QUANTIZATION CONDITION FOR MULTI-BARRIER RESONANCES

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Abstract. We give the quantization condition and the semiclassical distribution of resonances of the Schrödinger operator in a general setting where the trapped set of the underlying classical mechanics makes a finite graph consisting of hyperbolic fixed points and associated homoclinic and heteroclinic trajectories. This is one of the results in our paper [3]. We give some examples and a rough sketch of the proof.

1. Introduction

We consider the Schrödinger operator on the n-dimensional Euclidean space $\mathbb{R}^n$

$$P = -h^2\Delta + V(x) = -h^2\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right) + V(x_1, \dots, x_n), \tag{1}$$

where $V(x)$ is a real-valued smooth potential decaying at infinity and h is a small positive parameter coming from the Planck constant.

This self-adjoint operator has continuous spectrum on the real positive axis and has no positive eigenvalue. But instead there may be so called resonances close to the real positive axis, which are the poles of the meromorphic extension from the upper half complex plane to the lower one of the cutoff resolvent. Resonances are associated with semi-bound states (resonant state), which are outgoing near infinity, and the imaginary part of resonances represents the reciprocal of the lifetime of the quantum particles in the resonant state.

More precisely, let $\xi = (\xi_1, \dots, \xi_n)$ be the dual variable of x and

$$p(x, \xi) = \xi^2 + V(x) = \sum_{j=1}^{n} \xi_j^2 + V(x) \tag{2}$$

the classical Hamiltonian corresponding to P. Then the existence and the asymptotic distribution in the semiclassical limit $h \to 0$ of resonances in a complex neighborhood of a positive energy E_0 is closely related to the existence and the geometrical properties of the trapped set $K(E_0)$, which is the set of points in $p^{-1}(E_0) \subset \mathbb{R}^n_x \times \mathbb{R}^n_\xi$ from which the integral curve of the Hamiltonian vector field $H_p := \partial_\xi p \cdot \partial_x - \partial_x p \cdot \partial_\xi$ stays bounded.

In fact, the non-existence of resonance in a certain complex neighborhood of a non-trapping energy was proved in [6], [12], [14]. Precise asymptotic distribution of resonances was given in the case where the trapped set consists of a unique critical point ([17], [7], [16]), in the case of a unique closed trajectory of hyperbolic type ([10]) and in the case of so-called shape resonances created by a well in an island ([12], [8]).

In this report, we give the quantization condition and the precise asymptotic distribution of resonances near an energy E_0 whose trapped set consists of a finite number of hyperbolic fixed points and a finite number of associated homoclinic and heteroclinic trajectories. This is the case typically when E_0 is the non-degenerate maximal value of multiple bumps of the potential. The trapped set can be regarded as an oriented graph. The quantization condition is given by a transfer matrix whose size is the number of edges. Each element corresponding to a couple of edges is given by a geometrical quantity describing the evolution from one edge to the other of the microlocal solution. The principal part of the condition is determined by the cycle which minimizes the damping of the microlocal solution, and the imaginary part of the resonances is asymptotically $-D_0 h$, where D_0 is the damping index along the minimal cycle given in terms of the eigenvalues of the fundamental matrix at each vertex (fixed point) in the cycle.

This is a generalization of [9], where the one-dimensional case was studied using a complex WKB method. Here we employ a microlocal method assuming only the smoothness for the potential in arbitrarily large compact domain. A brief sketch of the proof will be given in the last section.

2. Result

We assume the following conditions (A1)–(A5) on the potential $V(x)$.

(A1) $V(x)$ is a real-valued smooth function on $\mathbb{R}^n$, analytic in a complex sector at infinity

$$\mathcal{S} = \big\{ x \in \mathbb{C}^n;\ |\operatorname{Im} x| \leq (\tan \theta_0) |\operatorname{Re} x|,\ |\operatorname{Re} x| > R \big\}, \quad 0 < \theta_0 < \frac{\pi}{2},\ R > 0,$$

and tends to 0 as $|x| \to \infty$.

This assumption enables us to define resonances in the complex sector $\{E \in \mathbb{C} \setminus \{0\}; -2\theta_0 < \arg E < 0\}$ by the complex distortion introduced by [1] and [13] (see Section 4).

(A2) There exists a finite set $\mathscr{V}$ in $\mathbb{R}^n$ such that $V(x)$ admits a non-degenerate maximum $E_0 > 0$ at each point $v \in \mathscr{V}$.

