Pseudo-rotations of the open annulus

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Abstract

In this paper, we study pseudo-rotations of the open annulus, *i.e.* conservative homeomorphisms of the open annulus whose rotation set is reduced to a single irrational number (the angle of the pseudo-rotation). We prove in particular that, for every pseudo-rotation h of angle ρ, the rigid rotation of angle ρ is in the closure of the conjugacy class of h. We also prove that pseudo-rotations are not persistent in C^r topology for any $r \geq 0$.

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Introduction

0.1 Some motivations

The concept of rotation number was introduced by H. Poincaré [28] to compare the dynamics of orientation preserving homeomorphisms of the circle to the dynamics of rigid rotations. To any orientation-preserving homeomorphism h is associated a unique rotation number $\rho(h)$, measuring in some sense the average speed of rotation of the orbits of h around the circle. In the case where $\rho(h)$ is rational, the dynamics of h may degenerate dramatically: h may present only one periodic orbit (whereas, for the rigid rotation $R_{\rho(h)}$, all the orbits are periodic). On the contrary, in the case where $\rho(h)$ is irrational, h is always semi-conjugate to the rigid rotation $R_{\rho(h)}$, and the closure of the conjugacy class of h always coincides with the closure of the conjugacy class of the rotation $R_{\rho(h)}$.

The notion of rotation number was generalized by Misiurewicz, Ziemian, and Franks in order to describe the dynamics of homeomorphisms of the closed annulus and of the two-torus (see e.g. [27]). More recently, it was used by P. Le Calvez in order to describe the dynamics of conservative homeomorphisms of the open annulus. Given a homeomorphism h of the (closed or open) annulus isotopic to the identity, one can define the *rotation set* of h, which is in some sense the set of all the possible asymptotic speeds of rotation of the orbits of h around the annulus. This is a subset of $\mathbb{R}$, defined up to the addition of an integer. In general, the rotation set of h is not reduced to a single point, and the dynamics of h is much richer than the dynamics of a single rotation. However, one can address the following problem:

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Problem. *Consider a homeomorphism h of the annulus, such that the rotation set of h is reduced to a single number ρ which is irrational (such an homeomorphism will be called a* pseudo-rotation of angle *ρ). To what extend does the dynamics of h looks like the rigid rotation with angle ρ ?*

In the case of the closed annulus $\mathbb{S}^1 \times [-1, 1]$, the above problem has been studied in [3], starting from a generalization of a theorem of J. Kwapisz [22]. We would like to deal here with the case of the open annulus $\mathbb{S}^1 \times \mathbb{R}$.

Results on homeomorphisms of the open annulus are usually much harder to prove than their analogs on the compact annulus. However, the open annulus setting has a particular interest: it is related to the conservative dynamics on the two-sphere. Indeed, any orientation-preserving conservative homeomorphism h of the two-sphere $\mathbb{S}^2$ has at least two distinct fixed points N and S; removing these two points, one gets a homeomorphism of the open annulus $\mathbb{S}^2 \setminus \{N, S\} \simeq \mathbb{S}^1 \times \mathbb{R}$. Moreover, the rotation set of this homeomorphism is reduced to a single irrational number if and only if h has no other periodic points than N and S (see proposition 0.2). This is the reason the above-mentionned problem is connected to the following conjecture of G. Birkhoff (see [4, page 712] and [19]).

Conjecture (Birkhoff's sphere conjecture). *Let h be an orientation preserving real-analytic conservative diffeomorphism of the two-sphere, and having only two periodic (necessarily fixed) points. Then, h is conjugate to a rigid rotation.*

This conjecture is still open. An example of M. Handel, improved by M. Herman, shows that the real-analyticity assumption is necessary: there exists a C^∞ diffeomorphism of the two-sphere, having only two periodic (fixed) points, that is not conjugate to a rigid rotation ([17, 18]). Note that, in Handel-Herman construction, the rotation number of the diffeomorphism is necessarily a Liouville number. On the contrary, in the case were the rotation number is assumed to be diophantian, some partial results towards the conjecture, based on KAM theory and working for C^∞ diffeomorphisms, were proposed by Herman and written in [8]. Our results, far from proving the conjecture, give some kind of qualitative and topological motivation for it.

0.2 The line translation theorem

Let us denote by $\mathbb{A} = \mathbb{S}^1 \times \mathbb{R}$ the open annulus. We can identify $\mathbb{A}$ with the sphere $\mathbb{S}^2$ minus two points N and S. We call *Lebesgue probability measure* on $\mathbb{A}$ the measure induced by the Lebesgue measure on $\mathbb{S}^2$. We call *essential topological line in* $\mathbb{A}$ every simple curve, parametrized by $\mathbb{R}$, properly embedded in $\mathbb{A}$, joining one of the ends of $\mathbb{A}$ to the other. We recall that a *Farey interval* is an interval of the form $]\frac{p}{q}, \frac{p'}{q'}[$ with $p, q, p', q' \in \mathbb{Z}$ and $qp' - pq' = 1$. Here is our main result.

Theorem 0.1 (Line translation theorem). *Let $h \colon \mathbb{A} \to \mathbb{A}$ be a homeomorphism of the open annulus which is isotopic to the identity, which preserves the Lebesgue measure. Assume that the closure of the rotation set of some lift $\widetilde{h} : \mathbb{R}^2 \to \mathbb{R}^2$ of h is contained in a Farey interval $]\frac{p}{q}, \frac{p'}{q'}[$.*

Then, there exists an essential topological line γ of $\mathbb{A}$ such that the topological lines $\gamma, h(\gamma), \dots, h^{q+q'-1}(\gamma)$ are pairwise disjoint. Moreover, the cyclic order of these topological lines is the same as the cyclic order of the $q + q' - 1$ first iterates of a vertical line $\{\theta\} \times \mathbb{R}$ under the rigid rotation with angle ρ, for any $\rho \in]\frac{p}{q}, \frac{p'}{q'}[$.

Very roughly speaking, theorem 0.1 asserts that, if the the rotation set of a homeomorphism $h : \mathbb{A} \to \mathbb{A}$ is included in a Farey interval $]\frac{p}{q}, \frac{p'}{q'}[$, then the dynamics of h is similar to those of a rigid rotation of angle $\rho \in]\frac{p}{q}, \frac{p'}{q'}[$, provided that one does not wait for more than $q + q' - 1$ iterates.

Although the statement of theorem 0.1 is the natural generalization of the arc translation theorem of [3], the proofs of these two results are completely different. Indeed, most of the arguments used in [3] are specific to the compact annulus ; here, we will have to use some technics coming from Brouwer theory, that are typical from topological dynamics on non-compact surfaces.

The line translation theorem implies the following useful corollary: *if the rotation set of h is bounded, then h is conjugate to a homeomorphism whose displacement function is bounded* (see proposition 5.1 below). This corollary plays a key role in the proof of the perturbation theorem 0.5 below.

0.3 Results on pseudo-rotations

We call *pseudo-rotation* of the open annulus any homeomorphism which is isotopic to the identity, which preserves the Lebesgue measure, and whose rotation set is reduced to a single number α. This number α (defined up to the addition of an integer) is called the *angle* of the pseudo-rotation. The following proposition provides an alternative definition of pseudo-rotations with irrational angles:

Proposition 0.2 (Characterization of pseudo-rotations). *Let h be a homeomorphism of the open annulus $\mathbb{A}$, isotopic to the identity and preserving the Lebesgue probability measure. then h is a pseudo-rotation with irrational angle if and only if it does not have any periodic orbit.*

This result does not seem to appear in the literature. It can be seen as a straightforward application of a generalisation of Poincaré-Birkhoff theorem by J. Franks, together with an ergodic theoretical argument of P. Le Calvez. We will provide a proof in section 2.3.

As an immediate corollary of the line translation theorem 0.1, we get:

Corollary 0.3 (Line translation theorem for pseudo-rotations). *Let $h : \mathbb{A} \to \mathbb{A}$ be a pseudo-rotation of irrational angle ρ. Then, for every $n \in \mathbb{N} \setminus \{0\}$, there exists an essential topological line γ in $\mathbb{A}$, such that the topological lines $\gamma, h(\gamma), \dots, h^n(\gamma)$ are pairwise disjoint. The cyclic order of the lines $\gamma, h(\gamma), \dots, h^n(\gamma)$ is the same as the cyclic order of the n first iterates of a vertical line under the rigid rotation of angle ρ.*

Corollary 0.3 can be seen as an analogue of the following well-known property for the dynamics on the circle: if h is an orientation-preserving homeomorphism of the circle with irrational rotation number ρ, then the cyclic order of the points of any orbit of h is the same as the cyclic order of the points of any orbit of the rigid rotation with angle ρ. However, note that, in corollary 0.3, the essential simple line γ does depend on the integer n. Indeed, one can construct a pseudo-rotation $h : \mathbb{A} \to \mathbb{A}$ with irrational angle such that no essential topological line in $\mathbb{A}$ is disjoint from all its iterates under h (see the examples of Handel [17] and Herman [19]).

Using corollary 0.3, one can prove the following:

Theorem 0.4 (Closure of the conjugacy class of a pseudo-rotation). *Let h be a pseudo-rotation of the open annulus with irrational angle ρ. The rigid rotation of angle ρ is in the closure (for the compact-open topology) of the conjugacy class*[1] *of h.*

In other words, for every pseudo-rotation h of angle ρ, there are conjugates of h which are arbitrarily close (for the compact-open topology) to a rigid rotation. We do not know if the same result holds if one allows only conservative conjugacies. We also do not know if any pseudo-rotation of angle ρ is in the closure of the rigid rotation of angle ρ.

Corollary 0.3 and theorem 0.4 show some common features between the dynamics of any pseudo-rotation with irrational angle and the dynamics on a rigid rotation. Nevertheless, there are examples of pseudo-rotations whose dynamics is quite different from those of a rotation. Indeed, using techniques developed by D. Anosov and A. Katok (see [1, 6, 7, 9]), one can construct C^∞ pseudo-rotations for which the Lebesgue probability measure is ergodic; in particular, such pseudo-rotations are not semi-conjugate to a rigid rotation.

We end this discussion on pseudo-rotations by noting that irrational pseudo-rotation are not robust under perturbations: for each $r \geq 0$, the set of irrational pseudo-rotations is meagre in the space of C^r conservative diffeomorphisms isotopic to the identity (see Corollary 6.3). This will be a consequence of the following perturbation result, where the perturbation is chosen *a priori*, and does not depend on the map one wants to perturb.

Theorem 0.5 (Perturbation of pseudo-rotation). *For every homeomorphism $h\colon \mathbb{A} \to \mathbb{A}$ isotopic to the identity and preserving the Lebesgue probability measure, there exists a rigid rotation R of arbitrarily small angle such that $h \circ R$ has a periodic orbit.*

Theorem 0.5 answers a question of J. Franks, who also proved that the same statement holds in the compact annulus (see [12] pages 18–19). To cope with the lack of compactness, we have to use the line translation theorem and some continuity results of P. Le Calvez. Note that the analogue of theorem 0.5 for non-conservative homeomorphism of the annulus was shown to be false G. Hall and M. Turpin [16]. Moreover, it is not known (see [19]) if, for $r \geq 2$, the space of C^r diffeomorphisms of the two-torus (in the non-conservative case) or of compact manifolds with dimension larger or equal to 3 (in the conservative and non-conservative cases) has a dense subset of diffeomorphisms that present a periodic orbit.

In a forthcoming paper, we shall prove that any irrational pseudo-rotation h possesses a circle compactification in the following sense : there exists a homeomorphism $\hat{h}$ of the compact annulus $\mathbb{S}^1 \times [0,1]$ whose restriction to the open annulus $\mathbb{S}^1 \times]0,1[$ is conjugate to h. In other words, if we see h as a homeomorphism of the sphere fixing the North and South poles, one can construct a blow-up of h at each fixed point.

