) and (34) that for all $s \in S_\delta$ and $s \geq -\log T + \sigma_0$,

$$I_a(s) = \|g_a(s)\|_{L^2_\rho} \leq C e^{-\frac{s}{2}} s^{C_0}. \tag{48}$$

In this step, we use the equations of Lemma 2.2 to refine estimate (48) and catch the term of order $e^{-\frac{s}{2}}$ in the expansion of w_a. Using the geometric constraint technique of Proposition 2.3, we derive better regularity for the blow-up set from this sharper estimate.

Let us first catch the the term of order $e^{-\frac{s}{2}}$ in the following:

Lemma 3.2 (Term of order $e^{-\frac{s}{2}}$ in the expansion of w) *There exist $\nu \in (0, \frac{1}{2})$ and continuous functions $a \to \lambda_\beta(a)$ for all $\beta \in \mathbb{N}^N$ with $|\beta| = 3$ and $\beta_1 = 0$ such that for all $a \in S_\delta$ and $s \geq s_1$,*

$$\left\| g_a(y,s) - e^{-\frac{s}{2}} \sum_{|\beta|=3,\ \beta_1=0} \lambda_\beta(a) h_\beta(y) \right\|_{L^2_\rho} \leq C e^{-\frac{s}{2}} s^{-\nu}$$

where s_1 is introduced in Lemma 2.2.

Remark: Note that in 2 dimensions, $\beta = (0,3)$ is the only admissible β in the sum above.
Proof: Using (48), we claim the following:

$$\text{If } C_0 > \frac{1}{2}, \quad \text{then } \forall s \geq s_1, \qquad I_a(s) \leq C e^{-\frac{s}{2}} s^{C_0 - \frac{1}{2}}, \tag{49}$$

$$\exists \gamma \in (0, \frac{1}{2}) \text{ such that } \forall s \geq s_1, \qquad I_a(s) \leq C e^{-\frac{s}{2}} s^{\gamma}. \tag{50}$$

It is clear that (50) follows from (48) and (49) by a finite induction. Hence, we focus on the proof of (49). Recalling from (34) that

$$\|g_a(s)\|^2_{L^2_\rho} = I_a(s)^2 = \sum_{n\le 2} l_{a,n}(s)^2 + \sum_{|\beta|=3} g_{a,\beta}(s)^2\|h_\beta(y)\|^2_{L^2_\rho} + r_{a,4}(s)^2, \tag{51}$$

we claim the following:

Lemma 3.3 (Smallness of high and low frequencies and equation on resonant frequencies) *Assume that for some $d\in\mathbb{R}$,*

$$\forall s\ge s_1,\quad \|g_a(y,s)\|_{L^2_\rho} \le Ce^{-\frac{s}{2}}s^d. \tag{52}$$

$$\textit{Then,}\quad \forall s\ge s_1,\quad \sup_{n\le 2}|l_{a,n}(s)| + r_{a,4}(s) \le Ce^{-\frac{s}{2}}s^{d-1} \tag{53}$$

$$\textit{and}\quad \forall |\beta|=3,\quad \left|\frac{d}{ds}\left(g_{a,\beta}(s)e^{\frac{s}{2}}s^{\beta_1}\right)\right| \le Cs^{\beta_1+d-\frac{3}{2}}. \tag{54}$$

Proof: Using (52) and Lemma 2.2, we write for all $s\ge s_1$,

$$\forall n\le 2,\quad \left|\frac{d}{ds}\left(l_{a,n}e^{(\frac{n}{2}-1)s}\right)\right| \le Ce^{(\frac{n}{2}-\frac{3}{2})s}s^{d-1},\ \text{and}\ \frac{d}{ds}\left(r_{a,4}e^s\right) \le Ce^{\frac{s}{2}}s^{d-1}.$$

Integrating the first equation between s and ∞ and the second between s_1 and s yields (53). Using (53) (remember that $l_{a,5}\le r_{a,4}$), (52) and (iv) in Lemma 2.2 yields (54). ■

Using Lemma 3.3, we see that the conclusion of (49) follows from (51), (53) and the integration of (54) between s_1 and s. Thus, (49) and then (50) hold.