For each $v \in \mathscr{V}$, the point $(v, 0)$ on the phase space is a hyperbolic fixed point of the Hamiltonian vector field H_p. Identifying v in the configuration space and $(v, 0)$ in the phase space, we denote by $\mathscr{V}$ the set of the hyperbolic fixed points. Let $0 < \lambda_1^v \leq \lambda_2^v \leq \cdots \leq \lambda_n^v$ be the positive eigenvalues of the fundamental matrix at v, and $\Lambda_\pm^v$ the outgoing and incoming stable manifolds associated with v.

Let $v, \tilde{v} \in \mathscr{V}$ and assume that $\Lambda_-^v \cap \Lambda_+^{\tilde{v}}$ is not empty. The classical trajectory $e(t) := \exp tH_p(x_0, \xi_0)$ starting from a point (x_0, ξ_0) in this intersection tends to v as time t tends to $+\infty$ and to $\tilde{v}$ as t tends to $-\infty$. We write $v = e^+$, $\tilde{v} = e^-$. This trajectory is called homoclinic when $v = \tilde{v}$ and heteroclinic when $v \neq \tilde{v}$. We denote $\mathscr{E}$ the set of homoclinic and heteroclinic trajectories associated with $\mathscr{V}$.

We assume that the trapped set $K(E_0)$ consists of the hyperbolic fixed points and the associated homoclinic and heteroclinic trajectories:

(A3) $K(E_0) = \mathscr{V} \cup \mathscr{E}$.

Moreover, for each $e \in \mathscr{E}$, $\Lambda_+^{e^-}$ and $\Lambda_-^{e^+}$ intersect cleanly along e, i.e.

(A4) $T_\rho \Lambda_+^{e^-} \cap T_\rho \Lambda_-^{e^+} = T_\rho e$ at each point ρ on e.

The final assumption is imposed on the asymptotic direction of the homoclonic or heteroclinic trajectories at the fixed points. For $e, \tilde{e} \in \mathscr{E}$ with $e^- = \tilde{e}^+ =: v \in \mathscr{V}$, the asymptotic direction ω_+^e (resp. $\omega_-^{\tilde{e}}$) at v is the limit as $t \to -\infty$ (resp. as $t \to +\infty$) of the properly normalized tangent vector of the x-space projection of e (resp. $\tilde{e}$).

(A5) For each couple $e, \widetilde{e} \in \mathscr{E}$ satisfying $e^- = \widetilde{e}^+ = v$, ω_+^e (resp. $\omega_-^{\widetilde{e}}$) is in the x-space projection of the eigenspace associated with the eigenvalue λ_1 (resp. $-\lambda_1$) of the fundamental matrix of p at v, and moreover $\omega_+^e \cdot \omega_-^{\widetilde{e}} \neq 0$.

The pair of sets $(\mathscr{V}, \mathscr{E})$ is a graph when $\mathscr{V}$ is regarded as the set of vertices and $\mathscr{E}$ the set of edges. The edges are oriented by the Hamiltonian vector field. The assumption (A4) implies in particular that $(\mathscr{V}, \mathscr{E})$ is a finite graph. A finite sequence $(e_1, \dots, e_k)$ is called path if $e_\ell^+ = e_{\ell+1}^-$, $\ell = 1, \dots, k-1$, and called cycle if moreover $e_k^+ = e_1^-$. We identify the cycles $(e_1, \dots, e_k)$ and $(e_2, \dots, e_k, e_1)$. A cycle is called primitive when it does not contain any non-trivial subcycle.

Suppose that $(\mathscr{V}, \mathscr{E})$ has at least one cycle. For each vertex v and each cycle γ, we define

$$\alpha_v = \frac{1}{2\lambda_1^v} \sum_{j=2}^{n} \lambda_j^v, \quad \beta_v = \frac{1}{\lambda_1^v}, \quad \alpha(\gamma) = \sum_{v \in \mathscr{V}(\gamma)} \alpha_v, \quad \beta(\gamma) = \sum_{v \in \mathscr{V}(\gamma)} \beta_v, \tag{3}$$

where $\mathscr{V}(\gamma)$ is the set of vertices belonging to γ. And we set

$$D(\gamma) = \frac{\alpha(\gamma)}{\beta(\gamma)}, \qquad D_0 = \min_{\gamma \text{ cycle}} D(\gamma). \tag{4}$$

This D_0 will give the principal asymptotics of the imaginary part of resonances. We call minimal cycle any cycle γ with $D(\gamma) = D_0$.