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1 Preliminaries (I) : rotation numbers

1.1 The open annulus

We denote by $\mathbb{A} = \mathbb{T}^1 \times \mathbb{R}$ the infinite annulus and by $\widetilde{\mathbb{A}} = \mathbb{R} \times \mathbb{R}$ its universal cover. We denote by π the canonical projection of $\widetilde{\mathbb{A}}$ onto $\mathbb{A}$. We denote by p_1 the projection

[1] Here, the conjugating homeomorphisms are not assumed to be conservative.

defined on $\mathbb{A}$ or $\widetilde{\mathbb{A}}$ by $p_1(x,y) = x$. We denote by $T\colon \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ the translation defined by $T(x,y) = (x+1,y)$. Note that the annulus $\mathbb{A}$ is the quotient space $\widetilde{\mathbb{A}}/T$. We will sometimes consider the annulus $\mathbb{A}_q = \mathbb{R}^2/T^q$ for some $q \geq 2$.

By the two points compactification, one can identify the annulus $\mathbb{A}$ to the punctured sphere $\mathbb{S}^2 \setminus \{N,S\}$, where N and S are two distinct points of $\mathbb{S}^2$. The Lebesgue measure on $\mathbb{S}^2$ induces on $\mathbb{A}$ a probability measure on $\mathbb{A}$ that we call the *Lebesgue probability measure of* $\mathbb{A}$ and denote by Leb.

The set of the homeomorphisms of the annulus (resp. of the two-sphere) that are isotopic to the identity is denoted by $\mathrm{Homeo}^+(\mathbb{A})$ (resp by $\mathrm{Homeo}^+(\mathbb{S}^2)$). We will mostly consider the subsets $\mathrm{Homeo}^+_{\mathrm{Leb}}(\mathbb{A})$ and $\mathrm{Homeo}^+_{\mathrm{Leb}}(\mathbb{S}^2)$ of $\mathrm{Homeo}^+(\mathbb{A})$ and $\mathrm{Homeo}^+(\mathbb{S}^2)$ made of the homeomorphisms which preserve the Lebesgue probability measure.

1.2 Rotation numbers of points and measures, rotation set of a homeomorphism

Consider a homeomorphism $h \in \mathrm{Homeo}^+(\mathbb{A})$, and a lift $\widetilde{h}\colon \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ of h. Since $\mathbb{A}$ is not compact, the definitions of the rotation number of a point under $\widetilde{h}$, of the rotation set of $\widetilde{h}$, *etc.* cannot be as simple as in the case of the closed annulus. We follow here the definitions proposed by Le Calvez in [23].

Let us consider a recurrent point $z \in \mathbb{A}$ of h. We say that the *rotation number of z under $\widetilde{h}$* is well-defined and equal to $\rho(z,\widetilde{h}) \in \mathbb{R} \cup \{\pm\infty\}$ if, for every lift $\widetilde{z}$ of z and for any subsequence $(h^{n_k})_{k\geq 0}$ of $(h^n)_{n\geq 0}$ and of $(h^n)_{n\leq 0}$ such that $h^{n_k}(z)$ converges to z, we have

$$\frac{p_1 \circ \widetilde{h}^{n_k}(\widetilde{z})}{n_k} \longrightarrow \rho(z,\widetilde{h}).$$

The *rotation set* $\mathrm{Rot}(\widetilde{h})$ of $\widetilde{h}$ is the set of all rotation numbers of recurrent points of $\widetilde{h}$. As it is discussed in [23], we consider only recurrent points in order to get a definition which is invariant by conjugacy. Note that the rotation set may be empty.

Now, consider a probability measure m on $\mathbb{A}$ which is invariant under h. Note that m-almost every point is recurrent under h. Suppose that

- m-almost every point $z \in \mathbb{A}$ has a rotation number $\rho(z,\widetilde{h})$;
- the function $z \mapsto \rho(z,\widetilde{h})$ is integrable (with respect to the measure m).

Then, we say that *the rotation number of the measure m under $\widetilde{h}$ is well-defined* and equal to

$$\rho(m,\widetilde{h}) = \int_{\mathbb{A}} \rho(z,\widetilde{h})\,dm.$$

In the case where m is the Lebesgue probability measure [2], Le Calvez found a nice condition implying that the rotation number of m is well-defined. First note that, if z is a fixed point of h, then the rotation number of z is always well-defined and is an integer. Consider the set $\mathrm{Rot}_{\mathrm{Fix}}(\widetilde{h})$ of the rotation numbers of all the fixed points of h. Then, one has the following result.

[2] Or, more generally, in the case where m is a probability measure such that $m(U) > 0$ for every open subset U of $\mathbb{A}$.

Theorem 1.1 (P. Le Calvez, existence of the mean rotation number). *Suppose that h preserves the Lebesgue probability measure, and that the set* $\mathrm{Rot}_{\mathrm{Fix}}(\widetilde{h})$ *is bounded. Then Lebesgue almost every point $\widetilde{x}$ has a rotation number, and the rotation set of $\widetilde{h}$ is bounded. In particular, the rotation number $\rho(\mathrm{Leb}, \widetilde{h})$ of the Lebesgue probability measure under $\widetilde{h}$ is well-defined.*

The rotation set, the rotation numbers of the points, and the rotation numbers of the measures satisfy the following elementary properties. [3]

Proposition 1.2.

1. *The rotation set, the rotation number of a point, and the rotation number of a measure are invariant by conjugacy in* $\mathrm{Homeo}^+_{\mathrm{Leb}}(\mathbb{A})$.
2. *The rotation set of $T^k \circ \widetilde{h}$ is obtained by translating by k the rotation set of $\widetilde{h}$. Similarly, for the rotation number of a point, or the rotation number of an invariant measure.*
3. *The rotation set of $\widetilde{h}^q$ is $q\mathrm{Rot}(\widetilde{h})$. Similarly for the rotation number of a point, and for the rotation number of an invariant measure.*

1.3 The morphism property

The *horizontal displacement of* $\widetilde{h}$ is the function $r\colon \mathbb{A} \to \mathbb{R}$ defined as follows: given $z \in \mathbb{A}$, we choose a lift $\widetilde{z}$ of z, and we set $r(z) = p_1(\widetilde{h}(\widetilde{z})) - p_1(\widetilde{z})$. Note that $r(z)$ does depend on the choice of $\widetilde{z}$. If m is an h-invariant probability measure, and if r is m-integrable, Birkhoff's ergodic theorem implies that m has a rotation number equal to $\int r dm$. This shows that the rotation number of the Lebesgue probability measure satisfies some morphism property.

Proposition 1.3. *Let h, g be two homeomorphisms of $\mathbb{A}$ that are isotopic to the identity and preserve the Lebesgue probability measure. Let $\widetilde{h}$, $\widetilde{g}$, $\widetilde{h \circ g}$ be some lifts to $\widetilde{\mathbb{A}}$ of h, g and $h \circ g$.*

If the horizontal displacement of h, g and $h \circ g$ are integrable for the Lebesgue probability measure, then

$$\rho(\mathrm{Leb}, \widetilde{h \circ g}) = \rho(\mathrm{Leb}, \widetilde{h}) + \rho(\mathrm{Leb}, \widetilde{g}).$$

In general, the horizontal displacement of a homeomorphism is not integrable. Moreover, one should note that the property of the horizontal displacement being Leb-integrable is not invariant by conjugacy. We do not know if proposition 1.3 is true without the integrability assumptions (see the precise question and the results in paragraph 5).

2 Preliminaries (II) : Brouwer theory

Every annulus homeomorphism h lifts to a homeomorphism $\widetilde{h}$ of the plane. Thus results about the existence of fixed points can be obtained by considering *Brouwer homeomorphisms*, which are the orientation-preserving fixed point free homeomorphisms of the plane $\mathbb{R}^2$. In this section, we briefly recall some of the main results of the theory of Brouwer homeomorphisms.

[3]For item 3, note that a point which is recurrent for h is also recurrent for h^q for any q.

2.1 Brouwer lines and Brouwer theorem

A *topological line* in the plane is the image Γ of a proper continuous embedding from $\mathbb{R}$ to $\mathbb{R}^2$ (equivalently, using Schoenflies theorem, it is the image of a Euclidean line under a homeomorphism of the plane). Given a Brouwer homeomorphism H, a *Brouwer line* for H is a topological line Γ, disjoint from its image $H(\Gamma)$, and such that Γ separates $H(\Gamma)$ from $H^{-1}(\Gamma)$. We will say that Γ is an *oriented Brouwer line* if it is endowed with the orientation such that $H(\Gamma)$ is on the right of Γ (and thus $H^{-1}(\Gamma)$ is on the left of Γ). Then for every $k \in \mathbb{Z}$, we can endow the line $H^k(\Gamma)$ with the image by H^k of the orientation of Γ. Since H^k preserves the orientation, the line $H^{k+1}(\Gamma)$ is on the right of $H^k(\Gamma)$, and the line $H^{k-1}(\Gamma)$ is on the left of $H^k(\Gamma)$. By induction, we see that $H^q(\Gamma)$ is on the right of $H^p(\Gamma)$ if and only if $q > p$. In particular, the lines $(H^k(\Gamma))_{k\in\mathbb{Z}}$ are pairwise disjoint.

Now let U be the open region of $\mathbb{R}^2$ situated between the lines Γ and $H(\Gamma)$, and $\mathrm{Cl}(U) = \Gamma \cup U \cup H(\Gamma)$. The sets $(H^k(U))_{k\in\mathbb{Z}}$ are pairwise disjoint. As a consequence, the restriction of H to the open set $O = \bigcup_{k\in\mathbb{Z}} H^k(\mathrm{Cl}(U))$ is conjugate to a translation. In particular, if the iterates of $\mathrm{Cl}(U)$ cover the whole plane, then H itself is conjugate to a translation.

The main result of Brouwer theory is the *plane translation theorem*: *every point of* $\mathbb{R}^2$ *lies on a Brouwer line for H (see for example [14]).*

2.2 Guillou-Sauzet-Le Calvez theorem

In the case where the Brouwer homeomorphism H is a lift of a homeomorphism of the annulus $\mathbb{A}$, one would like to have an "equivariant version" of the plane translation theorem, *i. e.* one would like to find some Brouwer lines for H which project as "nice" curves in the annulus $\mathbb{A}$. This is the purpose of a result of L. Guillou (see [15]), which was improved by A. Sauzet in his PhD thesis (see [29]). We give below a foliated version of Guillou-Sauzet's result which relies on a recent and powerful theorem of P. Le Calvez (see [24]). For sake of simplicity, we restrict ourselves to the case of homeomorphisms without wandering points. Recall that an *essential topological line* is the image of the line $\{0\} \times \mathbb{R}$ under a homeomorphism of the annulus that is isotopic to the identity.

Theorem 2.1 (L. Guillou, A. Sauzet, P. Le Calvez). *Let $h : \mathbb{A} \to \mathbb{A}$ be a homeomorphism isotopic to the identity. Assume that:*

- *$\widetilde{h} : \mathbb{R}^2 \to \mathbb{R}^2$ is a fixed point free lift of h;*
- *the homeomorphism h does not have any wandering point (*i.e. *every open set must meet some of its iterates under h).*

Then there exists an oriented foliation $\mathcal{F}$ of the annulus $\mathbb{A}$ such that each oriented leaf of $\mathcal{F}$ is an essential topological line which lifts in $\mathbb{R}^2$ to an oriented Brouwer line for $\widetilde{h}$.

Note that any foliation of the annulus by essential topological lines is homeomorphic to the trivial foliation by vertical lines.

