Regularity of the blow-up set and singular behavior for semilinear heat equations

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Abstract : We consider $u(x,t)$ a blow-up solution of $u_t = \Delta u + |u|^{p-1}u$ where $u : \mathbb{R}^N \times [0,T) \to \mathbb{R}$, $p > 1$, $(N-2)p < N+2$ and either $u(0) \geq 0$ or $(3N-4)p < 3N+8$. The blow-up set $S \subset \mathbb{R}^N$ of u is the set of all blow-up points. Under a non degeneracy condition, we show that if S is continuous, then it is a C^1 manifold. The blow-up behavior of u near non isolated blow-up points is derived as well. If the codimension of the blow-up set is one, then S is $C^{1,\alpha}$ for any $\alpha \in (0, \frac{1}{2})$. If in addition $p > 3$, then u is *very* close to a superposition of one dimensional solutions as functions of the distance to S.

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We are concerned in this note with blow-up phenomena arising in the following semilinear problem :

$$\begin{aligned} u_t &= \Delta u + |u|^{p-1}u \\ u(.,0) &= u_0 \in L^\infty(\mathbb{R}^N), \end{aligned} \tag{1}$$

where $u(t) : x \in \mathbb{R}^N \to u(x,t) \in \mathbb{R}$ and Δ stands for the Laplacian in $\mathbb{R}^N$. We assume in addition the exponent $p > 1$ subcritical : if $N \geq 3$ then $1 < p < (N+2)/(N-2)$. Moreover, we assume that

$$u_0 \geq 0 \text{ or } (3N-4)p < 3N+8. \tag{2}$$

This problem has attracted a lot of attention because it captures features common to a whole range of blow-up problems arising in various physical situations, particularly the role of scaling and self-similarity. Without pretending to be exhaustive, we would like nonetheless to mention some related equations : the motion by mean curvature (Soner and Souganidis [20]), vortex dynamics in superconductors (Chapman, Hunton and Ockendon [6], Merle and Zaag [15]), surface diffusion (Bernoff, Bertozzi and Witelski [2]) and chemotaxis (Brenner *et al.* [4], Betterton and Brenner [3]). However, equation (1) is simple enough to be tractable in rigorous mathematical terms, unlike other physical equations.

A solution $u(t)$ to (1) blows-up in finite time if its maximal existence time T is finite. In this case,

$$\lim_{t\to T}\|u(t)\|_{H^1(\mathbb{R}^N)} = \lim_{t\to T}\|u(t)\|_{L^\infty(\mathbb{R}^N)} = +\infty.$$

Let us consider such a solution. T is called the blow-up time of u. A point $a \in \mathbb{R}^N$ is called a blow-up point if

$$|u(x,t)| \to +\infty \text{ as } (x,t) \to (a,T)$$

(this definition is equivalent to the usual local unboundedness definition, thanks to Corollary 2 in [18]). S denotes the blow-up set, that is the set of all blow-up points. From [18], we know that there exists *a blow-up profile* $u^* \in C^2_{\text{loc}}(\mathbb{R}^N\backslash S)$ such that

$$u(x,t) \to u^*(x) \text{ in } C^2_{\text{loc}}(\mathbb{R}^N\backslash S) \text{ as } t \to T. \tag{3}$$

The blow-up problem has been addressed in different ways in the literature. An important direction was developed by authors looking for sufficient blow-up conditions on initial data or on the nonlinear term (see Fujita [10], Ball [1], Levine [13] and the review paper by Deng and Levine [7]). The behavior near singular time is a major direction too. More precisely, given $a \in \mathbb{R}^N$ a blow-up point of u, two issues arise :

- the **blow-up behavior** of $u(x,t)$ near the singularity $(\hat{a},T)$.

- the **regularity** of the blow-up set near $\hat{a}$.

The blow-up behavior issue has been extensively addressed in the literature, when $\hat{a}$ is an isolated blow-up point (note that the second question is irrelevant then). See for example Weissler [25], Bricmont and Kupiainen [5], Herrero and Velázquez [12] and [22]. No relevant results were known when $\hat{a}$ is not isolated. As a matter of fact, we address in this note these two issues in a case where $\hat{a}$ is *not* isolated. These two issues are very closely related. See [26] and [27].

