Modes propres de l'opérateur $-\Delta$ avec conditions aux limites de Dirichlet

On cherche (λ, u) solution de (u non nul)

$$(E_D) \begin{cases} -\Delta u = \lambda u & \text{dans } \Omega \subset \mathbb{R}^d \\ u_{|\partial\Omega} = 0 \end{cases}$$

Théorème 24 (Formule de Green) *Soit Ω un ouvert borné connexe de $\mathbb{R}^d$ de frontière Γ, $\mathcal{C}^1$ par morceaux. Alors, $\forall u \in H^2(\Omega)$, $\forall v \in H^1(\Omega)$,*

$$-\int_\Omega (\Delta u) v dx = \sum_{i=1}^d \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} dx - \int_\Gamma \frac{\partial u}{\partial n} v d\sigma. \tag{1.4.38}$$

$$\int_\Omega -\Delta u \cdot u = \int_\Omega \nabla u \cdot \nabla u - \int_{\partial\Omega} \frac{\partial u}{\partial n} \overbrace{u}^{=0} = \|\nabla u\|^2_{L^2}$$

et

$$\int \lambda u \, u = \lambda \|u\|^2_{L^2} \quad \Rightarrow \quad \lambda \|u\|^2_{L^2} = \|\nabla u\|^2_{L^2}$$

donc $\lambda \geqslant 0$

Trouver v tq

$$(S) \begin{cases} -\Delta v + \alpha v = f & \text{dans } \Omega \\ v_{|\partial\Omega} = g \end{cases}$$

Si il existe (λ, u) sol. de (E_D) alors $\lambda \geqslant 0$ et $u \neq 0$ et $\begin{cases} -\Delta u = \lambda u & \text{dans } \Omega \\ u_{|\partial\Omega} = 0 \end{cases}$

On pose $\alpha = -\lambda$ et on suppose qu'il existe v sol de (S). On pose $w = v + \beta u$ $\beta \in \mathbb{R}^*$

alors

$$(-\Delta + \alpha I) w = (-\Delta - \mu I)(v + \beta u) = \overbrace{-\Delta v - \mu v}^{= f} - \beta \overbrace{(-\Delta u - \mu u)}^{= 0}$$
$$= f$$

et

$$w_{|\partial\Omega} = (v + \beta u)_{|\partial\Omega} = \overbrace{v_{|\partial\Omega}}^{= g} + \beta \overbrace{u_{|\partial\Omega}}^{= 0} = g$$

$\Rightarrow$ $w = v + \beta u$ est aussi solution de (S) !

Calculs explicites en dimension 1

$u \neq 0$

Les modes propres (μ, u) du problème aux limites

$$\begin{cases} u''(t) = \mu\, u(t) & \forall t \in [0,1] \quad (1) \\ u(0) = u(1) = 0 & \quad (2) \end{cases}$$

Sont $(\mu_k, u_k)_{k \in \mathbb{N}^*}$ avec $\mu_k = -(k\pi)^2$ et $u_k(x) = c \sin(k\pi t)$
$c \in \mathbb{R}^*$

Preuve

Rappels

https://math.libretexts.org/Bookshelves/Differential_Equations/Introduction_to_Partial_Differential_Equations_(Herman)

Constant Coefficient Equations

The simplest second order differential equations are those with constant coefficients. The general form for a homogeneous constant coefficient second order linear differential equation is given as

$$ay''(x) + by'(x) + cy(x) = 0, \tag{12.2.5}$$

where a, b, and c are constants.

Solutions to (12.2.5) are obtained by making a guess of $y(x) = e^{rx}$. Inserting this guess into (12.2.5) leads to the characteristic equation

$$ar^2 + br + c = 0. \tag{12.2.6}$$

The roots of this equation, r_1, r_2, in turn lead to three types of solutions depending upon the nature of the roots.

Classification of Roots of the Characteristic Equation for Second Order Constant Coefficient ODEs

1. **Real, distinct roots** r_1, r_2. In this case the solutions corresponding to each root are linearly independent. Therefore, the general solution is simply $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$.
2. **Real, equal roots** $r_1 = r_2 = r$. In this case the solutions corresponding to each root are linearly dependent. To find a second linearly independent solution, one uses the Method of Reduction of Order. This gives the second solution as xe^{rx}. Therefore, the general solution is found as $y(x) = (c_1 + c_2 x)\, e^{rx}$.
3. **Complex conjugate roots** $r_1, r_2 = \alpha \pm i\beta$. In this case the solutions corresponding to each root are linearly independent. Making use of Euler's identity, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, these complex exponentials can be rewritten in terms of trigonometric functions. Namely, one has that $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$ are two linearly independent solutions. Therefore, the general solution becomes $y(x) = e^{\alpha x}(c_1\cos(\beta x) + c_2 \sin(\beta x))$.

l'équation caractéristique de (1) est $r^2 - \mu = 0$ (E)

- si $\mu = 0$ alors (E) a pour racine double $r = 0$ et les solutions de (1) s'écrivent sous la forme

$$u(t) = (c_1 + c_2 t)\, e^{rt} = c_1 + c_2 t$$

- si $\mu = \xi^2 > 0$ alors (E) a pour racines réelles et distinctes ξ et $-\xi$ et donc

$$u(t) = c_1 e^{\xi t} + c_2 e^{-\xi t}$$

- si $\mu = -\xi^2 < 0$ alors (E) a deux racines complexes conjuguées $i\xi$ et $-i\xi$ et donc

$$u(t) = c_1 \cos(\xi t) + c_2 \sin(\xi t)$$

Dans les 3 cas $c_1 \in \mathbb{R}$, $c_2 \in \mathbb{R}$ $c_1 c_2 \neq 0$ (pour sol. non nulle)