Using (50) and Lemma 3.3, we see from (54) that for any $|\beta|=3$ with $\beta_1=0$, there exists a continuous function $a\to\lambda_\beta(a)$ such that

$$|g_{a,\beta}(s) - \lambda_\beta(a)e^{-\frac{s}{2}}| \le Ce^{-\frac{s}{2}}s^{\gamma-\frac{1}{2}}. \tag{55}$$

If $|\beta|=3$ and $\beta_1\ge 1$, then we integrate (54) between s and ∞ to get

$$|g_{a,\beta}(s)| \le Ce^{-\frac{s}{2}}s^{\gamma-\frac{1}{2}}. \tag{56}$$

Since (53) holds too (with $d=\gamma$), the conclusion of Lemma 3.2 follows from (53), (55) and (56) by the decomposition (34). ■

Now, we are in a position to gain more regularity on φ_a, the local chart defined in (16).

Lemma 3.4 (Almost $C^{1,1}$ regularity for S_δ) *There exists $\xi_0>0$ such that for each $a\in S_\delta$, the local chart defined in (16) satisfies for all $i\ge 2$ and $|\xi|<\xi_0$,*

$$\left|\frac{\partial\varphi_a}{\partial\xi_i}(\xi)\right| \le C|\xi||\log|\xi||.$$

Proof: Using Lemma 3.2 and parabolic regularity (remember that $\frac{\partial \tilde w}{\partial y_i}\equiv 0$ for $i\geq 2$ and from (23) that $g_a = w_a-\tilde w$), we see that for all $i\geq 2$ and $s\geq s_1+1$

$$\sup_{a\in S_\delta,\ |y|<2}\left|\frac{\partial w_a}{\partial y_i}(y,s)\right|\leq Ce^{-\frac{s}{2}}.$$

Consider $a\in S_\delta$, $i\geq 2$, $y=(1,\tilde y)$ where $\tilde y$ is arbitrary in $\partial B_{N-1}(0,1)$ and $s\geq \max(s_1+1,s_2)$, and consider $b=b(a,y,s)$ defined in (38). Using Proposition 2.3, we write

$$\left|\frac{\partial \varphi_a}{\partial \xi_i}(e^{-\frac{s}{2}}\tilde y)\frac{\kappa}{2ps}\right|\leq C\left|\frac{\partial \varphi_a}{\partial \xi_i}(e^{-\frac{s}{2}}\tilde y)\right|\frac{\log s}{s^2}+Ce^{-\frac{1+\alpha}{2}s}s^{C_0}+Ce^{-\frac{s}{2}},$$

therefore, $\left|\frac{\partial \varphi_a}{\partial \xi_i}(e^{-\frac{s}{2}}\tilde y)\right|\leq Cse^{-\frac{s}{2}}$. If $\xi=e^{-\frac{s}{2}}\tilde y$, then $|\xi|=e^{-\frac{s}{2}}$ and $|\log|\xi||=|\frac{s}{2}-\log|\tilde y||$, hence

$$\left|\frac{\partial \varphi_a}{\partial \xi_i}(\xi)\right|\leq C|\xi||\log|\xi||, \tag{57}$$

Since $\tilde y=(y_2,...,y_N)$ is arbitrary in $\partial B_{N-1}(0,1)$, $\xi=e^{-\frac{s}{2}}\tilde y$ covers a whole neighborhood of 0, $B(0,\xi_0)$ where $\xi_0=e^{-\frac{1}{2}\max(s_2,s_1+1)}$ and (57) concludes the proof of Lemma 3.4. ■

This refined regularity for φ_a implies a constraint on the asymptotic behavior of $w_a(y,s)$: all the coefficients $\lambda_\beta(a)$ with $|\beta|=3$ and $\beta_1=0$ in Lemma 3.2 have to be identically zero. More precisely:

Proposition 3.5 (Vanishing of the term of order $e^{-\frac{s}{2}}$) *There exists $\nu\in(0,\frac{1}{2})$ such that for all $a\in S_\delta$ and $s\geq s_1$, $\|g_a(y,s)\|_{L^2_\rho}\leq Ce^{-\frac{s}{2}}s^{-\nu}$.*