Proof of theorem 2.1. Let h be a homeomorphism of the annulus $\mathbb{A}$, and let $\widetilde{h} : \mathbb{R}^2 \to \mathbb{R}^2$ be a fixed point free lift of h. Le Calvez has proved that there exists a C^0 oriented foliation $\mathcal{F}$ of the annulus $\mathbb{A}$, which lifts as an oriented foliation $\widetilde{\mathcal{F}}$ of $\mathbb{R}^2$ such that every oriented leaf of $\widetilde{\mathcal{F}}$ is an oriented Brouwer line Γ for $\widetilde{h}$, with $\widetilde{h}(\Gamma)$ on the right of Γ (see [24]). Now we see the annulus $\mathbb{A}$ as the sphere minus the two points N, S, and we see $\mathcal{F}$ as a foliation of $\mathbb{S}^2$ with two singularities N and S.

Suppose that $\mathcal{F}$ has a leaf γ which is homeomorphic to a circle. Since it lifts to a topological line Γ in the universal covering of $\mathbb{A}$, this leaf must separate N and S. Since Γ is a Brouwer line, the leaf γ is disjoint from its image, and the open annular region U between γ and $h(\gamma)$ is disjoint from its iterates under h, which contradicts the second assumption of the theorem.

Similarly, we see that $\mathcal{F}$ does not admit a leaf which is closed in $\mathbb{A}$ and whose endpoints in $\mathbb{S}^2$ are both equal to N, or both equal to S. Nor does $\mathcal{F}$ admits any cycle of oriented leaves γ_1, γ_2 that are closed in $\mathbb{A}$ and goes respectively from N to S and from S to N. Now Poincaré-Bendixson theory tells us that all the leaves of $\mathcal{F}$ are closed in $\mathbb{A}$, and either they all go from N to S, or they all go from S to N. $\square$

Remark 2.2. *In most situations, we will not need the whole foliation provided by theorem 2.1 but only one leaf of this foliation.*

2.3 Application to the existence of periodic orbits

In this section, we use some Guillou-Sauzet-Le Calvez theorem to prove classical results about the existence of periodic orbits. In particular, we provide the characterisation of irrational pseudo-rotations announced in the introduction, namely that an annulus homeomorphism does not have any periodic orbit if and only if its rotation set is reduced to a single irrational number (proposition 0.2).

Theorem 2.3 (Franks [12], Le Calvez [23]). *Let $h \in \mathrm{Homeo}^+_{\mathrm{Leb}}(\mathbb{A})$, and let $\widetilde{h}$ be a lift of h. Suppose that $\widetilde{h}$ does not have any fixed point. Then the rotation set $\mathrm{Rot}(\widetilde{h})$ is either contained in $[-\infty, 0]$ or in $[0, +\infty]$. Furthermore, Lebesgue almost every recurrent point has a non zero rotation number.*

We do not know if the statement can be improved by proving that the rotation set does not contain zero.

Proof. Let $h \in \mathrm{Homeo}^+_{\mathrm{Leb}}(\mathbb{A})$, and let $\widetilde{h}$ be a lift of h that has no fixed point. Let $\widetilde{\mathcal{F}}$ be the lift to $\mathbb{R}^2$ of the oriented foliation $\mathcal{F}$ provided by theorem 2.1. Either all the leaves of $\mathcal{F}$ are oriented from S to N, or they are all oriented from N to S. In the remainder, we assume that we are in the first situation. We will prove that the rotation set of $\widetilde{h}$ is contained in $[0, +\infty]$ and that Lebesgue almost every point has a positive rotation number.

Let Γ, Γ' be lifts in $\mathbb{R}^2$ of essential topological lines (oriented from S to N). We denote by $L(\Gamma)$ the connected component of $\mathbb{R}^2 \setminus \Gamma$ on the left of Γ, and by $R(\Gamma)$ the connected component of $\mathbb{R}^2 \setminus \Gamma$ on the right of Γ. We will write $\Gamma < \Gamma'$ if Γ' is included in $R(\Gamma)$.

Observe that, due to the orientations, for every $\Gamma \in \widetilde{\mathcal{F}}$, and every $p, q \geq 0$,

$$T^{-p}(\Gamma) < \Gamma < T^p(\Gamma) \qquad \text{and} \qquad \widetilde{h}^{-q}(\Gamma) < \Gamma < \widetilde{h}^q(\Gamma).$$

Consider a point $x \in \mathbb{R}^2$ and a leaf Γ of $\widetilde{\mathcal{F}}$ such that $x \in R(\Gamma) \cap L(T(\Gamma))$. On the one hand, for every $q \geq 0$, the point $\widetilde{h}^q(x)$ is in $\widetilde{h}^q(R(\Gamma)) \subset R(\Gamma)$. On the other hand, for every $p > 0$, the point $T^{-p}(x)$ is in $T^{-p}(L(T(\Gamma)) = T^{-p+1}(L(\Gamma)) \subset L(\Gamma)$. This implies that, the point x cannot have a negative rotation number. This proves that the rotation set of $\widetilde{h}$ is included in $[0, +\infty]$.

We are left to prove that Lebesgue almost every point in $\mathbb{R}^2$ has a positive rotation number. For this purpose, we use some ergodic theoretical arguments due to P. Le Calvez (see [23, page 3227]). Consider a leaf Γ of $\widetilde{\mathcal{F}}$. Let

$$\widetilde{U} = \widetilde{U}_\Gamma = R(\Gamma) \cap L(\widetilde{h}(\Gamma)) \cap L(T(\Gamma)),$$

and $U = U_\Gamma$ be the projection in $\mathbb{A}$ of $\widetilde{U}$. Note that, by definition, $\widetilde{U}$ is disjoint from its images under $\widetilde{h}$ and T. Consider the *return time function* $\nu = \nu_\Gamma : U \to \mathbb{N} \setminus \{0\}$, the *first return map* $\Phi = \Phi_\Gamma : U \to U$, and the *displacement function* $\tau = \tau_\Gamma : U \to \mathbb{Z}$ defined as follows:

- $\nu(x) = \inf\{n > 0 \mid h^n(x) \in U\}$;
- $\Phi(x) = h^{\nu(x)}(x)$;
- $\tau(x)$ is the unique integer such that $\widetilde{h}^{\nu(x)}(\widetilde{x}) \in T^{\tau(x)}(\widetilde{U})$, where $\widetilde{x}$ is the (unique) lift of x in $\widetilde{U}$.

By classical arguments, the function ν is integrable. Hence, by Birkhoff ergodic theorem, the quantity

$$\nu^*(x) = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} \nu(\Phi^k(x))$$

exists, is finite and positive for Lebesgue almost every x in U. We claim that $\tau(x)$ is a positive integer for every $x \in U$: indeed, for every $\widetilde{x} \in \widetilde{U}$, the point $\widetilde{h}^{\nu(x)}(\widetilde{x})$ is in $\widetilde{h}^{\nu(x)}(R(\Gamma))$, which is included in $R(\widetilde{h}(\Gamma))$, and, for every $p \geq 0$, the set $T^{-p}(\widetilde{U})$ is contained in $L(\widetilde{h}(\Gamma))$. Hence, by Birkhoff ergodic theorem for positive functions, the quantity

$$\tau^*(x) = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} \tau(\Phi^k(x))$$

exists and is greater than or equal to 1 (maybe equal to $+\infty$) for Lebesgue almost every x in U. Since U is open, the recurrent points of h in U are exactly the recurrent points of Φ. Hence, the rotation number of Lebesgue almost every point x of U is equal to

$$\lim_{n\to+\infty} \frac{\tau(x) + \cdots + \tau(\Phi^{n-1}(x))}{\nu(x) + \cdots + \nu(\Phi^{n-1}(x))} = \frac{\tau^*(x)}{\nu^*(x)},$$

which is positive (maybe equal to $+\infty$) for Lebesgue almost every point in U. Since $\mathbb{R}^2 = \bigcup_{\Gamma\in\widetilde{\mathcal{F}}} U_\Gamma$, and since U_Γ is a non-empty open set for every Γ, this implies that almost every point in $\mathbb{R}^2$ has a non-zero rotation number. □

Corollary 2.4. *Let $h \in \mathrm{Homeo}^+_{\mathrm{Leb}}(\mathbb{A})$, and let $\widetilde{h}$ be a lift of h. Let $\frac{p}{q}$ be a rational number in $]\rho^-, \rho^+[$, where ρ^- and ρ^+ belong to the rotation set of $\widetilde{h}$. Then $\frac{p}{q}$ also belongs to the rotation set, and is the rotation number of a q-periodic point of h.*

Proof. Apply the previous theorem to $T^{-p}\widetilde{h}^q$ (using proposition 1.2). □

Proof of proposition 0.2. Let $h \in \mathrm{Homeo}^+_{\mathrm{Leb}}(\mathbb{A})$, and let $\widetilde{h}$ be a lift of h. Any periodic point of h has a rational rotation number, which proves the easy part of the proposition. So assume that h does not have any periodic orbit. According to the previous corollary, the rotation set of $\widetilde{h}$ is reduced to a single number α. Furthermore, the second part of theorem 2.3 (applied to the homeomorphisms $T^{-p}\widetilde{h}^q$) implies that α cannot be a rational number. This completes the proof. □

3 Proof of the line translation theorem

The purpose of this section is to prove the line translation theorem 0.1. Let us explain briefly the strategy of the proof. In subsection 3.1, we prove a preliminary result which ensures that a homeomorphism whose rotation set is contained in $[\varepsilon, +\infty[$ for some $\varepsilon > 0$ is conjugate to a translation. In subsection 3.2, we introduce the first return maps $\widetilde{\varphi} = T^{-p} \circ \widetilde{h}^q$ and $\widetilde{\psi} = T^{p'} \circ \widetilde{h}^{-q'}$, and we state a proposition saying that, to prove theorem 0.1, it is enough to find an essential simple line γ in $\mathbb{A}$ and a lift of γ which is disjoint from its images under $\widetilde{\varphi}$ and $\widetilde{\psi}$. This proposition is a classical consequence of arithmetical properties of Farey intervals. Subsection 3.3 contains the core of the proof of theorem 0.1. The results of subsection 3.1 implies that the homeomorphism $\widetilde{\varphi}$ is conjugate to a translation, so that the quotient $\mathbb{A}' := \mathbb{R}^2/\widetilde{\varphi}$ is homeomorphic to an annulus. The homeomorphism $\widetilde{\psi}$ induces an homeomorphism ψ' of the annulus $\mathbb{A}'$. So, we can apply Guillou-Sauzet-Le Calvez theorem to the homeomorphism ψ'. It provides us with a line Γ in $\mathbb{R}^2$, which is a Brouwer line for $\widetilde{\psi}$, and projects in $\mathbb{A}'$ as an essential topological line. Thus Γ is is also a Brouwer line for $\widetilde{\varphi}$. Then, we prove that Γ is also a Brouwer line for the translation T, and that it projects to an essential topological line in our original annulus $\mathbb{A}$.

Note that the we do not know if one can strengthen the statement of theorem 0.1 by removing the word *closure*. The example described in appendix A only shows that our strategy fails to prove this stronger result, since the first step of the proof (proposition 3.1 below) does not work anymore.

3.1 Homeomorphisms with positive rotation sets

The purpose of this subsection is to prove the following.

Proposition 3.1. *Let $g \in \mathrm{Homeo}^+_{\mathrm{Leb}}(\mathbb{A})$, and $\widetilde{g} : \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of g. Assume that the closure of the rotation set of $\widetilde{h}$ is included in $]0, +\infty]$. Then $\widetilde{g}$ is conjugate to a translation.*

Note that the above statement is sharp: one can construct an example of a measure-preserving homeomorphism $g : \mathbb{A} \to \mathbb{A}$ isotopic to the identity, such that, for some lift $\widetilde{g}$ of g, the rotation set of $\widetilde{g}$ is included in $]0, +\infty]$, but $\widetilde{g}$ is not conjugate to a translation (see appendix A).