1 The regularity of the blow-up set

By definition, the blow-up set is closed, and if the initial data is sufficiently decaying at infinity, then it is bounded as well (see Giga and Kohn [11]). Two questions arise :

- **A constructive question** : Given a compact set $\hat{S} \subset \mathbb{R}^N$, can one construct $\hat{u}$ a solution of (1) blowing up at some time $\hat{T}$ exactly on $\hat{S}$? The answer is affirmative if $\hat{S}$ is a sphere (see Giga and Kohn [11] for example) or a collection of k points (see Merle [14] and Merle and Zaag [16]). The techniques of [16] give a solution when $\hat{S}$ is a union of k concentric spheres (which reduces to the case of k points in the radial setting). The question remains open otherwise.

- **A descriptive question** : Given u a solution of (1) that blows up at time T on a set S, consider $\hat{a}$ a non isolated blow-up point. What is the regularity of S near $\hat{a}$? We know from Velázquez [23] that the $(N-1)$-dimensional Hausdorff measure of S is bounded on compact sets (as a matter of fact, this provides a necessary condition on $\hat{S}$ in the constructive question above). No further information was available.

The description question is our first concern in this note. Given $\hat{a} \in S$, we know from Velázquez [22] that up to some scalings, u approaches a particular explicit function near the singularity $(\hat{a}, T)$. We consider the case where for all $K_0 > 0$,

$$\sup_{|z| \le K_0} \left| (T-t)^{\frac{1}{p-1}} u\left(\hat{a} + Q_{\hat{a}} z \sqrt{(T-t)|\log(T-t)|}, t\right) - f_{l_{\hat{a}}}(z) \right| \to 0 \quad (4)$$

as $t \to T$, where $Q_{\hat{a}}$ is an orthonormal $N \times N$ matrix, $l_{\hat{a}} = 1, ..., N$, and

$$f_l(z) = \left(p - 1 + \frac{(p-1)^2}{4p} \sum_{i=1}^{l} z_i^2 \right)^{-\frac{1}{p-1}}. \quad (5)$$

Other behaviors with the scaling $(T-t)^{-\frac{1}{2k}}(x - \hat{a})$ where $k = 2, 3, ..$ may occur (see [22]). We suspect them to be unstable.

If $l_{\hat{a}} = N$, then $\hat{a}$ is an isolated blow-up point. An extensive literature is devoted to this case (Weissler [25], Bricmont and Kupiainen [5], Herrero and Velázquez [12] and [22],...). We have proved the stability of such a behavior with Fermanian and Merle in [8]. The key argument in our proof was the following Liouville Theorem proved by Merle and Zaag in [17] and [18]:

Consider U a solution of (1) defined for all $(x,t) \in \mathbb{R}^N \times (-\infty, T)$ such that for all $(x,t) \in \mathbb{R}^N \times (-\infty, T)$, $|U(x,t)| \le C(T-t)^{-\frac{1}{p-1}}$. Then, either $U \equiv 0$ or $U(x,t) = [(p-1)(T^* - t)]^{-\frac{1}{p-1}}$ for some $T^* \ge T$.

The case $l_{\hat{a}} < N$ is known to occur, namely when u is invariant with respect to some coordinates. However, when $l_{\hat{a}} < N$, we cannot even tell whether $\hat{a}$ is isolated or not, or whether S is continuous near $\hat{a}$. Therefore, we assume that $\hat{a}$ is non isolated and that S contains a continuum that goes through $\hat{a}$. To make our presentation clearer, we restrict to the case $N = 2$ and assume that $\hat{a} = a(0) \in \operatorname{Im} a \subset S$ where $a \in C((-1,1), \mathbb{R}^2)$ and for some α_0,

$$\begin{array}{l}\forall \epsilon > 0,\ a(-\epsilon, \epsilon) \text{ intersects the complimentary of any} \\ \text{connected closed cone with vertex at } \hat{a} \text{ and angle } \alpha \in (0, \alpha_0]\end{array} \tag{6}$$