Proof: Consider $a\in S_\delta$. From Lemma 3.2, it is enough to prove that $\lambda_\beta(a)=0$ for all $|\beta|=3$ with $\beta_1=0$. Using Lemma 3.2 and parabolic regularity, we see that for all $i\geq 2$ and $s\geq s_1+1$,

$$\sup_{a\in S_\delta,\ |y|<2}\left|\frac{\partial w_a}{\partial y_i}(y,s)-e^{-\frac{s}{2}}\sum_{|\beta|=3,\beta_1=0}\lambda_\beta(a)\frac{\partial h_\beta}{\partial y_i}(y)\right|\leq Ce^{-\frac{s}{2}}s^{-\nu}. \tag{58}$$

Take $y=(0,\tilde y)$ where $\tilde y$ is arbitrary in $B_{N-1}(0,1)$. We would like to apply Proposition 2.3 for this choice of y (note that $y_1=0$) and make $s\to\infty$ in order to get algebraic cancellations and conclude. Using (58) for $\dfrac{\partial w_a}{\partial y_i}$ and $\dfrac{\partial w_b}{\partial y_i}$, (31) for $\dfrac{\partial h_\beta}{\partial y_i}(y)$, (40) and Lemma 3.4 for $\dfrac{\partial \varphi_a}{\partial \xi_i}$, and the fact that $y_1=0$ (note also from (29) that $h_{\beta_1}(y_1)=h_0(0)=1$), we see that Proposition 2.3 yields

$$\Bigg|e^{-\frac{s}{2}}\sum_{|\beta|=3,\ \beta_1=0}\lambda_\beta(a)\,\beta_i h_{\beta_i-1}(y_i)\prod_{j=2,\ j\neq i}^{N}h_{\beta_j}(y_j)$$
$$-e^{-\frac{s}{2}}\sum_{|\beta|=3,\ \beta_1=0}\lambda_\beta(b)\beta_i h_{\beta_i-1}(0)\prod_{j=2,\ j\neq i}^{N}h_{\beta_j}(0)\Bigg|\leq Ce^{-\frac{s}{2}}s^{-\nu}$$

for all $s \geq \max(s_1+1, s_2)$. Since $b \to a$ as $s \to \infty$ (see (38)) and $a \to \lambda_\beta(a)$ is continuous (see Lemma 3.2), we get at the limit as $s \to \infty$: for all $\tilde{y} \in B_{N-1}(0,1)$,

$$\sum_{|\beta|=3,\ \beta_1=0} \lambda_\beta(a)\beta_i h_{\beta_i-1}(y_i) \prod_{j=2,\ j\neq i}^{N} h_{\beta_j}(y_j) = \sum_{|\beta|=3,\ \beta_1=0} \lambda_\beta(a)\beta_i h_{\beta_i-1}(0) \prod_{j=2,\ j\neq i}^{N} h_{\beta_j}(0)$$

By orthogonality of the polynomials h_k, this yields for all $i \geq 2$ and $|\beta| = 3$ with $\beta_1 = 0$, $\beta_i\lambda_\beta(a) = 0$. Now, take any β with $|\beta| = 3$ and $\beta_1 = 0$. Since there exists $i \geq 2$ such that $\beta_i \geq 1$, this implies that $\lambda_\beta(a) = 0$. Thus, Lemma 3.2 yields the conclusion of Proposition 3.5. ■

3.2 Term of order $\frac{e^{-\frac{s}{2}}}{s}$

Now, we are ready to catch the term of order $\frac{e^{-\frac{s}{2}}}{s}$ in the expansion of g_a. In fact, in this subsection, we prove Proposition 3.1.

Proof of Proposition 3.1:
Using Proposition 3.5 and Lemma 3.3, we see that for all $s \geq s_1$,

$$\sup_{n\leq 2} |l_{a,n}(s)| + r_{a,4}(s) \leq Ce^{-\frac{s}{2}} s^{-\nu-1} \tag{59}$$

$$\text{and} \quad \forall |\beta| = 3, \quad \left| \frac{d}{ds}\left(g_{a,\beta}(s) e^{\frac{s}{2}} s^m \right) \right| \leq C s^{\beta_1-\nu-\frac{3}{2}} \tag{60}$$

where $\nu \in (0, \frac{1}{2})$. When $|\beta| = 3$, we integrate (60) between s and ∞ if $\beta_1 = 0$ and between s_1 and s if $\beta_1 \geq 2$ to get

$$|g_{a,\beta}(s)| \leq Ce^{-\frac{s}{2}} s^{-\nu-\frac{1}{2}}. \tag{61}$$

If $|\beta| = 3$ and $\beta_1 = 1$, (60) implies the existence of a continuous function $a \to \lambda_\beta(a)$ such that

$$|g_{a,\beta}(s) - \lambda_\beta(a) \frac{e^{-\frac{s}{2}}}{s}| \leq Ce^{-\frac{s}{2}} s^{-\nu-\frac{1}{2}}. \tag{62}$$