Proof of proposition 3.1. Choose a positive integer k such that the rotation set of $\widetilde{g}$ is included in $]\frac{1}{k}, +\infty]$. Consider the homeomorphism $\widetilde{g}' := \widetilde{g}^k \circ T^{-1}$, which is a lift of the homeomorphism $g' = g^k$. The rotation set of $\widetilde{g}'$ is included in $]0, +\infty]$ (see proposition 1.2). In particular, the homeomorphism $\widetilde{g}'$ is fixed point free. Furthermore, since g preserves the Lebesgue probability measure on $\mathbb{A}$, so does g', and in particular no point is wandering under the action of g'. Thus we can apply Guillou-Sauzet theorem 2.1, which provides us with an essential topological line γ in $\mathbb{A}$, such that some lift Γ of γ is disjoint from its image $\widetilde{g}'(\Gamma)$.

Using the conservative version of Schoenflies theorem (see appendix B), we can assume that Γ is the vertical line $\{0\} \times \mathbb{R}$ in $\mathbb{R}^2$, oriented from bottom to top. The image $\widetilde{g}'(\Gamma)$ is disjoint from Γ. If it was on the left side of Γ, then the rotation set of $\widetilde{g}'$ would be contained in $[-\infty, 0[$ (by the same argument as in the proof of theorem 2.3). Thus $\widetilde{g}'(\Gamma)$ is on the right side of Γ. Applying the covering translation T, we get that $\widetilde{g}^k(\Gamma)$ is on the right side of $T(\Gamma)$. By induction, for any positive integer n, $\widetilde{g}^{nk}(\Gamma)$ is on the right of $T^n(\Gamma)$. Similarly, the topological line $\widetilde{g}^{-nk}(\Gamma)$ is on the left of $T^{-n}(\Gamma)$. Let $\mathrm{Cl}(U)$ denote the closed band delimited by Γ and $\widetilde{g}^k(\Gamma)$; we get that the iterates of $\mathrm{Cl}(U)$ by $\widetilde{g}^k$ cover the whole plane. Thus $\widetilde{g}^k$ is conjugate to a translation (see paragraph 2.1).

Now it follows from a standard argument that $\widetilde{g}$, having a power conjugate to a translation, is also conjugate to a translation (the quotient $\widetilde{\mathbb{A}}/\widetilde{g}^k$ is an annulus, thus $\widetilde{\mathbb{A}}/\widetilde{g}$ is the quotient of an annulus by a map of finite order: this is a topological surface whose fundamental group is infinite cyclic, so (using the classification of surfaces) it is again an annulus, so that $\widetilde{g}$ is conjugate to a translation). □

3.2 The "first return maps" $\widetilde{\varphi} = T^{-p} \circ \widetilde{h}^q$ and $\widetilde{\psi} = T^{p'} \circ \widetilde{h}^{-q'}$

We consider a homeomorphism $h \in \mathrm{Homeo}^+_{\mathrm{Leb}}(\mathbb{A})$, and a lift $\widetilde{h} : \mathbb{R}^2 \to \mathbb{R}^2$ of h. We assume that the rotation set of $\widetilde{h}$ is included in a Farey interval $]\frac{p}{q}, \frac{p'}{q'}[$. We consider the homeomorphisms $\widetilde{\varphi} := T^{-p} \circ \widetilde{h}^q$ and $\widetilde{\psi} := T^{p'} \circ \widetilde{h}^{-q'}$, sometimes called *the first return maps associated with* h. These two homeomorphisms play a fundamental role in the proof of the line translation theorem, via the following proposition.

Proposition 3.2. *Let γ be an essential topological line in the annulus $\mathbb{A}$. Assume that some lift Γ of γ is disjoint from its images under the first return maps $\widetilde{\varphi}$ and $\widetilde{\psi}$.*

Then the $q + q' - 1$ first iterates of γ under h are pairwise disjoint, and ordered as the $q + q' - 1$ first iterates of a vertical line under a rigid rotation of angle $\alpha \in]\frac{p}{q}, \frac{p'}{q'}[$.

In other words, to prove the line translation theorem, it is enough to find an essential topological line γ in $\mathbb{A}$, and a lift of γ which is disjoint from its images under $\widetilde{\varphi}$ and $\widetilde{\psi}$. The analogue of proposition 3.2 in the context of homeomorphisms of the circle is well-known. The proof of the proposition relies on arithmetical properties of Farey intervals. The reader can find a proof in [3, appendix A] (the proof is written in the context of the closed annulus, but also works in the infinite annulus setting).

3.3 Proof of the line translation theorem

The closures of the rotation sets of the homeomorphisms $\widetilde{\varphi} = T^{-p} \circ \widetilde{h}^q$ and $\widetilde{\psi} = T^{p'} \circ \widetilde{h}^{-q'}$ are included respectively in $]0, \frac{1}{q'}[$ and $]0, \frac{1}{q}[$ (see proposition 1.2). In particular, according to proposition 3.1, the homeomorphism $\widetilde{\varphi}$ is conjugate to a translation, and thus, the quotient $\mathbb{A}' := \mathbb{R}^2/\widetilde{\varphi}$ is an open annulus. We denote by π' the natural projection of $\mathbb{R}^2$ onto $\mathbb{A}'$.

Since $\widetilde{\varphi}$ and $\widetilde{\psi}$ commute, $\widetilde{\psi}$ induces a homeomorphism ψ' of the open annulus $\mathbb{A}'$. The lift $\widetilde{\psi}$ of ψ' is fixed point free. The next task is to check that ψ' satisfies the second hypothesis of theorem (2.1).

Claim 1. No point of the annulus $\mathbb{A}'$ is wandering under the iteration of ψ'.

Proof. We shall prove that a dense set of points of $\mathbb{A}'$ are recurrent for the homeomorphism ψ'; the claim will follow.

Poincaré recurrence theorem implies that a dense set of points of $\mathbb{A}$ are recurrent for h; this set lifts to a dense set in $\mathbb{R}^2$, which again projects to a dense set in $\mathbb{A}'$. We prove that this last set consists of recurrent points for ψ'.

Consider a point $\widetilde{x} \in \mathbb{R}^2$, such that the point $x := \pi(\widetilde{x})$ in $\mathbb{A}$ is recurrent for h. There exists two sequences of integers $(i_n)_{n\in\mathbb{N}}$ and $(j_n)_{n\in\mathbb{N}}$, such that $j_n \to +\infty$ and $T^{-i_n} \circ \widetilde{h}^{j_n}(\widetilde{x}) \to \widetilde{x}$ when n goes to $+\infty$. For every n, we set

$$k_n := j_n p' - i_n q' \quad \text{and} \quad l_n := j_n p - i_n q,$$

so that

$$T^{-i_n} \circ \widetilde{h}^{j_n} = \widetilde{\psi}^{l_n} \circ \widetilde{\varphi}^{k_n}.$$

Hence, $\widetilde{\psi}^{l_n} \circ \widetilde{\varphi}^{k_n}(\widetilde{x}) \to \widetilde{x}$ when n goes to $+\infty$, which implies that ${\psi'}^{l_n}(x) \to x$. Moreover, since $\widetilde{\varphi}$ is conjugate to a translation, it has no recurrent point, so l_n cannot be equal to zero for n large enough (since $\widetilde{\psi}^{l_n} \circ \widetilde{\varphi}^{k_n}(\widetilde{x}) \to \widetilde{x}$, this would imply $k_n = 0$ for large n; since we also have $j_n = k_n q - l_n q'$, and $j_n \to +\infty$, this would be a contradiction.). Thus the point $x' := \pi'(\widetilde{x})$ is recurrent for the homeomorphism ψ'. This complete the proof of claim 1. □

We are now in a position to apply Guillou-Sauzet-Le Calvez theorem 2.1; it provides us with a Brouwer line Γ for $\widetilde{\psi}$, such that the projection γ' of Γ in the annulus $\mathbb{A}' = \mathbb{R}^2/\widetilde{\varphi}$ is an essential topological line. This implies that Γ is also a Brouwer line for $\widetilde{\varphi}$. According to proposition 3.2, we are left to prove that the projection γ of the line Γ *in the original annulus* $\mathbb{A} = \mathbb{R}^2/T$ is again an essential topological line.

Claim 2. The lines $\widetilde{\psi}(\Gamma)$ *and* $\widetilde{\varphi}(\Gamma)$ *belongs to the same connected component of* $\mathbb{R}^2 \setminus \Gamma$.

Proof. We choose an orientation of Γ in such a way that $\widetilde{\varphi}(\Gamma)$ is on the right of Γ (see subsection 2.1). For every $k, l \in \mathbb{Z}$, the line $\widetilde{\varphi}^k \circ \widetilde{\psi}^l(\Gamma)$ is endowed with the image by $\widetilde{\varphi}^k \circ \widetilde{\psi}^l$ of the orientation of Γ. We denote by U be the connected open region of $\mathbb{R}^2$ bounded by the lines Γ and $\widetilde{\varphi}(\Gamma)$.

We argue by contradiction: we assume that $\widetilde{\psi}(\Gamma)$ is on the left of Γ, or equivalently, that $\widetilde{\psi}^{-1}(\Gamma)$ is on the right of Γ. Under this assumption, the homeomorphisms $\widetilde{\varphi}$ and $\widetilde{\psi}^{-1}$ are both "pushing the line Γ towards the right". Hence, for every pair of positive integer (k, l), the region $\widetilde{\varphi}^k \circ \widetilde{\psi}^{-l}(U)$ is on the right of $\widetilde{\varphi}(\Gamma)$, and thus is disjoint from U.

According to Le Calvez theorem 1.1, almost every point of the annulus $\mathbb{A}$ is recurrent under h and has a well-defined rotation number. Thus we can find a point $\widetilde{x}$ in U and some positive integers m, n such that the point $\widetilde{h}^m \circ T^{-n}(\widetilde{x})$ is in U and such that n/m belongs to $]p/q, p'/q'[$. We have

$$\widetilde{h}^m \circ T^{-n} = \widetilde{\varphi}^k \circ \widetilde{\psi}^{-l},$$

$$\text{with } k = mp' - nq' \text{ and } l = -mp + nq.$$

Since n/m is in the Farey interval $]p/q, p'/q'[$, the integers $k = mp' - nq'$ and $l = -mp + nq$ are positive. Hence, the region $\widetilde{h}^m \circ T^{-n}(U)$ is disjoint from the region U. But this is absurd, since the point $h^m \circ T^{-n}(\widetilde{x})$ is in the intersection of these two regions. □

Claim 3. The line Γ *is a Brouwer line for* T*. Furthermore, let* V *be the connected open region of* $\mathbb{R}^2$ *bounded by the lines* Γ *and* $T(\Gamma)$*, and* $\mathrm{Cl}(V) = \Gamma \cup V \cup T(\Gamma)$*. Then* $\mathrm{Cl}(V)$ *is a fundamental domain for the covering map* $\mathbb{R}^2 \to \mathbb{A} = \mathbb{R}^2/T$.

Proof. By claim 2, both homeomorphisms $\widetilde{\varphi}$ and $\widetilde{\psi}$ "push the line Γ towards right". Hence, given four integers $k, l, k', l' \in \mathbb{Z}$, such that $k < k'$ and $l < l'$, the line $\widetilde{\varphi}^{k'} \circ \widetilde{\psi}^{l'}(\Gamma)$ is strictly on the right of the line $\widetilde{\varphi}^k \circ \widetilde{\psi}^l(\Gamma)$ (we call this "property $(\star)$").

In particular, $T(\Gamma) = \widetilde{\varphi}^q \circ \widetilde{\psi}^{q'}(\Gamma)$ is strictly on the right of Γ, and $T^{-1}(\Gamma)$ is strictly on the left of Γ. Therefore, Γ is a Brouwer line for T.

