Additional Structures on E_n-Cohomology

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Operads and algebras over operads

An operad is a tool for encoding operations abstractly.
An algebra over an operad is an object making these operations concrete.

Examples

- associative products
- associative commutative products
- Lie algebras /dg Lie algebras
- versions up to homotopy

Little n-cubes and loop spaces

The little n-cubes operad $\mathcal{C}_n$ is an operad in Top_* and plays an important role in understanding n-fold loop spaces:

- Every n-fold loop space $\Omega^n X$ is an algebra over $\mathcal{C}_n$ for $1 \leq n \leq \infty$.
- Conversely we get:

Theorem (Boardman-Vogt 1968, May 1972)

Let Y be a nice enough space and suppose $1 \leq n \leq \infty$. If Y is an algebra over $\mathcal{C}_n$ then Y has the homotopy type of an n-fold loop space.

E_n-operads in the algebraic setting

E_n-operads in dgmod are operads weakly equivalent to the operad formed by chains on little n-cubes.
We fix a certain E_n-operad from now on.

Example

- E_1-operads in dgmod encode A_∞-algebras, i.e. algebras that are associative up to all higher homotopies
- Algebras over E_∞-operads are E_∞-algebras, i.e. they are associative and commutative up to all higher homotopies
- For $1 < n < \infty$ algebras over an E_n-operad interpolate between these cases.

E_n-homology and -cohomology

For every sufficiently good operad $\mathcal{O}$ in dgmod one can define $\mathcal{O}$-(co)homology of $\mathcal{O}$-algebras with representations of the algebra as coefficients.
In particular we can define

$$H_*^{E_n}(A, M) \quad \text{and} \quad H^*_{E_n}(A, M).$$

Example

- E_1-(co)homology of an associative algebra is Hochschild (co)homology.
- E_∞-(co)homology of a commutative algebra coincides with Γ-(co)homology.
- E_n-homology of a commutative algebra coincides with higher Hochschild homology up to a degree shift.

The classical bar complex

Recall that for a nonunital dga algebra A the reduced bar construction $BA = (T^c(\Sigma A), \partial)$ is given by

- the tensor coalgebra on the suspension of A
- with differential twisted by utilising the multiplication in A.
- If A is graded commutative, BA is a graded commutative dga algebra and we can iterate.

The bar complex for E_n-algebras

Theorem (Fresse 2011)

(i) *There is a functor*

$$B^n \colon E_n\text{-}alg \to \mathrm{dgmod}$$

which coincides with the classical n-fold bar construction if restricted to graded commutative dga algebras.

(ii) *For any sufficiently good E_n-algebra A we have*

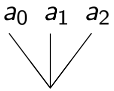
$$H_*^{E_n}(A, k) \cong H_*(\Sigma^{-n} B^n A)$$

for $1 \leq n \leq \infty$.

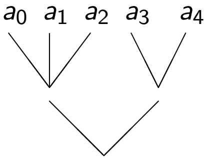
The iterated bar complex and trees

For $1 \leq n < \infty$ typical elements in $B^n(A)$ can be visualised as planar fully grown trees with n levels and leaves labeled by elements in A:

- Typical elements in $B(A)$ are elements in $(\Sigma A)^{\otimes k}$:



- Typical elements in $B^2(A)$ are elements in $B(A)^{\otimes j}$:



- ... and so on.

E_n-homology of functors

Definition

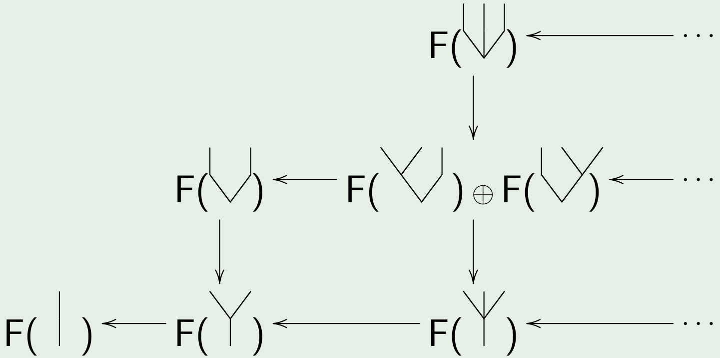
- Let Epi_n be the category with objects planar fully grown trees with n levels and morphisms generated by mimicking the differentials in $B^n(A)$.
- Let $F\colon \mathrm{Epi}_n \to k$-mod be a functor.

The E_n-homology of F is defined as the homology of the total complex associated to an n-fold complex indexed over trees with differentials like in $B^n(A)$.