Using the decomposition (34), we see that (59), (61) and (62) yield Proposition 3.1. ■

4 C^2 regularity linked to the refined uniform blow-up behavior

In this section, we use the refined asymptotic behavior of Proposition 3.1 and the geometric constraint of Proposition 2.3 to conclude the proofs of Theorem 1 and Proposition 2. We will also show how to get Theorem 3 from Theorem 1 and [23].

Proof of Theorem 1:
Using Proposition 3.1, parabolic regularity and (31), we see that for all $i \geq 2$ and $s \geq s_1+1$,

$$\sup_{a\in S_\delta,\ |y|<2} \left| \frac{\partial w_a}{\partial y_i}(y,s) - \frac{e^{-\frac{s}{2}}}{s} \sum_{|\beta|=3,\ \beta_1=1} \lambda_\beta(a) h_1(y_1) \beta_i h_{\beta_i-1}(y_i) \prod_{j=2,\ j\neq i}^{N} h_{\beta_j}(y_j) \right|$$
$$\leq Ce^{-\frac{s}{2}} s^{-1-\nu}.$$

Since $h_k(0) = 0$ when k is odd, it is easy to see from (29) that for all $\omega = \pm 1$, $j \geq 2$ and $s \geq s_1 + 1$,

$$\begin{aligned}\left|\frac{\partial w_a}{\partial y_i}(e_1 + \omega e_j, s) \quad - \quad \omega \frac{e^{-\frac{s}{2}}}{s}(1 + \delta_{i,j})\lambda_{e_1+e_i+e_j}(a)\right| &\leq Ce^{-\frac{s}{2}}s^{-1-\nu},\\ \left|\frac{\partial w_a}{\partial y_i}(e_1, s)\right| \quad &\leq \quad Ce^{-\frac{s}{2}}s^{-1-\nu}\end{aligned}$$

where e_k is the k-th vector of the canonical base of $\mathbb{R}^N$. Using the geometric constraint of Proposition 2.3, we write then for $y = e_1 + \omega e_j$ (note that $y_1 = 1$ and $y_k = \omega \delta_{k,j}$ for $k \geq 2$) and $s \geq \max(s_1 + 1, s_2)$,

$$\left|\omega \frac{e^{-\frac{s}{2}}}{s}(1 + \delta_{i,j})\lambda_{e_1+e_i+e_j}(a) - \frac{\partial \varphi_a}{\partial \xi_i}(\omega e^{-\frac{s}{2}} e_j)\frac{\kappa}{2ps}\right| \leq C\left|\frac{\partial \varphi_a}{\partial \xi_i}(\omega e^{-\frac{s}{2}} e_j)\right| \frac{\log s}{s^2} + Ce^{-\frac{s}{2}}s^{-1-\nu}.$$

Therefore, since $\frac{\partial \varphi_a}{\partial \xi_i}(0) = 0$ (see (15)), we have for all $j \geq 2$ and $\omega = \pm 1$:

$$\frac{\partial^2 \varphi_a}{\partial \xi_i \partial \xi_j}(0) = \lim_{s \to \infty} \frac{\frac{\partial \varphi_a}{\partial \xi_i}(\omega e^{-\frac{s}{2}} e_j)}{\omega e^{-\frac{s}{2}}} = \frac{2p}{\kappa}(1 + \delta_{i,j})\lambda_{e_1+e_i+e_j}(a).$$

Since $\varphi_a(0) = 0$ and $\nabla \varphi_a(0) = 0$, we have just computed the second fundamental form $(\Lambda_{i,j}(a))_{2 \leq i,j \leq N}$ of the blow-up set at the point a in the basis $(Q(a)e_2, ..., Q(a)e_N)$ of the tangent space: for all $2 \leq i, j \leq N$,

$$\Lambda_{i,j}(a) = \frac{\partial^2 \varphi_a}{\partial \xi_i \partial \xi_j}(0) = \frac{2p}{\kappa}(1 + \delta_{i,j})\lambda_{e_1+e_i+e_j}(a). \tag{63}$$