We are left to prove that the iterates of $\mathrm{Cl}(V)$ under T cover the whole plane, *i.e.* that $\bigcup_{k \in \mathbb{Z}} T^k(\mathrm{Cl}(V)) = \mathbb{R}^2$. As above, we denote by U the connected open region of $\mathbb{R}^2$ bounded by the lines Γ and $\widetilde{\varphi}(\Gamma)$. Since the projection of Γ in the annulus $\mathbb{A}' = \mathbb{R}^2/\widetilde{\varphi}$ is an essential simple line, $\mathrm{Cl}(U) = \Gamma \cup U \cup \widetilde{\varphi}(\Gamma)$ is a fundamental domain for the covering map $\mathbb{R}^2 \to \mathbb{A}'$, and thus, we have

$$\bigcup_{k\in\mathbb{Z}} \widetilde{\varphi}^k(\mathrm{Cl}(U)) = \mathbb{R}^2.$$

According to property $(\star)$, for every $n > 0$, the line $T^{-n}(\Gamma) = \widetilde{\varphi}^{-nq} \circ \widetilde{\psi}^{-nq'}(\Gamma)$ is on the left of the line $\widetilde{\varphi}^{-n}(\Gamma)$, and the line $T^n(\Gamma) = \widetilde{\varphi}^{nq} \circ \widetilde{\psi}^{nq'}(\Gamma)$ is on the right of the line $\widetilde{\varphi}^n(\Gamma)$ (remember that q and q' are greater than 1). Now observe that the set $\bigcup_{k=-n}^{n-1} T(\mathrm{Cl}(V))$ is the region situated between the lines $T^{-n}(\Gamma)$ and $T^n(\Gamma)$, and the set $\bigcup_{k=-n}^{n-1} \widetilde{\varphi}(\mathrm{Cl}(U))$ is the region situated between the lines $\widetilde{\varphi}^{-n}(\Gamma)$ and $\widetilde{\varphi}^n(\Gamma)$. As a consequence, for every $n > 0$, we have

$$\bigcup_{k=-n}^{n-1} T(\mathrm{Cl}(V)) \supset \bigcup_{k\in\mathbb{Z}=-n}^{n-1} \widetilde{\varphi}^k(\mathrm{Cl}(U)),$$

and thus $\bigcup_{k\in\mathbb{Z}} T^k(\mathrm{Cl}(V)) = \mathbb{R}^2$. This completes the proof of the claim. □

Claim 4. The line Γ *projects in* $\mathbb{A}$ *to an essential line* γ

Proof. What remains to be proved is that, with respect to the translation $T : (x, y) \mapsto (x + 1, y)$, the Brouwer line Γ is equivalent to the "trivial" Brouwer line $\Gamma_0 := \{0\} \times \mathbb{R}$. That is, that Γ is proper in $\mathbb{A}$. For that, it suffices to construct a homeomorphism G of the plane that commutes with T, and such that $G(\Gamma_0) = \Gamma$. This is very classical, as we have already mentioned in paragraph 2.1. By Schoenflies theorem, there exists a homeomorphism G from the band $[0, 1] \times \mathbb{R}$ onto the region $\mathrm{Cl}(V)$, such that

$$T \circ G_{|\{0\}\times\mathbb{R}} = G_{|\{1\}\times\mathbb{R}} \circ T.$$

Then we extend G by conjugacy, that is, we set

$$G(p + \alpha, t) = T^p(G(\alpha, t))$$

for any real number t, any integer p and any number α between 0 and 1. The map G is continuous. It is one-to-one (because Γ and V are disjoint from their iterates under T). It is onto (because of claim 3). Clearly, G is an open map; hence, it is an homeomorphism. □

This completes the proof of the line translation theorem.

4 Closure of the conjugacy class of a pseudo-rotation

Recall that theorem 0.4 states that, for any pseudo-rotation $h : \mathbb{A} \to \mathbb{A}$ of irrational angle ρ, the rigid rotation of $(x, y) \mapsto (x + \rho, y)$ is in the closure (for the compact open topology) of the conjugacy class of h. A similar result was proved in [3, corollary 0.2] in the compact annulus setting. Actually, the proof given in [3, section 5] applies to the open annulus setting, with the following modifications:

– replace the notion of *essential simple arc* used in [3] by the notion of *essential topological line* defined in the present article,

– instead of using the *arc translation theorem* of [3], use the *line translation theorem* of the present article.

5 Integrability of the displacement function

The aim of this section is to show that, under the hypothesis of Le Calvez theorem 1.1, up to a suitable change of coordinates, the horizontal displacement function is bounded, and hence integrable (the horizontal displacement function has been defined in paragraph 1.3).

5.1 Statements

Proposition 5.1 (Integrability of the displacement function). *Consider a homeomorphism $h \in \mathrm{Homeo}^+_{\mathrm{Leb}}(\mathbb{A})$. Assume that the set $\mathrm{Rot}_{\mathrm{Fix}}(\widetilde{h})$ of rotation numbers of the fixed points of h is bounded (for some lift $\widetilde{h}$).*

Then there exists $g \in \mathrm{Homeo}^+_{\mathrm{Leb}}(\mathbb{A})$, such that the horizontal displacement function r of any lift $\widetilde{h}_1$ of the homeomorphism $h_1 = g \circ h \circ g^{-1}$ is bounded.

Note that as a consequence of Birkhoff ergodic theorem, the mean rotation number of $\widetilde{h}_1$ is equal to the integral of r over the annulus $\mathbb{A}$. As a classical consequence, we get a more geometrical definition.

Proposition 5.2. *Let $h_1 \in \mathrm{Homeo}^+_{\mathrm{Leb}}(\mathbb{A})$, and $\widetilde{h}_1 : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ be a lift of h_1. Suppose that the horizontal displacement function r of $\widetilde{h}_1$ is bounded. Then the mean rotation number of $\widetilde{h}_1$ is equal to the algebraic area (for the lift of the Lebesgue probability measure on $\mathbb{A}$) of the region of $\widetilde{\mathbb{A}} = \mathbb{R} \times \mathbb{R}$ situated between any vertical line $\widetilde{D} = \{\theta\} \times \mathbb{R}$ and its image $\widetilde{h}_1(\widetilde{D})$.*

In view to proposition 5.1, it seems natural to hope that (under suitable assumptions) the mean rotation number "defines a morphism", as in the case of the compact annulus (see 1.3). For example, the following question may be asked.

Question 5.3. *Let f, g be two homeomorphisms of the annulus, which are isotopic to the identity and preserve the Lebesgue probability measure. Consider some lifts $\widetilde{f}, \widetilde{g}$ of f, g, and assume that the mean rotation numbers of $\widetilde{f}$, $\widetilde{g}$ and $\widetilde{f} \circ \widetilde{g}$ are well-defined.*

Is the mean rotation number of $\widetilde{f} \circ \widetilde{g}$ equal to the sum of the mean rotation numbers of $\widetilde{h}$ and $\widetilde{g}$?

We briefly explain the idea of the proof of proposition 5.1. The easy case is when the closure of the rotation set of $\widetilde{h}$ is contained in some interval $]p, p+1[$ with $p \in \mathbb{Z}$ (e.g. when h is an irrational pseudo-rotation). In this case, since $]p, p+1[$ is a Farey interval, we can directly apply the line translation theorem 0.1, and we get an essential topological line in $\mathbb{A}$ which is disjoint from its image under h. The conservative version of Schoenflies theorem gives a $g \in \mathrm{Homeo}^+_{\mathrm{Leb}}(\mathbb{A})$ that maps this topological line on the straight line $\{0\} \times \mathbb{R}$. The conjugated homeomorphism ghg^{-1} now maps this straight line off itself, and we see easily that the horizontal displacement function of any lift is bounded. In the general case, we will use this easy case by considering intermediate coverings.

5.2 Rotation numbers for intermediate coverings

As usual, take $h \in \mathrm{Homeo}^+_{\mathrm{Leb}}(\mathbb{A})$ and $\widetilde{h} : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ a lift of h. Remember that T denotes the covering translation of $\widetilde{\mathbb{A}}$ (which commutes with $\widetilde{h}$). Given an integer $q \geq 2$, we may consider the intermediate covering $\mathbb{A}_q = \widetilde{\mathbb{A}}/T^q$, which is again an annulus. The homeomorphism $\widetilde{h}$ induces a homeomorphism h' of $\mathbb{A}_q$. In addition to the previously defined notions of rotation numbers of $\widetilde{h}$ as a lift of h, one can consider the rotation numbers of $\widetilde{h} = \widetilde{h}'$ *as a lift of* h'. These numbers are linked in the following way. If z is a recurrent point for h, and z' is any lift of z in $\mathbb{A}_q$, then one easily proves that z' is a recurrent point for h'. If z has a well-defined rotation number $\rho(z, \widetilde{h})$ under h, then the rotation number of z' under h' is also well defined and equal to $\frac{1}{q}\rho(z, \widetilde{h})$.

5.3 Proofs

The core of the proof of proposition 5.1 is contained in the proposition given below. We use the notations of the previous paragraph. Assume that the closure of the rotation set of $\widetilde{h}' = \widetilde{h}$ as a lift of h' is contained in the Farey interval $]0, 1[$. Then we can apply the line translation theorem 0.1, which provides us with an essential topological line γ' of $\mathbb{A}_q$, which is disjoint from its image $h'(\gamma')$. Note that in general, the projection of γ' in $\mathbb{A}$ is not a topological line (it may have self-intersections).

Proposition 5.4. *We can choose the topological line γ' so that its projection in $\mathbb{A}$ is again a topological line.*

We will also call *essential topological line in* $\mathbb{R}^2$ an oriented simple curve $\Gamma : \mathbb{R} \to \mathbb{R}^2$ such that the second coordinate of $\Gamma(t)$ tends to $-\infty$ (resp. $+\infty$) when t tends to $-\infty$ (resp. $+\infty$). Remember that we denote by $R(\Gamma)$ (resp. $L(\Gamma)$) the connected component of $\mathbb{R}^2 \setminus \Gamma$ on the right (resp. on the left) of Γ. If Γ_1 and Γ_2 are two essential topological lines in $\mathbb{R}^2$, we write $\Gamma_1 \leq \Gamma_2$ when Γ_2 is contained in $R(\Gamma_1)$; we write $\Gamma_1 < \Gamma_2$ if $\Gamma_1 \leq \Gamma_2$ and the two lines are disjoint.

Lemma and notation 5.5. *Let Γ_1 and Γ_2 be two essential topological lines in $\mathbb{R}^2$, and let U be the unique connected component of the set $L(\Gamma_1) \cap L(\Gamma_2)$ which contains half lines of the form $]-\infty, a[\times\{b\}$. Then the boundary of U is an essential topological line in $\mathbb{R}^2$, that we denote by $\Gamma_1 \vee \Gamma_2$.*

The proof of lemma 5.5 is similar to that of lemma 3.2 in [3] and uses a classical result by B. Kerékjártó ([21]).

Remark 5.6. *Let Γ_1, Γ_2, Γ_3 be three essential topological lines in $\mathbb{R}^2$. The following properties are immediate consequences of the definition of the line $\Gamma_1 \vee \Gamma_2$.*

- **(i)** *The line $\Gamma_1 \vee \Gamma_2$ is included in the union of the lines Γ_1 and Γ_2. Hence, if $\Gamma_3 < \Gamma_1$ and $\Gamma_3 < \Gamma_2$, then $\Gamma_3 < \Gamma_1 \vee \Gamma_2$.*
- **(ii)** *The sets $R(\Gamma_1)$ and $R(\Gamma_2)$ are included in the set $R(\Gamma_1 \vee \Gamma_2)$. In other words, we have $\Gamma_1 \vee \Gamma_2 \leq \Gamma_1$ and $\Gamma_1 \vee \Gamma_2 \leq \Gamma_2$.*

Proof of proposition 5.4. By theorem 0.1, there exists an essential topological line γ_0 of $\mathbb{A}_q$ which is disjoint from its image $h'(\gamma_0)$. We consider some lift Γ_0 of γ_0 to $\mathbb{R}^2$. Since γ_0 is simple in $\mathbb{A}_q$, the arc Γ_0 is disjoint from $T^q(\Gamma_0)$. Note that since the rotation set of $\widetilde{h}' = \widetilde{h}$ as a lift of h' is contained in $]0, 1[$, we have $T^{-q}(\Gamma_0) < \widetilde{h}^{-1}(\Gamma_0) < \Gamma_0$.