The quantization condition of resonances is given by a square matrix $\mathscr{Q}(E, h)$ whose size is the number of edges $\sharp\mathcal{E}$ of the graph. For $e, \tilde{e} \in \mathscr{E}$, we define the $(e, \tilde{e})$ component of the matrix by

$$\mathscr{Q}_{e,\widetilde{e}}(E, h) = \begin{cases} h^{\alpha_v - i\beta_v z} \mathcal{Q}_{e,\widetilde{e}}(\alpha_v - i\beta_v z) & \text{if } e^- = \widetilde{e}^+ =: v, \\ 0 & \text{if } e^- \neq \widetilde{e}^+, \end{cases} \tag{5}$$

with

$$z = (E - E_0)/h, \quad \mathcal{Q}_{e,\widetilde{e}}(\zeta) = C_{e,\widetilde{e}}\, e^{iA_e/h} \Gamma\Big(\zeta + \frac{1}{2}\Big) \big(i\lambda_1^v \omega_+^e \cdot \omega_-^{\widetilde{e}}\big)^{-(\zeta + \frac{1}{2})},$$

where $A_e := \int_e \xi \cdot dx$ is the action along e and $C_{e,\widetilde{e}}$ is a constant independent of h and E that can be explicitly given in terms of geometric quantities near e and $\tilde{e}$.

Definition 2.1 (Quantization condition). *We call E pseudo-resonance if it satisfies*

$$\det(I - \mathscr{Q}(E, h)) = 0. \tag{6}$$

We denote the set of pseudo-resonances by $\mathrm{Res}_0(P)$.

The following theorem shows that, for positive constants C, ϵ, the resonances are "close" to pseudo-resonances in the rectangular complex domain near E_0 given by $[E_0 - Ch, E_0 + Ch] + i[-D_0 h - C\frac{h}{|\log h|}, 0]$ except in the εh-neighborhood of the set $E_0 - ih\Gamma_{\mathscr{V}}$, with $\Gamma_{\mathscr{V}} = \bigcup_{v \in \mathscr{V}} \Gamma_v$ and

$$\Gamma_v = \left\{ \sum_{j=1}^{n} \lambda_j^v \left(\alpha_j + \frac{1}{2} \right) ;\ \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n \right\}$$

is the set of eigenvalues of the multidimensional harmonic oscillator $-\Delta+\sum\frac{(\lambda_j^v)^2}{4}x_j^2$. We denote this set $B(C,\varepsilon)$.

Theorem 2.2. *Assume (A1)–(A5). Fix $C>0$ arbitrarily large and $\varepsilon>0$ arbitrarily small. In the domain $B(C,\varepsilon)$, it holds that, as $h\to 0$,*

$$\operatorname{dist}\big(\operatorname{Res}(P),\operatorname{Res}_0(P)\big)=o\Big(\frac{h}{|\log h|}\Big),$$

which means that for any $E\in\operatorname{Res}(P)\cap B(C,\varepsilon)$, there exists $\tilde{E}\in\operatorname{Res}_0(P)$ such that $|E-\tilde{E}|=o(h/|\log h|)$, and conversely, for any $\tilde{E}\in\operatorname{Res}_0(P)\cap B(C,\varepsilon)$, there exists $E\in\operatorname{Res}(P)$ such that $|E-\tilde{E}|=o(h/|\log h|)$.

In order to study the asymptotic distribution of (pseudo-)resonances defined by the condition (6), let us fix $\tau\in[-C,C]$ arbitrarily, and set

$$E=E_0+\tau h-iD_0h+Z\frac{h}{|\log h|},$$

to consider Z as a new spectral parameter instead of E. Then the power of h in each element of the matrix $\mathscr{Q}(E,h)$ is written

$$h^{\alpha_v-i\beta_v z}=h^{\alpha_v-D_0\beta_v}e^{i\beta_v Z}h^{-i\beta_v\tau}.$$