Note that $\Lambda_{i,j}(a)$ is symmetric. Since $a \to \lambda_\beta(a)$ is continuous, this implies that the blow-up set is of class C^2. This concludes the proof of Theorem 1. ■

As a matter of fact, we have proved more than stated in Theorem 1. We claim the following:

Proposition 4.1 (Link between the refined uniform blow-up behavior of the solution and the second fundamental form of its blow-up set) *For all $a \in S_\delta$:*

(i) For all $s \geq s_1$,

$$\left\|W_a(Q(a)y, s) - \tilde{w}(y_1, s + \sigma(a)) - \frac{\kappa e^{-\frac{s}{2}}}{4ps} y_1 \sum_{2 \leq i,\, j \leq N} \Lambda_{i,j}(a)(y_i y_j - 2\delta_{i,j})\right\|_{L^2_\rho} \leq Ce^{-\frac{s}{2}}s^{-1-\nu} \tag{64}$$

where W_a is defined in (2) and $(\Lambda_{i,j}(a))_{2 \leq i,j \leq N}$ is a continuous symmetric matrix representing the second fundamental form of the blow-up set at a in $(Q(a)e_2, ..., Q(a)e_N)$, a basis of the tangent space.

(ii) It holds that

$$\Lambda_{i,j}(a) = \lim_{t\to T}\frac{p}{4\kappa}|\log(T-t)|^{-1}(T-t)^{\frac{1}{p-1}-\frac{1}{2}}\times \tag{65}$$

$$\int_{\mathbb{R}^N} u(x,t)(x-a).Q(a)e_1\left((x-a).Q(a)e_i\times(x-a).Q(a)e_j-2\delta_{i,j}\right)\frac{e^{-\frac{|x-a|^2}{4(T-t)}}}{(4\pi)^{N/2}}dx.$$

Remark: Note that W_a defined in (2) is different from w_a defined in (18) by a rotation of coordinates.

Proof of Proposition 4.1:
Part (i) follows directly form Proposition 3.1. Indeed, from (23), (2) and (18), we have

$$g_a(y,s) = W_a(Q(a)y,s) - \tilde{w}(y_1, s+\sigma(a)) \tag{66}$$

on one hand. On the other hand, it is easy to check that the sum in Proposition 3.1 can be indexed as follows

$$\{\beta\in\mathbb{N}^N \ \mid\ |\beta|=3,\ \beta_1=1\} = \{e_1+e_i+e_j\ \mid\ 2\le i\le j\le N\}$$

where e_k is the k-th vector of the canonical basis of $\mathbb{R}^N$. Therefore, using (63), (28) and (29), we write

$$\begin{aligned}\sum_{|\beta|=3,\ \beta_1=1}\lambda_\beta(a)h_\beta(y) &= \sum_{2\le i\le j\le N}\lambda_{e_1+e_i+e_j}(a)h_{e_1+e_i+e_j}(y)\\ &= \frac{\kappa}{2p}y_1\sum_{2\le i\le j\le N}\frac{\Lambda_{i,j}(a)}{1+\delta_{i,j}}\left(y_iy_j-2\delta_{i,j}\right) = \frac{\kappa}{4p}y_1\sum_{2\le i,j\le N}\Lambda_{i,j}(a)\left(y_iy_j-2\delta_{i,j}\right).\end{aligned} \tag{67}$$

Using Proposition 3.1, (66) and (67), we obtain (64). This concludes the proof of (i) in Proposition 4.1.

(ii) From (63), (62) and the definition of $g_{a,\beta}(s)$ (33), we write for all $2\le i,j\le N$,

$$\begin{aligned}\Lambda_{i,j}(a) &= \frac{2p}{\kappa}(1+\delta_{i,j})\lambda_{e_1+e_i+e_j}(a)\\ &= \frac{2p}{\kappa}(1+\delta_{i,j})\lim_{s\to\infty}se^{\frac{s}{2}}g_{a,e_1+e_i+e_j}(s)\\ &= \frac{2p}{\kappa}(1+\delta_{i,j})\lim_{s\to\infty}se^{\frac{s}{2}}\int_{\mathbb{R}^N}g_a(y,s)\frac{h_{e_1+e_i+e_j}(y)}{\|h_{e_1+e_i+e_j}\|^2_{L^2_\rho}}\rho(y)dy.\end{aligned} \tag{68}$$