Now, we choose some essential topological lines $\Gamma_1, \dots, \Gamma_{q-1}$ in $\mathbb{R}^2$ such that

$$T^{-q}(\Gamma_0) < \Gamma_{q-1} < \Gamma_{q-2} < \cdots < \Gamma_1 < \Gamma_0.$$

Consider the essential topological line

$$\Gamma = \Gamma_0 \vee T(\Gamma_1) \vee \cdots \vee T^{q-1}(\Gamma_{q-1}) = \bigvee_{i=0}^{q-1} T^i(\Gamma_i).$$

For every $i \in \{0, \ldots, q-2\}$, we have $T^{i+1}(\Gamma_{i+1}) < T^{i+1}(\Gamma_i)$ (by definition of the Γ_i's) and $\Gamma \leq T^{i+1}(\Gamma_{i+1})$ (by definition of Γ and by item (ii) of remark 5.6). Hence for every $i \in \{0, \ldots, q-2\}$, we get

$$\Gamma < T^{i+1}(\Gamma_i).$$

Moreover, we have $\Gamma_0 < T^q(\Gamma_{q-1})$ and $\Gamma \leq \Gamma_0$. Hence

$$\Gamma < T^q(\Gamma_{q-1}).$$

Finally, using item (i) of remark 5.6, we get

$$\Gamma < \bigvee_{i=0}^{q-1} T^{i+1}(\Gamma_i) = T(\Gamma).$$

In particular, Γ is disjoint from its image under T. Moreover, we may assume that the lines $\Gamma_1, \ldots, \Gamma_{q-1}$ were chosen such that

$$\widetilde{h}^{-1}(\Gamma_0) < \Gamma_1, \ldots, \Gamma_{q-1} < \Gamma_0.$$

Using the definition of Γ and remark 5.6, this easily implies that

$$\Gamma < \widetilde{h}'(\Gamma).$$

Similarly, since $\widetilde{h} \circ T^{-q}(\Gamma_0) < \Gamma_0$, we may assume that the lines $\Gamma_1, \ldots, \Gamma_{q-1}$ were chosen such that

$$\widetilde{h} \circ T^{-q}(\Gamma_0) < \Gamma_1, \ldots, \Gamma_{q-1} < \Gamma_0.$$

This easily implies that

$$\widetilde{h}'(\Gamma) < T^q(\Gamma).$$

Let γ' be the projection of Γ in the annulus $\mathbb{A}_q$. Since $\Gamma < \widetilde{h}(\Gamma) < T^q(\Gamma)$, the curve γ' is an essential topological line in $\mathbb{A}_q$ which is disjoint from its image $h'(\gamma')$. Furthermore, since $\Gamma < T(\Gamma)$, the projection of γ' in the annulus $\mathbb{A}$ is again simple, thus it is an essential topological line. □

We are now able to prove proposition 5.1.

Proof of proposition 5.1. Let $h \in \mathrm{Homeo}^+_{\mathrm{Leb}}(\mathbb{A})$, and $\widetilde{h}$ be a lift of h. Assume that the set of rotation numbers of the fixed points of h is bounded. Applying theorem 2.3 to the homeomorphisms $T^{-p} \circ \widetilde{h}$, we see that the rotation set of $\widetilde{h}$ is also bounded. Up to a change of lift, we may assume that $\mathrm{Rot}(\widetilde{h})$ is included in an interval $[1, q-1]$ for some integer q.

Consider the homeomorphism h' induced by $\widetilde{h}$ on the intermediate covering $\mathbb{A}_q$. The rotation set of $\widetilde{h}$, seen as a lift of h', is included in $[\frac{1}{q}, \ldots, \frac{q-1}{q}]$ (see paragraph 5.2). Hence, by proposition 5.4, there exists an essential topological line γ in $\mathbb{A}$ and a lift γ' of γ in $\mathbb{A}_q$ which is disjoint from its image $h'(\gamma')$.

Using the conservative Schoenflies theorem, we get some homeomorphism $g \in \text{Homeo}^+_{\text{Leb}}(\mathbb{A})$ which sends γ on the vertical line $D = \{0\} \times \mathbb{R}$. Take any lift $\widetilde{g}$ of g to $\mathbb{R}^2$ and denote by g' the induced map on $\mathbb{A}_q$. We consider the conjugates $h_1 = g \circ h \circ g^{-1}$, $h'_1 = g' \circ h' \circ (g')^{-1}$ and $\widetilde{h}_1 = \widetilde{g} \circ \widetilde{h} \circ \widetilde{g}^{-1}$. Consider some lift $\widetilde{D}$ of D in $\mathbb{R}^2$, and its projection D' in $\mathbb{A}_q$. Since $h'(\gamma')$ is disjoint from γ', the line D' is disjoint from $h'_1(D')$. Thus the image $\widetilde{h}_1(\widetilde{D})$ is between $T^{qk}(\widetilde{D})$ and $T^{qk+q}(\widetilde{D})$ for some integer k. This easily implies that the image of the displacement function $r = p_1 \circ \widetilde{h}_1 - p_1$ is bounded. The proof is complete. □

Proof of proposition 5.2. We may assume that $\widetilde{D}$ is the vertical line $\{0\} \times \mathbb{R}$. We also note that if the proposition is satisfied for some lift $\widetilde{h}_1$ of h_1, then it is true for any other lift. Since the displacement functions of the lifts of h_1 are bounded, one may choose a lift $\widetilde{h}_1$ and an integer $q \geq 1$ such that $\widetilde{D} < \widetilde{h}_1(\widetilde{D}) < T^q(\widetilde{D})$. Hence, there exists a vertical line $\widetilde{D}_1$ between $\widetilde{D}$ and $T^q(\widetilde{D})$ such that the area of the strip bounded by $\widetilde{D}$ and $\widetilde{h}_1(\widetilde{D})$ is equal to the area of the strip between $\widetilde{D}$ and $\widetilde{D}_1$. We will work on the intermediate covering $\mathbb{A}_q$ with the homeomorphism h'_1 induced by $\widetilde{h}_1$. In $\mathbb{A}_q$, the projection of $\widetilde{D}_1$ is a vertical line D'_1 of the form $\{\theta\} \times \mathbb{R}$ and the projection of $\widetilde{D}$ is a vertical line D'.

By the conservative Schoenflies theorem, there exists some homeomorphism $\psi \in \text{Homeo}^+_{\text{Leb}}(\mathbb{A}_q)$ which is the identity on D' and which maps the line $h'_1(D')$ on D'_1: one can require moreover that each point $(0,t)$ in D' is mapped by $\psi \circ h'_1$ on the point (θ, t) in D'_1. We denote by $\widetilde{\psi}$ the lift of ψ to $\widetilde{\mathbb{A}}_q = \widetilde{\mathbb{A}}$ which fixes $\widetilde{D}$. Note that the displacement function of $\widetilde{\psi}$ is bounded. We also introduce the translation $R_\theta \colon \widetilde{\mathbb{A}} \to \mathbb{A}$ by θ. It is the lift of the rotation R' of $\mathbb{A}_q$ with angle $\frac{\theta}{q}$.

Since $\widetilde{\psi}$ and $R_\theta^{-1} \circ \widetilde{\psi} \circ \widetilde{h}_1$ are the identity on $\widetilde{D}$, their rotation sets (as lift of homeomorphism on $\mathbb{A}_q$) are $\{0\}$ and their mean rotation number are 0. By the morphism property (proposition 1.3), one deduces that the mean rotation number of $\widetilde{h}_1$ (as lift of homeomorphisms of $\mathbb{A}_q$) is equal to $\frac{\theta}{q}$. As it is explained at paragraph 5.2, this implies that the mean rotation number of $\widetilde{h}_1$, as lift of the homeomorphism h_1 of $\mathbb{A}$, is equal to θ. By construction, θ is also the area between $\widetilde{D}$ and its image $\widetilde{h}_1(\widetilde{D})$ in $\widetilde{\mathbb{A}}$. This concludes the proof. □

6 Periodic orbits in one-parameter families

In this section, we consider perturbations of conservative homeomorphisms by hamiltonian vector flows, and we prove a general statement that implies theorem 0.5.

6.1 General statement and some consequences

Let X be a smooth vector field on $\mathbb{A}$ such that

1. X is bounded (in the coordinates system $\mathbb{A} = \mathbb{S}^1 \times \mathbb{R}$);
2. the flow $(\Phi^t)_{t \in \mathbb{R}}$ generated by X preserves the Lebesgue probability measure on $\mathbb{A}$.

The vector field X lifts to a vector field $\tilde{X}$ on $\tilde{\mathbb{A}}$, which in turn generates a flow $(\tilde{\Phi}^t)_{t \in \mathbb{R}}$ which is a lift of the flow $(\Phi^t)_{t \in \mathbb{R}}$. The rotation number of the Lebesgue probability measure for $\tilde{\Phi}^t$ is well defined. Since the displacement function is bounded, the morphism property of proposition 1.3 is satisfied and the map $t \mapsto \rho(\mathrm{Leb}, \tilde{\Phi}^t)$ is continuous. Hence, we have

$$\rho(\mathrm{Leb}, \tilde{\Phi}^t) = t.\rho(\mathrm{Leb}, \tilde{\Phi}^1).$$

Our favourite example is of course the family of Euclidean rotations on $\mathbb{S}^2$ (given on $\mathbb{A} = \mathbb{S}^1 \times \mathbb{R}$ by the constant vector field $X = (1, 0)$). The following statement implies at once theorem 0.5.

Theorem 6.1. *Let X be a vector field as above, and suppose*

$$\rho(\mathrm{Leb}, \tilde{\Phi}^1) \neq 0.$$

Let $h \in \mathrm{Homeo}^+_{\mathrm{Leb}}(\mathbb{A})$. Then there exists arbitrarily small values of t such that $h \circ \Phi^t$ has a periodic orbit.

Let us first state and prove two interesting corollaries of this theorem.

Corollary 6.2. *Given any homeomorphism $h \in \mathrm{Homeo}^+_{\mathrm{Leb}}(\mathbb{A})$, the set*

$$D(h) := \{t \in \mathbb{R} | h \circ \Phi^t \textit{ has a periodic orbit}\}$$

is dense in $\mathbb{R}$.

For each $r \in [0, \infty]$, we consider the space $\mathrm{Diff}^{r,+}_{\mathrm{Leb}}(\mathbb{A})$ of C^r diffeomorphisms of the annulus $\mathbb{A}$ that preserves the Lebesgue probability measure, endowed either with the compact-open or the Whitney topology.