E_n-homology of functors

Example

$n = 2$:



E_n-homology of functors

Remark

Let A be a projective commutative algebra and $\mathcal{L}(A,k)$ the following functor from Epi_n to k-mod:
A tree with r leaves gets mapped to $A^{\otimes r}$, morphisms in Epi_n induce multiplication and permutations according to how they merge and permute the top leaves. Then...

- *... the total complex associated to $C_{*,\dots,*}(\mathcal{L}(A,k))$ coincides with $\Sigma^{-n}B^n(A)$*
- *E_n-homology of $\mathcal{L}(A,k)$ equals E_n-homology of A.*

E_n-homology as a derived functor

Theorem (Livernet-Richter 2011)

There exists a functor $b\colon Epi_n^{op} \to k\text{-}mod$ such that for all $F\colon Epi_n \to k\text{-}mod$ the equality

$$H_*^{E_n}(F) \cong Tor_*^{Epi_n}(b, F)$$

holds.

E_n-(co)homology with coefficients

Goal (1)

Construct

- *a suitable category* Epi_n^+,
- *a suitable definition of* E_n*-homology (cohomology) for functors* $F\colon Epi_n^+ \to k\text{-mod}$ *(functors* $G\colon {Epi_n^+}^{op} \to k\text{-mod}$*)*
- *a functor* $b^+\colon {Epi_n^+}^{op} \to k\text{-mod}$

such that

$$H_*^{E_n}(F) \cong Tor_*^{Epi_n^+}(b^+, F) \quad \text{and} \quad H^*_{E_n}(G) \cong Ext^*_{Epi_n^+}(b^+, G)$$

generalising E_n*-homology of functors as above and* E_n*-homology/cohomology of projective commutative algebras with coefficients in symmetric bimodules.*

Gerstenhaber structures

The Hochschild cohomology $HH^*(A, A)$ of an associative algebra A has the structure of a Gerstenhaber algebra, consisting of

- an associative graded commutative product $HH^*(A, A) \otimes HH^*(A, A) \to HH^*(A, A)$ given by the cup product
- a graded Lie bracket $HH^*(A, A) \otimes HH^*(A, A) \to HH^{*-1}(A, A)$
- such that $[x, -]$ is a graded derivation for every $x \in HH^*(A, A)$.

Gerstenhaber structures

The Lie bracket originates from the graded commutator of the homotopy

$$\cup_1 \colon C^*(A, A) \otimes C^*(A, A) \to C^{*-1}(A, A)$$

making the cup product on $HH^*(A, A)$ graded commutative.

- Compare with singular cohomology: There we have a full system of homotopys $\cup_i$.

Gerstenhaber structures

Remark

The operad $H_*(\mathcal{C}_2)$ encodes Gerstenhaber algebras.
The Deligne conjecture states that the Gerstenhaber structure on $HH^*(A, A)$ stems from an action of C_2 on $C^*(A, A)$ and has been proven to be true.

Higher cup products

Goal (2)

- *Use the definition of* $H^*_{E_n}(A, A)$ *by a multicomplex indexed over trees to define* $\cup, \cup_1, ..., \cup_n$.
- *Define a graded Lie bracket on* $H^*_{E_n}(A, A)$ *as the graded commutator of* $\cup_n$.

Cohomology operations

Recall that cohomology operations in singular cohomology can be defined by using that singular cohomology is a representable functor.
In the derived setting one has the Yoneda product

$$\mathrm{Ext}_k^i(M, N) \otimes \mathrm{Ext}_k^j(L, M) \to \mathrm{Ext}_k^{i+j}(L, N),$$

for k-modules L, M and N, defined by splicing extensions.

Cohomology operations

Goal (3)

Given $H^*_{E_n}(F) \cong Ext^*_{Epi_n}(b^+, F)$ *define a map*

$$Ext^j_{Epi_n}(b^+, F) \otimes Ext^i_{Epi_n}(b^+, b^+) \to Ext^{i+j}_{Epi_n}(b^+, F),$$

making $H^*_{E_n}(F)$ *a module over the algebra* $H^*_{E_n}(b^+)$.

Thanks

Thank you for your attention!

References

References:

- Benoit Fresse, *Iterated bar complexes of E-infinity algebras and homology theories*, Alg. Geom. Topol. **11** (2011), 747–838.
- Muriel Livernet, Birgit Richter, *An interpretation of E_n-homology as functor homology,*, Math. Z. **269 (1)** (2011), 193–219.