Recall that the determinant of a matrix is a sum over the symmetric group of products of elements corresponding to each permutation, and that a permutation is a product of cyclic permutations. Hence each permutation giving a non-zero term to $\det(I-\mathscr{Q}(E,h))$ corresponds naturally to a finite union of cycles of the graph, and $\det(I-\mathscr{Q}(E,h))$ is of the form

$$\det(I-\mathscr{Q}(E,h))=\sum_{a,b}F_{a,b}(\rho,\kappa,\frac{Z}{|\log h|})h^a e^{ibZ},$$

where a,b are constants with values in $\{\alpha(\gamma)-D_0\beta(\gamma)\}_{\gamma:\text{cycle}}$, $\{\beta(\gamma)\}_{\gamma:\text{cycle}}$ respectively, and the sum is finite. $F_{a,b}$ is a holomorphic function of $\rho:=\{e^{iA_e/h}\}_e$, $\kappa:=\{h^{-i\beta_v\tau}\}_v$ and Z. The power $\alpha(\gamma)-D_0\beta(\gamma)$ is non-negative and 0 if and only if γ consists only of minimal cycles, and hence minimal cycles give the principal part of the asymptotics of $\det(I-\mathscr{Q}(E,h))$. Set

$$f_\tau(Z,h):=\sum_b F_{0,b}(\rho,\kappa,0)e^{ibZ},$$

This is an exponential sum with respect to Z. We see from the general theory of such sums that, for any $C,\epsilon>0$, there exists a constant N such that the imaginary part of the zeros of $f_\tau(Z,h)$ is less than N and the number of the zeros is also locally finite uniformly for $\tau\in[-C,C]$, $h\in(0,1)$ satisfying $\operatorname{dist}(E_0+\tau h-iD_0h,E_0-ih\Gamma_{\mathscr{V}})\geq\epsilon h$. Thus we obtain

Corollary 2.3. *Fix $C,\epsilon>0$ and $\tau\in[-C,C]$. Then the resonances belonging to the rectangular set $[E_0+\tau h-C\frac{h}{|\log h|},E_0+\tau h+C\frac{h}{|\log h|}]+i[-D_0h-C\frac{h}{|\log h|},0]$ except in the εh-neighborhood of the finite set $E_0-ih\Gamma_{\mathscr{V}}$ satisfy*

$$E=E_0+\tau h-iD_0h+Z\frac{h}{|\log h|}+o\Big(\frac{h}{|\log h|}\Big),\tag{7}$$

where Z is a zero of $f_\tau(\cdot,h)$.

3. Examples

3.1. **Unique homoclinic trajectory.** The first example is the simplest case where the trapped set consists of a unique hyperbolic fixed point and a unique associated homoclinic trajectory e. This e is of course the unique minimal primitive cycle and $D_0 = \frac{1}{2}\sum_{j=2}^{n} \lambda_j$. The matrix $\mathscr{Q}$ is a function of $\zeta = \frac{1}{\lambda_1}(D_0 - iz) = \alpha - i\beta z$ and the set of pseudo-resonances is given by

$$\mathrm{Res}_0(P) := \left\{E \in \mathbb{C};\ h^{\zeta} e^{iA/h} \mu(\zeta) = 1\right\}, \tag{8}$$

where

$$\mu(\zeta) := C\Gamma(\zeta + \frac{1}{2})(i\lambda_1\omega^+ \cdot \omega^-)^{-(\zeta+1/2)}$$

is independent of h. The constant $C \neq 0$ is independent of E, h and A is the action along e. If a pseudo-resonance $E(h)$ satisfies $\mathrm{Re}\, E = E_0 + \tau h + o(h)$ for some real number τ, there exists an integer k such that

$$E = E_0 + \lambda_1\Big(2k\pi - \frac{A}{h}\Big)\frac{h}{|\log h|} - i\bigg(D_0 h - \lambda_1 \log \mu\Big(\frac{\tau}{i\lambda_1}\Big)\frac{h}{|\log h|}\bigg) + o\Big(\frac{h}{|\log h|}\Big). \tag{9}$$