Corollary 6.3. *The space $\mathrm{Diff}^{r,+}_{\mathrm{Leb}}(\mathbb{A})$ contains an open and dense subset of diffeomorphisms having periodic orbits. In particular, the set of irrational pseudo-rotations is meagre.*

Proof of corollary 6.2 assuming theorem 6.1. Fix $t_0 \in \mathbb{R}$. Applying theorem 6.1 to the homeomorphism $h \circ \Phi^{t_0}$, we get arbitrarily small numbers η such that the homeomorphism $h \circ \Phi^{t_0+\eta} = h \circ \Phi^{t_0} \circ \Phi^\eta$ has a periodic orbit. □

Proof of corollary 6.3 assuming theorem 6.1. Consider the set $\mathcal{U}$ of all the elements of $\mathrm{Diff}^{r,+}_{\mathrm{Leb}}(\mathbb{A})$ that have at least one hyperbolic periodic orbit. Note that $\mathcal{U}$ is an open subset of $\mathrm{Diff}^{r,+}_{\mathrm{Leb}}(\mathbb{A})$ (since hyperbolic periodic orbits are persistent) for the compact-open toplogy, and so for the Whitney topology. Consider an element h of $\mathrm{Diff}^{r,+}_{\mathrm{Leb}}(\mathbb{A})$, and an open neighbourhood $\mathcal{V}$ of h in $\mathrm{Diff}^{r,+}_{\mathrm{Leb}}(\mathbb{A})$, for the Whitney topology. There exists $h' \in \mathcal{V}$ that has a periodic orbit: this follows from theorem 6.1 since there exists a smooth vector field X of $\mathbb{A}$ with compact support such that the time-one map Φ^1 has a non-zero rotation number. We can perturb h' in order to get a diffeomorphism $h'' \in \mathcal{V}$ that has a hyperbolic periodic orbit. This proves that $\mathcal{U}$ is dense in $\mathrm{Diff}^{r,+}_{\mathrm{Leb}}(\mathbb{A})$ for the Whitney topology, hence also for the compact-open topology. □

6.2 Idea of the proof

Let us explain briefly the idea of the proof of theorem 6.1, as developed in paragraph 6.5. We suppose h is an irrational pseudo-rotation, with angle α (otherwise there is nothing to prove). There are two disjoint cases. Either the rotation sets of the homeomorphisms $h \circ \Phi^t$ "explode" (that is, there exists arbitrarily small values of t for which the rotation set of $h \circ \Phi^t$ contains numbers arbitrarily far from α), or the rotation set of $h \circ \Phi^t$ is uniformly bounded for t close to 0. Surprisingly, the first case is the easiest: because of the lower semi-continuity property of the rotation set (see below), the rotation set of $h \circ \Phi^t$ must also contain some numbers arbitrarily close to α; so we can apply Poincaré-Birkhoff-Franks's theorem to get a periodic orbit of rotation number close to α. In the second case, we use proposition 5.1 that allows us to suppose that the horizontal displacement of h is bounded. Approximating the maps Φ^t by compactly supported maps and using a continuity property (see below), we see that the "morphism property" holds: $\rho(\mathrm{Leb}, \tilde{h} \circ \tilde{\Phi}^t) = \rho(\mathrm{Leb}, \tilde{h}) + \rho(\mathrm{Leb}, \tilde{\Phi}^t)$. When $\alpha + t$ is rational, this again gives rise to periodic orbits.

6.3 The compactly supported case

Before addressing the general issue, it is useful to deal with a restricted problem. *We first prove theorem 6.1 assuming that the vector field X is compactly supported in $\mathbb{A}$.* The following argument is essentially due to J. Franks ([11]).

Proof (compactly supported case). If h is not an irrational pseudo-rotation, then it has a periodic orbit (proposition 0.2), hence we can take $\theta = 0$, and there is nothing to prove. So from now on we assume that h is an irrational pseudo-rotation.

We fix a lift $\widetilde{h}$ of h. We denote by α the rotation number of $\widetilde{h}$. Since h is a pseudo-rotation, its rotation set is certainly bounded. Thus we can apply proposition 5.1[4]: by performing a change of coordinates given by a homeomorphism g, we may assume that the horizontal displacement function of $\widetilde{h}$ is bounded. Note that the change of coordinates does not affect the fact that the flow $(\Phi^t)_{t\in\mathbb{R}}$ is compactly supported[5].

We now deal with $\widetilde{h}$ and $\widetilde{\Phi}^t$ having bounded (integrable) horizontal displacement functions. Thus we have the morphism property (proposition 1.3):

$$\rho(\mathrm{Leb}, \widetilde{h} \circ \widetilde{\Phi}^t) = \rho(\mathrm{Leb}, \widetilde{h}) + \rho(\mathrm{Leb}, \widetilde{\Phi}^t) = \alpha + t.\rho(\mathrm{Leb}, \widetilde{\Phi}^1).$$

By hypothesis, $\rho(\mathrm{Leb}, \widetilde{\Phi}^1)$ is non null, so there exists arbitrarily small values of t such that the mean rotation number $\rho(\mathrm{Leb}, \widetilde{h} \circ \widetilde{\Phi}^t)$ is rational. For any such value, $h \circ \Phi^t$ is not an irrational pseudo-rotation, so it must have periodic orbits according to proposition 0.2. This solves the compactly supported case. □

6.4 Some continuity results by P. Le Calvez

We need some more tools before coping with the general case. Le Calvez has proved the following continuity property for the rotation number of the Lebesgue probability measure (see [23, theorem 2]). Remember that $\mathrm{Rot}_{\mathrm{Fix}}(\widetilde{h})$ denotes the set of rotation numbers of the fixed points of $\widetilde{h}$ (see section 1.2).

Theorem 6.4 (P. Le Calvez, continuity of the mean rotation number). *Consider a sequence $(h_n)_{n\in\mathbb{N}}$ of $\mathrm{Homeo}^+_{\mathrm{Leb}}(\mathbb{A})$ converging towards some homeomorphism $h \in \mathrm{Homeo}^+_{\mathrm{Leb}}(\mathbb{A})$ for the compact-open topology. Consider also a sequence $(\widetilde{h}_n)$ of lifts converging towards a lift $\widetilde{h}$ of h.*

If the sets $\mathrm{Rot}_{\mathrm{Fix}}(\widetilde{h}_n)$ are uniformly bounded, then the set $\mathrm{Rot}_{\mathrm{Fix}}(\widetilde{h})$ is bounded and the sequence of rotation numbers $\rho(\mathrm{Leb}, \widetilde{h}_n)$ converges towards the rotation number $\rho(\mathrm{Leb}, \widetilde{h})$.

[4] Note that this is the easy case of the proposition, as explained at the end of paragraph 5.1.

[5] The conjugated flow is not smooth anymore, which will not do any harm.

Actually, the proof of the above theorem uses another continuity property proved in the same paper (see [23, proposition 3]). We consider a sequence of lifts $(\widetilde{h}_n)$ converging towards a lift $\widetilde{h}$ as in the previous statement.

Theorem 6.5 (P. Le Calvez, lower semi-continuity of the rotation set). *If the closures of the rotation sets* $\mathrm{Rot}(\widetilde{h}_n)$ *converge towards some interval* $[a,b] \subset [-\infty,+\infty]$*, then the closure of the rotation set* $\mathrm{Rot}(\widetilde{h})$ *is contained in* $[a,b]$.

There are some easy remarks in this footnote[6].

6.5 The general case

We now cope with the general case, without assuming that X is compactly supported.

Proof (general case). As before, it suffices to consider an irrational pseudo-rotation. We use the notations introduced for the compactly supported case. We consider two disjoint subcases.

First subcase: the rotation set of $\widetilde{h} \circ \widetilde{\Phi}^t$ is not uniformly bounded for t close to 0. More precisely, there exists a sequence of numbers $t_n \to 0$ such that the set $\mathrm{Rot}(\widetilde{h} \circ \widetilde{\Phi}^{t_n})$ contains a number β_n with $\beta_n \to +\infty$ or $\beta_n \to -\infty$.

According to the lower semi-continuity of the rotation set (theorem 6.5), there must exist another sequence $(\alpha_n) \to \alpha$ such that $\alpha_n \in \mathrm{Rot}(\widetilde{h} \circ \widetilde{\Phi}^{t_n})$. For each value of n, choose a rational number α'_n strictly between α_n and β_n, in such a way that the sequence (α'_n) tends to α. Then, Franks' version of Poincaré-Birkhoff theorem (first part of theorem 2.3) provides for each n a periodic orbit for the map $h \circ \Phi^{t_n}$ with rotation number α'_n, and we are done.

Second case: the rotation set of $\widetilde{h} \circ \widetilde{\Phi}^t$ is uniformly bounded for t close to 0. We now assume that there exists a bound $M > 0$ and an angle t_0 such that

$$\mathrm{Rot}(\widetilde{h} \circ \widetilde{\Phi}^t) \subset [-M, M]$$

for all $t \in [-t_0, t_0]$. In particular, the rotation numbers of all the fixed points of $h \circ \Phi^t$ are included in $[-M, M]$ for every $t \in [-t_0, t_0]$.

Let $(\varphi_s)_{s \in [1,+\infty]}$ be a continuous family of smooth functions from $\mathbb{A} = \mathbb{S}^1 \times \mathbb{R}$ to $[0,1]$ such that

1. φ_s is equal to 1 on the compact annulus $\mathbb{S}^1 \times [-s, s]$;

2. φ_s is equal to 0 outside the compact annulus $\mathbb{S}^1 \times [-2s, 2s]$.

[6] In particular, the closure of the conjugacy class of an irrational pseudo-rotation with angle α contains only pseudo-rotations with angle α. However, the closure of the conjugacy class of some homeomorphism whose rotation set is not reduced to a point may contain some homeomorphism whose rotation set is smaller. For instance, since $\mathbb{A}$ is open, it is easy to build an example where each homeomorphism $\widetilde{h}_n$ has rotation set equal to $[0,1]$ and h has a rotation set reduced to $\{0\}$. This shows that the rotation set is not upper semi-continuous.

Theorem 6.5 is false if one does not assume that the h_n's preserve the Lebesgue probability measure. Note that in the context of the compact annulus, the rotation set is always upper semi-continuous (even without assuming that the Lebesgue measure is preserved).

For each value of s we consider the vector field $X_s = \varphi_s.X$. We denote by $(\Phi_s^t)_{t\in\mathbb{R}}$ the corresponding flow. The lifted flow (generated by the pullback on $\widetilde{\mathbb{A}}$ of the vector field X_s) is denoted by $(\widetilde{\Phi}_s^t)_{t\in\mathbb{R}}$. For each finite value of s, the flow $(\Phi_s^t)_{t\in\mathbb{R}}$ is compactly supported; whereas it coincides with the original flow $(\Phi^t)_{t\in\mathbb{R}}$ for $s = +\infty$.

Since we have slowed down the flow, we have the following easy but crucial property: *for any positive t, for any point $\widetilde{x} \in \widetilde{\mathbb{A}}$ and any $s \geq 1$ there exists a time t' between 0 and t such that $\widetilde{\Phi}^{t'}(\widetilde{x}) = \widetilde{\Phi}_s^t(\widetilde{x})$.* In particular, for every $t \in [-t_0, t_0]$ and every $s \geq 1$, each rotation number of some fixed point of $h \circ \Phi_s^t$ is equal to a rotation number of some fixed point of $h \circ \Phi^{t'}$ for $t' \in [-t_0, t_0]$. So the sets $\mathrm{Rot}_{\mathrm{Fix}}(\widetilde{h} \circ \widetilde{\Phi}_s^t)$ of rotation numbers of the fixed points are all included in $[-M, M]$.

With this property we can apply the continuity theorem 6.4. We get for each $t \in [-t_0, t_0]$,

$$\lim_{t\to+\infty} \rho(\mathrm{Leb}, \widetilde{h} \circ \widetilde{\Phi}_s^t) = \rho(\mathrm{Leb}, \widetilde{h} \circ \widetilde{\Phi}^t).$$

Moreover, from the compactly supported case we know that the morphism property holds for the compactly supported flow $(\Phi_s^t)_{t\in\mathbb{R}}$:

$$\rho(\mathrm{Leb}, \widetilde{h} \circ \widetilde{\Phi}_s^t) = \rho(\mathrm{Leb}, \widetilde{h}) + \rho(\mathrm{Leb}, \widetilde{\Phi}_s^t).$$

When s tends towards $+\infty$ we get

$$\rho(\mathrm{Leb}, \widetilde{h} \circ \widetilde{\Phi}^t) = \rho(\mathrm{Leb}, \widetilde{h}) + \rho(\mathrm{Leb}, \widetilde{\Phi}^t) = \alpha + t.\rho(\mathrm{Leb}, \widetilde{\Phi}^1).$$

Thus in that subcase the morphism property also holds for the flow $(\Phi^t)_{t\in\mathbb{R}}$, and we conclude as in the compactly supported case. □

Remark

1. Note that the proof provides a periodic orbit whose rotation number is close to the rotation number α of the irrational pseudo-rotation h. However, in the case where the (R_θ)'s are the Euclidean rotations, we do not know if there must be a periodic orbit of rotation number $\alpha + \theta$ every time this number is rational. This is linked to the morphism property, see question 5.3.
2. One sees on examples (that may even be conjugate to an irrational rotation) that the two cases of the proof may actually occur.