(this is in a way to insure that $\hat{a}$ is not an endpoint). Assuming that u behaves according to (4) near the singularity $(\hat{a}, T)$, we have the following result :

Theorem 1 (Regularity of the blow-up set at a point with the behavior (4) assuming S contains a continuum) *Assume $N = 2$ and consider u a solution of (1) that blows-up at time T on a set S. Consider $\hat{a} = a(0) \in \operatorname{Im} a \subset S$ where $a \in C((-1,1), \mathbb{R}^2)$ and $\hat{a}$ is not an endpoint (in the sense (6)). If u behaves near $(\hat{a}, T)$ as stated in (4), then there are $\delta > 0$, $\delta_1 > 0$ and $\varphi \in C^1([-\delta_1, \delta_1], \mathbb{R})$ such that*

$$S \cap B(\hat{a}, 2\delta) = \operatorname{graph} \varphi \cap B(\hat{a}, 2\delta) = \operatorname{Im} a \cap B(\hat{a}, 2\delta). \tag{7}$$

In particular, S is a C^1 manifold near the point $\hat{a}$. More precisely, there exists $C_0 > 0$ and h_0 such that for all $|\xi| < \delta_1$ and $|h| < h_0$ such that $|\xi + h| < \delta_1$, we have :

$$|\varphi(\xi + h) - \varphi(\xi) - h\varphi'(\xi)| \leq C_0 |h| \sqrt{\frac{\log|\log|h||}{|\log|h||}}.$$

Remark : The function φ is actually $C^{1,\alpha}$ for any $\alpha \in (0, \frac{1}{2})$ (see Proposition 5 below). In higher dimensions, we proved $C^{1,\alpha}$ regularity only when the codimension of the blow-up set is one.
Remark : From [22], we know that the limit function at $(\hat{a}, T)$ stated in (4) has a degenerate direction, and that we can not have two curves of blow-up points intersecting transversally at $\hat{a}$. With our contribution, we eliminate the possibility of two curves meeting tangentially at $\hat{a}$. In particular, there is no cusp at $\hat{a}$, and there is no sequence of isolated blow-up points converging to $\hat{a} \in S$.

Theorem 1 also holds in higher dimensions. We claim the following :

Theorem 1' (Regularity of the blow-up set near a point with the behavior (4) assuming S contains a $N - l$ dimensional continuum)

Take $N \geq 2$ and $l \in \{1, ..., N-1\}$. Consider u a solution of (1) that blows-up at time T on a set S and take $\hat{a} \in S$ where u behaves locally as stated in (4) with $l_{\hat{a}} = l$. Consider $a \in C((-1,1)^{N-l}, \mathbb{R}^N)$ such that $\hat{a} = a(0) \in \operatorname{Im} a \subset S$ and $\operatorname{Im} a$ is at least $(N-l)$ dimensional. If $\hat{a}$ is not an endpoint, then there are $\delta > 0$, $\delta_1 > 0$ and $\varphi \in C^1([-\delta_1, \delta_1]^{N-l}, \mathbb{R}^l)$ such that (7) holds and S is a C^1 manifold near $\hat{a}$.

Remark : The rigorous definition of "endpoint" and "$(N-l)$ dimensional" in this theorem requires some technical notations. See section 6 in [26] for details.

