Convolution in perfect Lie groups

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Abstract

Let G be a connected perfect real Lie group G. We show that there exists $\alpha < \dim G$ and $p \in \mathbb{N}^*$ such that if μ is a compactly supported α-Frostman Borel measure on G, then the p-th convolution power μ^{*p} is absolutely continuous with respect to the Haar measure on G, with arbitrarily smooth density. As an application, we obtain that if $A \subset G$ is a Borel set with Hausdorff dimension at least α, then the p-fold product set A^p contains a non-empty open set.

1 Introduction

The original motivation for this article was to study the Hausdorff dimension of product sets in perfect Lie groups. Recall that a group G is called perfect if it is equal to its commutator group, i.e. $G = [G, G]$. Given a Borel subset A of G, we want to obtain non-trivial lower bounds on the Hausdorff dimension of the product set A^p of all elements that can be written as products of p elements of A. We prove the following.

Theorem 1.1 (Hausdorff dimension of product sets)**.** *Let G be a connected real perfect Lie group of dimension d. Then there exists $\alpha < d$ such that if A is any Borel subset of G of Hausdorff dimension at least α, then the product set A^p contains a non-empty open subset of G. In particular, G has no proper measurable dense subgroup of Hausdorff dimension larger than α.*

It was proved in [6] that in a connected simple Lie group of dimension d, there is no measurable dense subgroup of Hausdorff dimension strictly between 0 and d. The above result allows to conclude that in fact, any proper measurable dense subgroup of a simple Lie group must have Hausdorff dimension zero. This comes in contrast with what happens in nilpotent Lie groups, where it was shown by the second author in [5] that there always exist proper measurable dense subgroups of arbitrary Hausdorff dimension in $[0, \dim G]$.

It turns out that the techniques used for the proof of Theorem 1.1 also have interesting implications about regularity of convolution products of compactly supported measures on a connected perfect Lie group G. We say that a Borel measure μ on a Lie group G is *α-Frostman* if there exists $C \geq 0$ such that for any ball B_r of radius r in G, $\mu(B_r) \leq C r^{\alpha}$. Our main result is as follows.

Theorem 1.2 (Convolution of Frostman measures)**.** *Let G be a connected perfect Lie group. Given a positive integer k, there exists $\kappa > 0$ and $p \in \mathbb{N}^*$ such that if $\mu_1, \ldots, \mu_p$ are compactly supported $(d-\kappa)$-Frostman measure on G, then* *the k-fold convolution $\mu_p * \cdots * \mu_1$ is absolutely continuous with respect to Haar measure, with density k times differentiable.*

It is worth noting that this result already has a non-trivial consequence about convolution of compactly supported continuous functions on a perfect Lie group.

Corollary 1.3 (Convolution of continuous functions)**.** *Let G be a connected perfect Lie group and for $k \in \mathbb{N}$, denote by $C_c^k(G)$ the space of compactly supported C^k functions on G. Given a positive integer k, there exists $p \in \mathbb{N}^*$ such that*

$$C_c^0(G)^{*p} \subset C_c^k(G).$$

The main ingredient in the proof of Theorem 1.2 above originates in the work of Bourgain and his coauthors in [1, 2, 3], very recently developed in a more general context by Boutonnet, Ioana and Salehi-Golsefidi [4]. This ingredient is a bound on the coefficients of the regular representation of a connected perfect Lie group G. This bound will imply a convolution inequality for Frostman measures, which is the subject of Section 2. Then, we need to iterate this convolution inequality. For that, the main idea is to identify G locally with a torus of equal dimension, so as to be able to use Fourier analysis in this setting. This is done in Section 3. Finally, in Section 4, we derive Theorems 1.2 and 1.1.

2 A local convolution inequality for perfect groups

2.1 Bounds on coefficients of the regular representation

Let G be a Lie group of dimension d and denote by $\mathfrak{g}$ the Lie algebra of G. We fix a Euclidean structure on $\mathfrak{g}$ and consider a left-invariant Riemannian metric on G. We will assume that the exponential map induces a diffeomorphism from $B_{\mathfrak{g}}(0,1)$ to a neighborhood Ω of the identity in G; this can always be ensured by changing the norm on $\mathfrak{g}$ by a multiplicative constant.