On va maintenant étudier les solutions u non nulles satisfaisant les conditions aux limites (2) ie $u(0) = u(1) = 1$, dans les trois cas

- $\mu = 0$ ie $u(t) = c_1 + c_2 t$

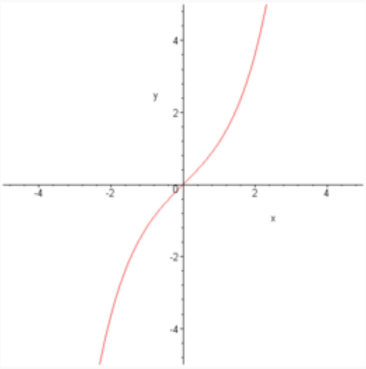
$$\begin{cases} u(0) = 0 \\ u(1) = 0 \end{cases} \Leftrightarrow \begin{cases} c_1 = 0 \\ c_1 + c_2 = 0 \end{cases} \Leftrightarrow c_1 = c_2 = 0$$

Il n'y pas de solution non nulle

- $\mu = \xi^2 > 0$ i.e. $u(t) = c_1 e^{\xi t} + c_2 e^{-\xi t}$

$$\begin{cases} u(0) = 0 \\ u(1) = 0 \end{cases} \Leftrightarrow \begin{cases} c_1 + c_2 = 0 \\ c_1 e^{\xi} + c_2 e^{-\xi} = 0 \end{cases} \Leftrightarrow \begin{cases} c_1 = -c_2 \\ c_1 \underbrace{(e^{\xi} - e^{-\xi})}_{= 2\sinh(\xi)} = 0 \end{cases} \Leftrightarrow \begin{cases} c_1 = -c_2 \\ c_1 \sinh(\xi) = 0 \end{cases}$$

Fonction sinus hyperbolique



Graphe de la fonction sinus hyperbolique sur une partie de $\mathbb{R}$.

On a donc $c_1 = 0$ ou $\sinh(\xi) = 0$

or $c_1 = 0 \Rightarrow c_2 = 0$

et $\sinh(\xi) = 0 \Leftrightarrow \xi = 0$ impossible car $\xi > 0$

Il n'y pas de solution non nulle

- $\mu = -\xi^2 < 0$ ie $u(t) = c_1 \cos(\xi t) + c_2 \sin(\xi t)$

$$\begin{cases} u(0) = 0 \\ u(1) = 0 \end{cases} \Leftrightarrow \begin{cases} c_1 = 0 \\ c_1 \cos(\xi) + c_2 \sin(\xi) = 0 \end{cases} \Rightarrow c_2 \sin(\xi) = 0 \Leftrightarrow \begin{cases} c_2 = 0 \\ \text{ou} \\ \sin(\xi) = 0 \end{cases}$$

$$\underset{\xi \neq 0}{\sin(\xi) = 0} \Leftrightarrow \xi = k\pi, \quad k \in \mathbb{Z}^*$$

Les modes propres sont donc $(\mu_k, u_k)_{k \in \mathbb{N}^*}$

avec $\mu_k = -(k\pi)^2$ et $u_k(t) = c \sin(k\pi t)$

$v \neq 0$

Les modes propres (λ, w) du problème aux limites

$$\begin{cases} \text{Trouver } (\lambda, w) \text{ tq} & \\ w''(x) = \lambda w(x) \quad \forall x \in [a,b] & (3) \\ (a) = (b) = 0 & (4) \end{cases}$$

Sont $(\lambda_k, w_k)_{k \in \mathbb{N}^*}$ avec $\lambda_k = -\left(\frac{k\pi}{b-a}\right)^2$ et $w_k(x) = c \sin\left(k\pi \frac{x-a}{b-a}\right)$

Preuve

On note $g(x) = \frac{x-a}{b-a}$ le changement de variable affine et $w = v \circ g$

avec $g(a) = 0$ et $g(b) = 0$

On a

$$w'(x) = g'(x)\, v'(g(x)) = \frac{1}{b-a} v'(g(x))$$

et

$$w''(x) = \left(\frac{1}{b-a}\right)^2 v''(g(x))$$

On note $t = g(x)$ et on a

$$w''(x) = \lambda w(x), \ \forall x \in [a,b] \iff \frac{1}{(b-a)^2} v''(t) = \lambda v(t), \quad \forall t \in [0,1]$$

$$\iff v''(t) = \lambda (b-a)^2 v(t) \quad \forall t \in [0,1]$$

Résoudre (3)-(4) est équivalent à résoudre

$$\begin{cases} v''(t) = \underbrace{\lambda (b-a)^2}_{=\mu} v(t) & \forall t \in [0,1] \\ v(0) = v(1) = 0 \end{cases} \iff (1)\text{-}(2)$$

avec $w = v \circ g$

Les modes propres de (1)-(2) étant

$(\mu_k, v_k)_{k \in \mathbb{N}^*}$ avec $\mu_k = -(k\pi)^2$ et $v_k(x) = c \sin(k\pi t)$

$c \in \mathbb{R}^*$

On en déduit que les modes propres de (3)-(4) sont

$(\lambda_k, w_k)_{k \in \mathbb{N}^*}$ avec $\lambda_k = \frac{\mu_k}{(b-a)^2} = -\left(\frac{k\pi}{b-a}\right)^2$

et $w_k(x) = v_k\left(\frac{x-a}{b-a}\right) = c \sin\left(k\pi \frac{x-a}{b-a}\right) \quad c \in \mathbb{R}^*$