2 The blow-up behavior near a non isolated blow-up point

The behavior of $u(x,t)$ near the singularity $(\hat{a}, T)$ is our second concern in this paper. We claim the following :

Theorem 2 (Blow-up behavior and profile near a blow-up point where u behaves as in (4) assuming S contains a continuum) *Under the hypotheses of Theorems 1 and 1', there exists $t_0 < T$ such that for all $K_0 > 0$, $t \in [t_0, T)$ and $x \in B(\hat{a}, \delta)$ s.t. $d(x,S) \leq K_0\sqrt{(T-t)|\log(T-t)|}$, we have*

$$\left|(T-t)^{\frac{1}{p-1}}u(x,t) - f_1\left(\frac{d(x,S)}{\sqrt{(T-t)|\log(T-t)|}}\right)\right| \leq C_0'(K_0)\frac{\log|\log(T-t)|}{|\log(T-t)|} \tag{8}$$

where f_1 is defined in (5). Moreover, $\forall x \in \mathbb{R}^N \backslash S$, $u(x,t) \to u^*(x)$ as $t \to T$ with

$$u^*(x) \sim U(d(x,S)) \text{ as } d(x,S) \to 0 \text{ and } x \in B(\hat{a}, \delta) \tag{9}$$

where $U(z) = \left(\frac{8p}{(p-1)^2}\frac{|\log z|}{z^2}\right)^{\frac{1}{p-1}}$ for $z > 0$.

Remark : This is the first time where the blow-up profile u^* is derived near a non-isolated point. Indeed, in the earlier work of Velázquez, the behavior along the "tangential" direction of S was not derived. Estimate (8) shows that in a tubular neighborhood of S, the main term in the blow-up asymptotics is the one dimensional blow-up profile f_1, function of only the normal coordinate $\pm d(x,S)$.

The major step towards Theorems 1, 1' and 2 is the proof of the stability of the behavior (4) in a neighborhood of $\hat{a}$ in S. Without such a stability, no further result could be obtained after Velázquez's result in [23] about the Hausdorff measure of S. The key argument in getting this stability is the Liouville Theorem of [18], stated on page 3.

The error term in (8) shows that we fall in logarithmic scales $\nu = -1/\log(T-t)$ of the blow-up small parameter $\epsilon = T-t$. Further refinements in this direction should give an expansion of the solution in terms of powers of ν, i.e., in logarithmic scales of ϵ (see Stewartson and Stuart [21]). Logarithmic scales also arise in some singular perturbation problems such as low Reynolds number fluids and some vibrating membranes studies (see Ward [24] and the references therein, see also Segur and Kruskal [19] for a Klein-Gordon equation). Since ν goes to zero slowly, infinite logarithmic series may be of only limited practical use in approximating the exact solution. Relevant approximations, i.e., approximations up to lower order terms such as ϵ^β for $\beta > 0$, lie beyond all logarithmic scales. When the codimension of the blow-up set is one, namely when

$$l_{\hat{a}} = 1,$$

we do better, and get to error terms of order $(T-t)^\beta$ with $\beta > 0$. Our idea to capture such relevant terms is to abandon the explicit profile function obtained as a first order approximation, and take a less explicit function as a first order description of the singular behavior. Both formulations agree to the first order. Through scaling and matching, we can reach the order ϵ^β by iterating the expansion around the less explicit function.

3 Further refinements when the codimension of the blow-up set is one

A natural candidate for this non explicit function is simply a one dimensional solution of (1) that has the same profile f_1. It is classical that there exists a one dimensional even function $\tilde{u}(x_1,t)$, solution of (1), which decays on $(0,\infty)$ and blows up at time T only at the origin, with the profile f_1, in the sense that for all $K_0 > 0$ and $t \in [t_0, T)$, if $|x_1| \le K_0\sqrt{(T-t)|\log(T-t)|}$, then

$$\left|(T-t)^{\frac{1}{p-1}}\tilde{u}(x_1,t) - f_1\left(\frac{x_1}{\sqrt{(T-t)|\log(T-t)|}}\right)\right| \le C_0'(K_0)\frac{\log|\log(T-t)|}{|\log(T-t)|} \tag{10}$$

(see Appendix A in [27] for a proof of this fact). Hence, it follows from (8) that for all $K_0 > 0$, $t \in [t_0, T)$ and $x \in B(\hat{a}, \delta)$ such that $d(x, S) \le K_0 \sqrt{(T-t)|\log(T-t)|}$, we have

$$(T-t)^{\frac{1}{p-1}} |u(x,t) - \tilde{u}(d(x,S),t)| \le C(K_0) \frac{\log|\log(T-t)|}{|\log(T-t)|}. \qquad (11)$$