Using (28) and (29), we see that

$$\frac{h_{e_1+e_i+e_j}(y)}{\|h_{e_1+e_i+e_j}\|^2_{L^2_\rho}} = \frac{y_1(y_iy_j-2\delta_{i,j})}{8(1+\delta_{i,j})}. \tag{69}$$

Using the definition of g_a (23) and the fact that $\tilde{w}(y_1,s)$ does not depend on y_i for $i\ge 2$, we derive from (68) and (69) that for all $2\le i,j\le N$,

$$\Lambda_{i,j}(a) = \frac{p}{4\kappa}\lim_{s\to\infty}se^{\frac{s}{2}}\int_{\mathbb{R}^N}w_a(y,s)y_1\left(y_iy_j-2\delta_{i,j}\right)\rho(y)dy.$$

Using the change of variables in (18) gives (65) and concludes the proof of Proposition 4.1. ■

Proof of Proposition 2:
We computed in Proposition 4.1 $(\Lambda_{i,j}(a))_{2\le i,j\le N}$ which is a $(N-1)\times(N-1)$ continuous symmetric matrix representing the second fundamental form of the blow-up set in the basis $(Q(a)e_2, ..., Q(a)e_N)$ of the tangent space. Therefore, it can be diagonalized to provide the principal curvatures and directions of the blow-up set. More precisely, there exists a $(N-1)\times(N-1)$ matrix $P(a) = (P_{i,j}(a))_{2\le i,j\le N}$ such that

$$P(a)^T P(a) = \mathrm{Id},\ \ P(a)^T \Lambda P(a) = \mathrm{diag}\,(l_2(a), ..., l_N(a))$$

and $a \to l_j(a)$ is continuous for $j \ge 2$. Note however that $a \to P(a)$ is not necessarily continuous. The $l_j(a)$ for $2 \le j \le N$ (continuous in terms of a) are the principal curvatures of the blow-up set at the point a, and the column vectors of P are its principal directions in the basis $(Q(a)e_2, ..., Q(a)e_N)$ of the tangent space. Now, if we introduce the (non necessarily continuous) $N\times N$ orthogonal matrix $M(a) = Q(a)\tilde P(a)$ where

$$\tilde P_{i,j}(a) = P_{i,j}(a) \text{ if } 2 \le i,j \le N \text{ and } \tilde P_{i,j}(a) = \delta_{i,j} \text{ otherwise,} \tag{70}$$

then it is easy to check that the principal vectors (which are by definition in the tangent space) take the simple form of $M(a)e_j$ for $2 \le j \le N$. Like the matrix $P(a)$, the principal vectors are not necessary continuous in terms of a. Note from the definition (70) of $\tilde P$ that $\tilde P(a)e_1 = e_1$, therefore, the normal vector (see (17)) $Q(a)e_1 = Q(a)\tilde P(a)e_1 = M(a)e_1$. It is continuous because of the continuity of $Q(a)$ (see (17)).

Using (70), it is easy to check that (i) of Proposition 2 follows from (i) in Proposition 4.1 by a simple change of variables. The convergence in (64) holds also in $W_{2,\infty}(|y| < R)$ for any $R > 0$ by parabolic regularity if $s \ge s_0 \equiv s_1 + 1$, so does the convergence in (i) of Proposition 2.

(ii) Take $R_0 > 0$, $t \in [T - e^{-s_0}, , T)$ and $x \in B(\hat a, \delta)$ such that $\mathrm{dist}(x, S) < R_0\sqrt{T-t}$. Take $a = P_S(x)$, the orthogonal projection of x on the blow-up set S, and according to (2), introduce Y and s such that

$$u(x,t) = (T-t)^{-\frac{1}{p-1}} W_a(M(a)Y, s) \text{ where } M(a)Y = \frac{x-a}{\sqrt{T-t}},\ \ s = -\log(T-t). \tag{71}$$