From this expression, we see for example that the resonances in the domain $B(C, \epsilon)$ accumulate in the semiclassical limit along the curve defined by

$$\mathrm{Im}\, E = -D_0 h + \lambda_1 \log\left|\mu\Big(\frac{\mathrm{Re}E - E_0}{i\lambda_1 h}\Big)\right|\frac{h}{|\log h|}, \tag{10}$$

and that the distance between two neighboring resonances is

$$2\pi\lambda_1\frac{h}{|\log h|} + o\Big(\frac{h}{|\log h|}\Big).$$

Since the constant C is arbitrary, it is interesting to see the asymptotic behavior of the above curve when $(\mathrm{Re}E - E_0)/h \to \pm\infty$. Using the fact $\omega_+ \cdot \omega_- > 0$ in this particular case and the formula $\left|\Gamma\Big(\frac{1}{2} - it\Big)\right| = \sqrt{\pi/\cosh(\pi t)}$, we have

$$\left|\mu\left(\frac{\tau}{i\lambda_1}\right)\right| = c(e^{2\pi\tau/\lambda_1} + 1)^{-1/2} \sim \begin{cases} c\, e^{-\pi\tau/\lambda_1} & (\tau \to +\infty), \\ c & (\tau \to -\infty). \end{cases}$$

Hence the curve (10) satisfies, with $\tilde{c} = \lambda_1 \log c$,

$$\mathrm{Im}\, E = \begin{cases} -D_0 h + \dfrac{h}{|\log h|}\left(\tilde{c} - \pi\dfrac{\mathrm{Re}E - E_0}{h} + o(1)\right) & (\mathrm{Re}\, E \to +\infty), \\ -D_0 h + \dfrac{h}{|\log h|}(\tilde{c} + o(1)) & (\mathrm{Re}\, E \to -\infty). \end{cases} \tag{11}$$

This formula shows that the imaginary part of resonances which is almost constant below the level E_0 increases in a linear way above this level. This reflects the geometrical transition property of the classical mechanics that energies below E_0 are trapping while those above E_0 are non-trapping.

3.2. **Three bumps potential.** Next, we study the resonances near the barrier top of three bumps. Suppose the trapped set consists of three hyperbolic fixed points $\mathscr{V} = \{a, b, c\}$ and 6 heteroclinic trajectories $\mathscr{E} = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ joining these fixed points as in Figure 1. Then there are 5 primitive cycles

$$(e_1, e_2),\ (e_3, e_4),\ (e_5, e_6),\ (e_1, e_3, e_5),\ (e_2, e_6, e_4).$$

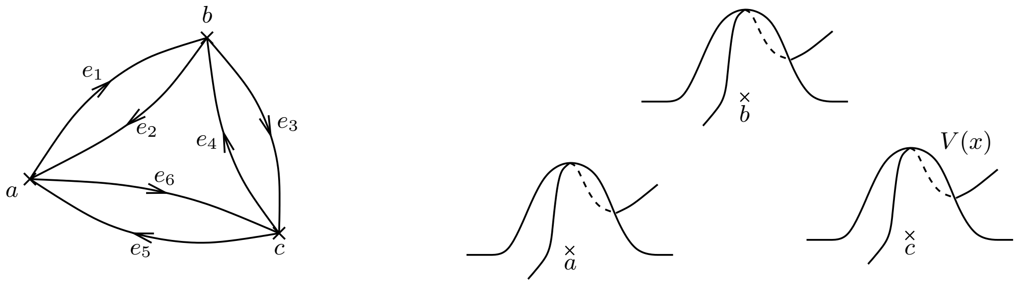

FIGURE 1. Three bumps potential and the graph of trapped set.

The matrix $\mathscr{Q}$ is a 6×6 matrix given by

$$\mathscr{Q}=\begin{pmatrix} 0 & \mathscr{Q}_{e_1,e_2} & 0 & 0 & \mathscr{Q}_{e_1,e_5} & 0\\ \mathscr{Q}_{e_2,e_1} & 0 & 0 & \mathscr{Q}_{e_2,e_4} & 0 & 0\\ \mathscr{Q}_{e_3,e_1} & 0 & 0 & \mathscr{Q}_{e_3,e_4} & 0 & 0\\ 0 & 0 & \mathscr{Q}_{e_4,e_3} & 0 & 0 & \mathscr{Q}_{e_4,e_6}\\ 0 & 0 & \mathscr{Q}_{e_5,e_3} & 0 & 0 & \mathscr{Q}_{e_5,e_6}\\ 0 & \mathscr{Q}_{e_6,e_2} & 0 & 0 & \mathscr{Q}_{e_6,e_5} & 0 \end{pmatrix}.$$

Assume for simplicity that $n=2$ and the three fixed points are all isotropic, i.e. $\lambda_1^a=\lambda_2^a=:\lambda_a$ etc. There are three cases up to symmetry:
(i) $\lambda_a=\lambda_b=\lambda_c=:\lambda$, (ii) $\lambda_a=\lambda_c>\lambda_b$, (iii) $\lambda_c>\lambda_a,\lambda_b$.