This estimate remains valid even if we replace $\tilde{u}(d(x,S),t)$ by any $\tilde{u}_{\sigma(x,t)}(d(x,S),t)$ where $\tilde{u}_\sigma$ is defined by

$$\tilde{u}_\sigma(x_1,t) = e^{-\frac{\sigma}{p-1}} \tilde{u}(e^{-\frac{\sigma}{2}} x_1, T - e^{-\sigma}(T-t)), \qquad (12)$$

provided that $|\sigma(x,t)| \le C(K_0)$. Indeed, for any $\sigma \in \mathbb{R}$, $\tilde{u}_\sigma$ is still a blow-up solution of (1) with the same properties and the same profile (10) as $\tilde{u}$. Moreover, $\tilde{u}_\sigma \ne \tilde{u}$, unless $\sigma = 0$, because $\tilde{u}$ is not self-similar (see Appendix A in [27]).

For each blow-up point a near $\hat{a}$, we will suitably choose this free scaling parameter $\sigma = \sigma(a)$ so that the difference $(T-t)^{\frac{1}{p-1}} \left(u(x,t) - \tilde{u}_{\sigma(a)}(d(x,S),t)\right)$ along the normal direction to S at a is minimum. Following the ideas of page 6, if we refine the expansion about this well chosen, though less explicit, function $\tilde{u}_{\sigma(a)}(d(x,S),t)$, then we escape logarithmic scales. In particular, if $p > 3$, then the difference $u(x,t) - \tilde{u}_{\sigma(a)}(d(x,S),t)$ is bounded and goes to zero as $t \to T$, although both functions blow up. This can be done only when

$$l_{\hat{a}} = 1$$

which corresponds to a codimension 1 blow-up set. We claim the following:

Theorem 3 (The N dimensional solution seen as a superposition of one dimensional solutions of the normal variable to the blow-up set, with a suitable dilation) *Under the hypotheses of Theorems 1 and 1' and if $l_{\hat{a}} = 1$ and $p > 3$, then for all $t \in [t_1, T)$ and $x \in B(\hat{a}, \delta)$ such that $d(x,S) < \epsilon_0$ for some $t_1 < T$, $\delta > 0$ and $\epsilon_0 > 0$, we have*

$$\left|u(x,t) - \tilde{u}_{\sigma(P_S(x))}(d(x,S),t)\right| \le h(x,t) < M < +\infty, \qquad (13)$$

where $P_S(x)$ is the projection of x over S and $h(x,t) \to 0$ as $d(x,S) \to 0$ and $t \to T$.

Thus, when $p > 3$, all the singular terms of u in a neighborhood of $(\hat{a}, T)$ are contained in the rescaled one dimensional solution $\tilde{u}_{\sigma(P_S(x))}(d(x,S),t)$, which shows that in a tubular neighborhood of the blow-up set S, the space variable splits into 2 independent variables:

- A primary variable, $d(x,S)$, normal to S. It accounts for the main singular term of u and gives the size of $u(x,t)$, as already shown in the formulation (11), which follows directly from Theorem 2.

- A secondary variable, $P_S(x)$, whose effect is sharper. Through the optimal choice of the dilation $\sigma(P_S(x))$, it absorbs *all* next singular terms in the normal direction to S at $P_S(x)$.

Similar ideas are used by Betterton and Brenner [3] in a chemotaxis model; see section 5 in [27] for a short discussion of connections with that work. We would like to mention that we have successfully used this idea of modulation of the dilation with Fermanian in [9] to prove that for $N=1$ and $p\geq 3$, there is only one blow-up solution of (1) with the profile (4), up to a bounded function and to the invariances of the equation (the dilation and translations in space and in time).

Theorem 3 is a direct consequence of the following result which is valid also for $1<p\leq 3$.