A A hairy example

In this appendix we describe an example that shows why proposition 3.1 is sharp.

The left part of figure A shows the dynamics of a conservative homeomorphism h of the disc. The disc is foliated by circles, apart from the central set which is the union of a sequence $(I_k)_{k\in\mathbb{Z}}$ of segments whose length tends to zero, having a single point N in common. The homeomorphism fixes N and sends I_k to I_{k+1}. Each circle is invariant, and the homeomorphism acts as a non-trivial rotation. Note that the rotation number must tends to zero when the circles get closer to the hairy set.

Next we see this disc as the upper half-sphere, and extend h to the lower half-sphere by a rotation. Thus we get a conservative homeomorphism (again called h) of the sphere, with the two poles N and S as the only fixed points. We see h as an element of $\mathrm{Homeo}^+_{\mathrm{Leb}}(\mathbb{A})$

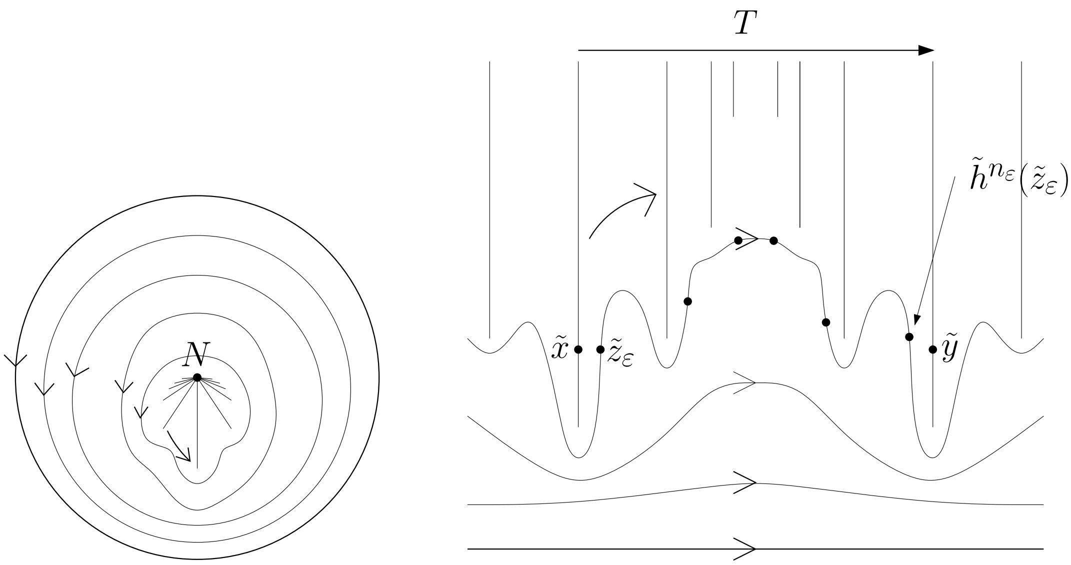

Figure 1: The hairy example

with no fixed point. Then h has a lift $\widetilde{h}$ whose rotation set is equal to $]0, \alpha]$ for some α. In particular, the points on the hairs are not recurrent, which explains why zero does not belong to the rotation set. Furthermore, one can prove that the homeomorphism $\widetilde{h}$ is not conjugate to a translation. Indeed, consider a point $\widetilde{x}$ that projects on a hair, and $\widetilde{y} = T(\widetilde{x})$ (see the right part of the figure). Then the couple $(\widetilde{x}, \widetilde{y})$ is *singular*: there exists points $\widetilde{z}_\varepsilon$ arbitrarily near $\widetilde{x}$, and arbitrarily large integers n_ε such that $\widetilde{h}^{n_\varepsilon}(\widetilde{z}_\varepsilon)$ is arbitrarily near $\widetilde{y}$. This is a dynamical feature that distinguishes $\widetilde{h}$ from a translation.

However there exists some essential topological lines γ in $\mathbb{A}$ that are disjoint from their image $h(\gamma)$ (take the projection of a vertical line in the right part of the figure). In general, when the rotation set is supposed to be included in the open interval $]0, 1[$, we do not know if there always exist such a line. The line translation theorem 0.1 only works under the stronger assumption that the *closure* of the rotation set is included in $]0, 1[$.

As explained in paragraph 2.3, we do not know either if a homeomorphism h must have a fixed point when some lift $\widetilde{h}$ has a rotation set equal to $[0, \alpha]$ for some positive α.

B Conservative version of Schoenflies theorem

Theorem B.1. *Let α_1, α_2 be two simple arcs in the sphere $\mathbb{S}^2$, and fix a homeomorphism φ from α_1 onto α_2. Then there exists an orientation-preserving homeomorphism Φ of $\mathbb{S}^2$ which is an extension of φ. In addition to this, if the Lebesgue measure of α_1, α_2 is zero, then Φ can be chosen so that it preserves the Lebesgue measure.*

In this paper we use the theorem to straighten some essential topological lines Γ of the annulus $\mathbb{A} = \mathbb{S}^1 \times \mathbb{R}$. However there is a slight difficulty coming from the fact that we have to deal with topological lines whose Lebesgue measure is not zero. We indicate here how to by-pass the problem. The topological line Γ comes with a homeomorphism h of the annulus such that $\Gamma \cap h(\Gamma) = \emptyset$. We choose a neighbourhood G of Γ such that $G \cap h(G) = \emptyset$. Then one can find an essential topological line Γ', included in G, whose

Lebesgue measure is zero (for example, Γ' can be piecewise affine). We replace Γ by Γ' before applying theorem B.1.

Idea of the proof. We only indicate how to get the second part from the first one. Let Φ be an orientation-preserving homeomorphism that is an extension of φ. Let m denotes the image of the Lebesgue measure under Φ. The measure m has the following property: it is positive on each open set, it has no atom, it gives measure zero to the arc α_2. According to a theorem of Oxtoby and Ulam ([13]), there exists an orientation-preserving homeomorphism Ψ, that is the identity on α_2, and that sends the measure m on the Lebesgue measure (actually, their theorem is stated on the square, but we can cut the sphere along the arc α_2 to match this setting). The homeomorphism $\Psi \circ \Phi$ preserves the Lebesgue measure and is still an extension of φ. □

References

[1] Anosov, Dmitri and Katok, Anatole. New examples in smooth ergodic theory. Ergodic diffeomorphisms. *Transactions of the Moscow Mathematical Society* **23** (1970), 1–35.

[2] Béguin, François; Le Roux, Frédéric. Ensemble oscillant d'un homomorphisme de Brouwer, homomorphismes de Reeb. *Bull. Soc. Math. France 131* **2** (2003), 149–210.

[3] Béguin, François ; Crovisier, Sylvain ; Le Roux, Frédéric and Patou, Alice. Pseudo-rotations of the closed annulus: variation on a theorem of J. Kwapisz. Nonlinearity 17 (2004), no. 4, 1427–1453.

[4] Birkhoff, George D. *Collected papers*, Vol II, Amer. Math. Soc., New York City, 1950.

[5] Cairns, Stewart S. An elementary proof of the Jordan-Schoenflies theorem. *Proc. Amer. Math. Soc.* **2** (1951), 860–867.

[6] Fathi, Albert and Herman, Michael. Existence de diffomorphismes minimaux. Dynamical systems, Vol. I Warsaw. *Astérisque* **49** Soc. Math. France, Paris (1977), 37–59.

[7] Fayad, Bassam and Katok, Anatole. Constructions in elliptic dynamics. *Ergod. Th. Dyn. Sys.* **24** (2004), no. 5, 1477–1520.

[8] Fayad, Bassam ; Krikorian, Raphael and Vivier, Thérèse. In preparation.

[9] Fayad, Bassam and Saprykina, Maria. Weak mixing diffeomorphisms on the disc and the annulus with arbitrary Liouvillean rotation number on the boundary. Preprint.

[10] Franks, John. Generalizations of the Poincaré-Birkhoff theorem. *Ann. of Math.* (2) **128** (1988), no. 1, 139–151.

[11] Franks, John. Rotation numbers for area preserving homeomorphisms of the open annulus. *Dynamical systems and related topics* (Nagoya, 1990), 123–127, *Adv. Ser. Dynam. Systems* **9**, World Sci. Publishing, River Edge, NJ (1991).

[12] Franks, John. Area preserving homeomorphisms of open surfaces of genus zero. *New York J. Math.* **2** (1996), 1–19.

[13] Goffman, Casper and Pedrick, George. A proof of the homeomorphism of Lebesgue-Stieltjes measure with Lebesgue measure. *Proc. Amer. Math. Soc.* **52** (1975), 196–198.

[14] Guillou, Lucien. Théorème de translation plane de Brouwer et généralisations du théorème de Poincaré-Birkhoff. *Topology* **33** (1994), no 2, 331–351.

[15] Guillou, Lucien. Unpublished text.

[16] Hall, Glen and Turpin, Mark. Robustness of periodic point free maps of the annulus. *Topology Appl.* **69** (1996), no. 3, 211–215.

[17] Handel, Michael. A pathological area preserving C^∞ diffeomorphism of the plane. *Proc. Amer. Math. Soc.* **86** (1982), no. 1, 163–168.

[18] Herman, Michael. Construction of some curious diffeomorphisms of the Riemann sphere. *J. London Math. Soc.* (2) **34** (1986), no. 2, 375–384.

[19] Herman, Michael. Some open problems in dynamical systems. *Proceedings of the International Congress of Mathematicians*, Vol. II (Berlin, 1998). Doc. Math. 1998, Extra Vol. II, 797–808.

[20] Kerékjártó, Béla. Sur le groupe des transformations topologiques du plan. *Ann. S.N.S. Pisa*, t. **II**, Ser. 3 (1934), 393–400.

[21] Kerékjártó, Béla. *Vorlesungen über Topologie (I), Flächentopologie.* Springer, Berlin, 1923.

[22] Kwapisz, Jaroslaw. A priori degeneracy of one-dimensional rotation sets for periodic point free torus maps. *Trans. Amer. Math. Soc.* **354** (2002), no 7, 2865–2895.

[23] Le Calvez, Patrice. Rotation numbers in the infinite annulus. *Proc. Amer. Math. Soc.* **129** (2001), no. 11, 3221–3230.

[24] Le Calvez, Patrice. Une version feuilletée quivariante du thorme de translation de Brouwer. Preprint, 2004.

[25] Le Roux, Frédéric. Étude topologique de l'espace des homéomorphismes de Brouwer. PhD thesis, Université Joseph Fourrier, Grenoble, 1997.

[26] Le Roux, Frédéric. Homéomorphismes de surfaces - Théorèmes de la fleur de Leau-Fatou et de la variété stable. *Astérisque* **292** (2004).

[27] Misiurewicz, Michal and Ziemian, Krystyna. Rotation sets for maps of tori. *J. London. Math Soc.* (2) **40** (1989), no.3, 490–506.

[28] Poincaré, Henri. Mémoire sur les courbes définies par une équation différentielle. *J. de Math Pures Appl.* Série III **7** (1881), 375–422, **8** (1882), 251–296, *J. de Math Pures Appl.* Série IV **1** (1885), 167–244, **2** (1886), 151–217.

Also in *Œuvres de Henri Poincaré*, tome I, Gauthier Villars, Paris (1928), 3–44, 44–84, 90–158 and 167–222.

[29] Sauzet, Alain. *Application des décompositions libres à l'étude des homéomorphismes de surfaces.* PhD thesis, Université Paris Nord, 2001.