In case (i), every cycle is minimal and $D_0=\lambda/2$. All the elements of the matrix $\mathscr{Q}$ have a common factor $h^{\alpha-i\beta z}=h^{\frac12-\frac{iz}{\lambda}}$. Hence, if we denote $\mu_m(\tau,h)$ $(m=1,\dots,6)$ the eigenvalues of the 6×6 matrix with elements

$$\widehat{\mathcal{Q}}_{e,\tilde e}(\tau,h):=\mathcal{Q}_{e,\tilde e}(E_0+\tau h-iD_0h,h),$$

instead of $\mathscr{Q}_{e,\tilde e}$ in $\mathscr{Q}$, then the resonances in $B(C,\varepsilon)$ are close to the set

$$E_{m,k}(\tau)=E_0+2k\pi\lambda\frac{h}{|\log h|}-i\frac{\lambda}{2}h+i\log(\mu_m(\tau,h))\lambda\frac{h}{|\log h|},\tag{12}$$

with $1\le m\le 6$ and $k\in\mathbb{Z}$. If moreover Δabc is equilateral, the action integral is the same for all e_j and μ_m can be explicitly computed. In particular there are two double eigenvalues, and the resonances accumulate to the 4 curves illustrated in Figure 2.

In case (ii), the minimal cycles are (e_1,e_2) and (e_3,e_4), and

$$D_0=\frac{2^{-1}+2^{-1}}{\lambda_a^{-1}+\lambda_b^{-1}}=\frac{\lambda_a\lambda_b}{\lambda_a+\lambda_b}.\tag{13}$$

In this case, the matrix giving the principal part of the resonances is reduced to

$$\begin{pmatrix} \widehat{\mathcal{Q}}_{e_1,e_2}\widehat{\mathcal{Q}}_{e_2,e_1} & \widehat{\mathcal{Q}}_{e_1,e_2}\widehat{\mathcal{Q}}_{e_2,e_4}\\ \widehat{\mathcal{Q}}_{e_4,e_3}\widehat{\mathcal{Q}}_{e_3,e_1} & \widehat{\mathcal{Q}}_{e_4,e_3}\widehat{\mathcal{Q}}_{e_3,e_4} \end{pmatrix},$$

and the resonances appear close to the points

$$E_{m,k}(\tau)=E_0+2k\pi\frac{\lambda_a\lambda_b}{\lambda_a+\lambda_b}\frac{h}{|\log h|}-i\frac{\lambda_a\lambda_b}{\lambda_a+\lambda_b}h+i\log(\mu_m(\tau,h))\frac{\lambda_a\lambda_b}{\lambda_a+\lambda_b}\frac{h}{|\log h|},$$

with $m=1,2,$, where $\mu_1(\tau,h),\mu_2(\tau,h)$ are the eigenvalues of this matrix.

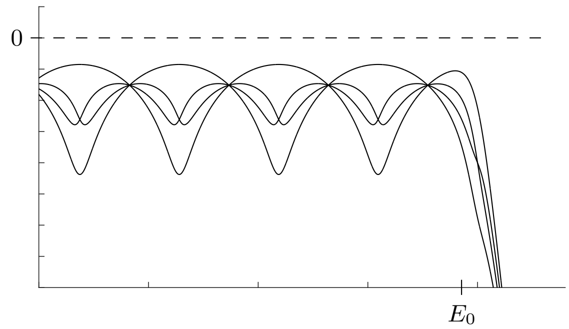

FIGURE 2. Accumulation curves of the resonances near the barrier top of three bumps in the equilateral case.

In case (iii), the minimal cycle is (e_1, e_2) and (13) still holds. The matrix reduces to a scalar $\mu(\tau, h) = \widehat{\mathcal{Q}}_{e_1,e_2}\widehat{\mathcal{Q}}_{e_2,e_1}$ and the resonances are close to

$$E_k(\tau) = E_0 + 2k\pi\frac{\lambda_a\lambda_b}{\lambda_a+\lambda_b}\frac{h}{|\log h|} - i\frac{\lambda_a\lambda_b}{\lambda_a+\lambda_b}h + i\log(\mu(\tau,h))\frac{\lambda_a\lambda_b}{\lambda_a+\lambda_b}\frac{h}{|\log h|}.$$

4. Outline of the proof

We do not prove the results in this short report, but give at least some comments necessary to understand the meaning of the results and the outline of the proof.