Since $M(a)e_1$ is a normal vector and $M(a)e_j$ for all $j \ge 2$ are tangent to the blow-up set at a (see (i)), we see that

$$\begin{aligned}
&|a - \hat a| \le |a - x| + |x - \hat a| \le 2|x - \hat a| < \delta,\\
&Y_1 = \frac{\tilde d(x,S)}{\sqrt{T-t}} \text{ and } Y_j = 0 \text{ for all } j \ge 2,\\
&|Y| \le \frac{d(x,S)}{\sqrt{T-t}} \le R_0 \text{ and } s \ge s_0,
\end{aligned}$$

where $\tilde{d}(x,S) = \pm\,\mathrm{dist}(x,S)$ (depending on the side of the blow-up set S where x is). Applying (i) of Proposition 2 and using (71), we write

$$\begin{aligned}&\left|(T-t)^{\frac{1}{p-1}}u(x,t) - \left\{\tilde{w}(y_1, s+\sigma(a)) - \frac{\kappa}{2ps}Y_1\sum_{j=2}^{N} l_j(a)\right\}\right|\\ \leq & C(R_0)(T-t)^{\frac{1}{2}}|\log(T-t)|^{-1-\nu}.\end{aligned}$$

Using (10), (9), (8) and the definition of the mean curvature

$$m(a) = \sum_{j=2}^{N} l_j(a),$$

this concludes the proof of (ii) in Proposition 2. ■

(iii) If we project the estimate of (i) on the polynomial $y_1(y_j^2-2)$, then we see from (31) and the fact that $\tilde{w}$ does not depend on y_j for all $j \geq 2$ that

$$l_j(a) = \lim_{s\to\infty} \frac{pse^{s/2}}{4\kappa}\int_{\mathbb{R}^N} w_a(M(a)y, s)y_1(y_j^2-2)\rho(y)dy.$$

Making the change of variables (2), we get to the conclusion. This concludes the proof of Proposition 2. ■

Proof of Theorem 3: Since $(T-t)^{\frac{1}{p-1}}u(\hat{a},t) \to \kappa$ with the speed $|\log(T-t)|^{-1}$, we know from Velázquez [21] and [20] (see also Filippas and Liu [8] and Filippas and Kohn [7]) that the local behavior of u near the blow-up point $\hat{a}$ is given by (12). Therefore, if we apply Theorem 4 of [23], we see that the blow-up set is a C^1 hypersurface, locally near $\hat{a}$. Therefore, we can apply Theorem 1 and derive the C^2 regularity as well as the behaviors described in Proposition 2. ■

A Projection of equation (24) on the different modes

We prove (v) of Lemma 2.2 here. For simplicity, we drop down the index a here. Since the projector R_k and the operator $\mathcal{L}$ commute, we write from equation (24) for all $a \in S_\delta$, $y \in \mathbb{R}^N$ and $s \geq -\log T$,

$$\partial_s R_k g = \mathcal{L}\left(R_k g\right) + R_k\left(\alpha g\right).$$

Next, we multiply this equation by $\rho R_k g$ and integrate over $\mathbb{R}^N$ to obtain

$$\frac{1}{2}\frac{d}{ds}\int \left(R_k g\right)^2\rho = \int R_k g\mathcal{L}\left(R_k g\right)\rho + \int R_k g R_k(\alpha g)\rho. \tag{72}$$

Since $R_k g$ is the projection of g on the spectrum of $\mathcal{L}$ below $1-\frac{k}{2}$, it holds that

$$\int R_k g\mathcal{L}\left(R_k g\right)\rho \leq (1-\frac{k}{2})\int \left(R_k g\right)^2\rho. \tag{73}$$

Since R_k is an orthogonal projector, we use Hölder's inequality to estimate the second term on the right-hand side of (72) by:

$$\left|\int R_k(\alpha g)R_k g\rho\right| \leq \|R_k(\alpha g)\|_{L^2_\rho}\|R_k g\|_{L^2_\rho} \leq \|\alpha g\|_{L^2_\rho}\|R_k g\|_{L^2_\rho} \leq \|\alpha\|_{L^4_\rho}\|g\|_{L^4_\rho}\|R_k g\|_{L^2_\rho}.$$

Note that equation (24) has the following regularizing property (control of the L^4_ρ norm by the L^2_ρ norm up to some delay in time, see Lemma 2.3 in [13]):

$$\left(\int g(y,s)^4\rho dy\right)^{1/4} \leq C\left(\int g(y,s-s_*)^2\rho dy\right)^{1/2} \tag{74}$$

for some $s_* > 0$. Combining (32), (72), (73) and (74) yields (v).

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