Theorem 4 (Blow-up behavior and profile near a blow-up point where u behaves as in (4) assuming S is locally a $(N-1)$-dimensional manifold) *Under the hypotheses of Theorems 1 and 1' and without the restriction $p>3$, if $l_{\hat a}=1$, then there exists $t_1<T$ and $\epsilon_0>0$ such that for all $x\in B(\hat a,\delta)$ such that $d(x,S)\leq\epsilon_0$, we have the following:*
i) For all $t\in[t_1,T)$,

$$\begin{aligned}&\left|u(x,t)-\tilde u_{\sigma(P_S(x))}(d(x,S),t)\right|\leq\\ &C\,\mathrm{mM}\left((T-t)^{\frac{p-3}{2(p-1)}}|\log(T-t)|^{\frac{3}{2}+C_0},d(x,S)^{\frac{p-3}{p-1}}|\log d(x,S)|^{\frac{p}{p-1}+C_0}\right),\end{aligned}\tag{14}$$

where $P_S(x)$ is the projection of x over S, $\mathrm{mM}=\min$ *if $1<p\leq 3$ and* $\mathrm{mM}=\max$ *if $p>3$.*
ii) If $x\notin S$, then $u(x,t)\to u^*(x)$ as $t\to T$ and

$$\begin{aligned}&\left|u^*(x)-e^{-\frac{\sigma(P_S(x))}{p-1}}\tilde u^*\left(e^{-\frac{\sigma(P_S(x))}{2}}d(x,S)\right)\right|\\ &\leq Cd(x,S)^{\frac{p-3}{p-1}}|\log d(x,S)|^{\frac{p}{p-1}+C_0},\end{aligned}$$

where $\tilde u^*(x_1)=\lim_{t\to T}\tilde u(x_1,t)$.

Remark: In view of Theorem 2, we see from our new estimate that up to a suitable dilation, all the next terms in the expansion of u^* up to the order $d(x,S)^{\frac{p-3}{p-1}}|\log d(x,S)|^{\frac{p}{p-1}+C_0}$ are the same as the particular one dimensional solution.

The splitting of the space variable x into $d(x,S)$ and $P_S(x)$, as shown in (14), induces a geometric constraint on the blow-up set S, leading to more regularity on S.

Proposition 5 ($C^{1,\frac{1}{2}-\eta}$ regularity for S and $C^{1-\eta}$ regularity for the dilation σ) *Under the hypotheses of Theorems 1 and 1' and if $l_{\hat{a}}=1$, then S is the graph of a function $\varphi \in C^{1,\frac{1}{2}-\eta}(B_{N-1}(0,\delta_1),\mathbb{R})$, locally near $\hat{a}$, and σ is a $C^{1-\eta}$ function, for any $\eta>0$. More precisely, there is a $h_0>0$ such that for all $|\xi|<\delta_1$ and $|h|<h_0$ such that $|\xi+h|<\delta_1$, we have*

$$\begin{aligned} |\varphi(\xi+h)-\varphi(\xi)-h\varphi'(\xi)| &\le C|h|^{3/2}|\log|h||^{\frac{1}{2}+C_0}, \\ |\sigma(\xi,\varphi(\xi))-\sigma(\xi+h,\varphi(\xi+h))| &\le C|h||\log|h||^{3+C_0}. \end{aligned}$$

The reader is referred to the papers [26] and [27] for proofs and details.

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References

[1] J. M. Ball. Remarks on blow-up and nonexistence theorems for nonlinear evolution equations. *Quart. J. Math. Oxford Ser. (2)*, 28(112):473–486, 1977.

[2] A. J. Bernoff, A. L. Bertozzi, and T. P. Witelski. Axisymmetric surface diffusion: dynamics and stability of self-similar pinchoff. *J. Statist. Phys.*, 93(3-4):725–776, 1998.

[3] M. D. Betterton and M. P. Brenner. Collapsing bacterial cylinders. preprint.

[4] M. P. Brenner, P. Constantin, L. P. Kadanoff, A. Schenkel, and S. C. Venkataramani. Diffusion, attraction and collapse. *Nonlinearity*, 12(4):1071–1098, 1999.