We first recall the definition of resonances. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth vector field, such that $F(x) = 0$ for $|x| \leq R$ and $F(x) = x$ for $|x| \geq R + 1$ for a large enough R. For $\mu \in \mathbb{R}$ small enough, we denote $U_\mu : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ the unitary operator defined by $U_\mu\varphi(x) = \big|\det(1 + \mu dF(x))\big|^{1/2}\varphi(x + \mu F(x))$. Then, the operator $U_\mu P(U_\mu)^{-1}$ is a differential operator with analytic coefficients with respect to μ, and can be analytically continued to small enough complex values of μ. For $\theta \in \mathbb{R}$ small enough, we denote $P_\theta = U_{i\theta}P(U_{i\theta})^{-1}$. The spectrum of P_θ is discrete in $\mathcal{E}_\theta = \{z \in \mathbb{C};\ -2\theta < \arg z \leq 0\}$, and the resonances of P are by definition the eigenvalues of P_θ in $\mathcal{E}_\theta$. The resonances do not really depend on θ.

Let E be an energy in a $\mathcal{O}(h)$-neighborhood of E_0 and let $u \in L^2(\mathbb{R}^n)$ satisfy $\|(P_\theta - E)u\| = \mathcal{O}(h^\infty)$. Then a first important fact due to [5] and the propagation of singularities is that if u is microlocally 0 on the trapped set $K(E_0)$, then $\|u\| = \mathcal{O}(h^\infty)$. Here, we say that u is microlocally 0 at a point (x_0, ξ_0) in the phase space if there exists a function $a(x, \xi) \in C_0^\infty(\mathbb{R}^{2n})$ with $a(x_0, \xi_0) \neq 0$ such that $a^w(x, hD)u = \mathcal{O}(h^\infty)$, where $a^w(x, hD)$ is the semiclassical pseudo-differential operator of symbol $a(x, \xi)$.

Let $v \in \mathscr{V}$. There are a finite number of $\tilde{e}$'s such that $\tilde{e}^+ = v$ and a finite number of e's such that $e^- = v$. Let u^e be the microlocal restriction of u near e. Then, another important fact ([2]) is that the functions $u^{\tilde{e}}$ with $\tilde{e}^+ = v$ (incoming data) determine uniquely and explicitly the functions u^e with $e^- = v$ (outgoing data) since $(P - E)u = 0$ microlocally near v, for E in a $\mathcal{O}(h)$-neighborhood of E_0 except in a εh-neighborhood of the set $E_0 - ih\Gamma_v$ for any small $\varepsilon > 0$. Furthermore, under the condition (A5), the u^e's are always Lagrangian distributions associated with the outgoing Lagrangian manifold Λ_+^v. If moreover the $u^{\tilde{e}}$'s are also Lagrangian distributions, then, at the principal level, the amplitude of each u^e is given as a linear combination of the amplitude of the $u^{\tilde{e}}$'s with coefficient $\mathscr{Q}_{e,\tilde{e}}(E,h)$.

Thus the matrix $\mathscr{Q}$ is a transfer matrix of the microlocal solutions on the edges of the graph of the trapped set, and the quantization condition is the condition under which there exists at least one non-trivial microlocal solution defined consistently on the whole graph.

The proof of the theorem is done by a contradiction argument. For example the existence of a pseudo-resonance near a resonance is proven by showing that if $\|(P_\theta - E)u\| = \mathcal{O}(h^\infty)$ for $\|u\| \leq 1$ and for E away from the set of pseudo-resonances with distance $\delta h/|\log h|$, $\delta > 0$, then $\|u\| = \mathcal{O}(h^\infty)$. Roughly speaking, using the invertibility of $I - \mathscr{Q}$ for such an E, we deduce that the amplitude of the Lagrangian distribution becomes smaller (more precisely, the order in h becomes greater) on each edge after a continuation beyond the vertices. Repeating this argument, we conclude that u is microlocally 0, which implies $\|u\| = \mathcal{O}(h^\infty)$ as mentioned above.

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