[5] J. Bricmont and A. Kupiainen. Universality in blow-up for nonlinear heat equations. *Nonlinearity*, 7(2):539–575, 1994.

[6] S. J. Chapman, B. J. Hunton, and J. R. Ockendon. Vortices and boundaries. *Quart. Appl. Math.*, 56(3):507–519, 1998.

[7] K. Deng and H. A. Levine. The role of critical exponents in blow-up theorems : the sequel. *J. Math. Anal. Appl*, 2000.

[8] C. Fermanian Kammerer, F. Merle, and H. Zaag. Stability of the blow-up profile of non-linear heat equations from the dynamical system point of view. *Math. Annalen*, 317(2):195–237, 2000.

[9] C. Fermanian Kammerer and H. Zaag. Boundedness up to blow-up of the difference between two solutions to a semilinear heat equation. *Nonlinearity*, 13(4):1189–1216, 2000.

[10] H. Fujita. On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. *J. Fac. Sci. Univ. Tokyo Sect. I*, 13:109–124, 1966.

[11] Y. Giga and R. V. Kohn. Nondegeneracy of blowup for semilinear heat equations. *Comm. Pure Appl. Math.*, 42(6):845–884, 1989.

[12] M. A. Herrero and J. J. L. Velázquez. Blow-up behaviour of one-dimensional semilinear parabolic equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 10(2):131–189, 1993.

[13] H. A. Levine. Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $Pu_t = -Au + F(u)$. *Arch. Rational Mech. Anal.*, 51:371–386, 1973.

[14] F. Merle. Solution of a nonlinear heat equation with arbitrarily given blow-up points. *Comm. Pure Appl. Math.*, 45(3):263–300, 1992.

[15] F. Merle and H. Zaag. Reconnection of vortex with the boundary and finite time quenching. *Nonlinearity*, 10(6):1497–1550, 1997.

[16] F. Merle and H. Zaag. Stability of the blow-up profile for equations of the type $u_t = \Delta u + |u|^{p-1}u$. *Duke Math. J.*, 86(1):143–195, 1997.

[17] F. Merle and H. Zaag. Optimal estimates for blowup rate and behavior for nonlinear heat equations. *Comm. Pure Appl. Math.*, 51(2):139–196, 1998.

[18] F. Merle and H. Zaag. A Liouville theorem for vector-valued nonlinear heat equations and applications. *Math. Annalen*, 316(1):103–137, 2000.

[19] H. Segur and M. D. Kruskal. Nonexistence of small-amplitude breather solutions in ϕ^4 theory. *Phys. Rev. Lett.*, 58(8):747–750, 1987.

[20] H. M. Soner and P. E. Souganidis. Singularities and uniqueness of cylindrically symmetric surfaces moving by mean curvature. *Comm. Partial Differential Equations*, 18(5-6):859–894, 1993.

[21] K. Stewartson and J. T. Stuart. A non-linear instability theory for a wave system in plane Poiseuille flow. *J. Fluid Mech.*, 48:529–545, 1971.

[22] J. J. L. Velázquez. Higher-dimensional blow up for semilinear parabolic equations. *Comm. Partial Differential Equations*, 17(9-10):1567–1596, 1992.

[23] J. J. L. Velázquez. Estimates on the $(n-1)$-dimensional Hausdorff measure of the blow-up set for a semilinear heat equation. *Indiana Univ. Math. J.*, 42(2):445–476, 1993.

[24] M. J. Ward. Topics in singular perturbations and hybrid asymptotic-numerical methods. In *ICIAM 95 (Hamburg, 1995)*, pages 435–462. Akademie Verlag, Berlin, 1996.

[25] F. B. Weissler. Single point blow-up for a semilinear initial value problem. *J. Differential Equations*, 55(2):204–224, 1984.

[26] H. Zaag. On the regularity of the blow-up set for semilinear heat equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 2001. to appear.

[27] H. Zaag. One dimensional behavior of singular N dimensional solutions of semilinear heat equations. 2001. preprint.

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