Fix a function $\varphi \in C_c^\infty(\mathfrak{g})$ supported on $B_{\mathfrak{g}}(0,1)$. For $1 > \delta > 0$, and $x \in G$ we let

$$P_\delta(x) = \begin{cases} c_\delta \varphi(\frac{\log x}{\delta}) & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega, \end{cases}$$

where c_δ is chosen so that $\int_G P_\delta = 1$.

We denote by $T_g : L^2(G) \to L^2(G)$ the left regular representation of G, defined by $T_g F(x) = F(g^{-1}x)$.

The next theorem is essentially due to Boutonnet, Ioana and Salehi-Golsefidi [4], generalizing work of Bourgain and Gamburd [2] in the case $G = SU(d)$. We will explain in this paragraph the minor modifications needed to get the version stated here. Recall that a group is G is called perfect if it is equal to its commutator subgroup, i.e. $G = [G, G]$.

Theorem 2.1. *[4, Formula (∗), page 35]. Let G be a connected perfect Lie group and B a compact subset of G. There exists positive constants a, b, κ such that for every $F_1, F_2 \in L^2(B)$, for all $\delta \in (0,1)$,*

$$\int_G |\langle T_g F_1, F_2\rangle|^2 \, \mathrm{d}g \le a\|P_\delta * F_1\|_2^2 \|P_\delta * F_2\|_2^2 + b\delta^\kappa \|F_1\|_2^2 \|F_2\|_2^2.$$

Remark 1. For the application presented in this article, we will only need the bound on the L^1-norm of the coefficient

$$\int_G |\langle T_g F_1, F_2 \rangle| \, \mathrm{d}g \leq a \|P_\delta * F_1\|_2 \|P_\delta * F_2\|_2 + b\delta^\kappa \|F_1\|_2 \|F_2\|_2,$$

but the proof yields the stronger inequality on the L^2-norm of the coefficient, and we do not know of an easier proof of this slightly weaker inequality.

In the proof described in [4], the assumption that G is simple is only used through the fact that there exists a one-dimensional compact subgroup H with Lie algebra $\mathfrak{h}$ such that the ideal generated by $\mathfrak{h}$ in $\mathfrak{g} = \operatorname{Lie} G$ is equal to $\mathfrak{g}$. We show now that this property is still valid as long as the Lie algebra $\mathfrak{g}$ is perfect. More precisely, we have the following.

Proposition 2.2 (Existence of a spanning torus)**.** *Let $\mathfrak{g}$ be a perfect real Lie algebra and $X \in \mathfrak{g}$ with non-zero projection in all the simple factors of $\mathfrak{g}$. Then the ideal $\mathcal{I}$ generated by X is equal to $\mathfrak{g}$.*

The proof will use the following lemma.

Lemma 2.3. *Let $\mathfrak{g}$ be a perfect real Lie algebra, then $\mathfrak{g}$ decomposes as a semi-direct product $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{u}$, where $\mathfrak{s}$ is a semisimple subalgebra and $\mathfrak{u}$ is a nilpotent ideal in $\mathfrak{g}$.*

Proof. By Lévi's theorem [7, Corollary 1, page 49] we know that $\mathfrak{g}$ is the semi-direct product of its solvable radical $\mathfrak{r}$ and a semisimple Lie algebra $\mathfrak{s}$. All we need to check is that $\mathfrak{r}$ is in fact nilpotent. We will show that for all $X \in \mathfrak{r}$, $\operatorname{ad} X$ is nilpotent; by Engel's theorem, this will prove the lemma.

By Lie's theorem applied to $\operatorname{ad} \mathfrak{r}$, the set of $X \in \mathfrak{r}$ such that $\operatorname{ad} X$ is nilpotent is an ideal of $\mathfrak{r}$. So we only need to check that $\mathfrak{r}$ is spanned by ad-nilpotent elements. Since $\mathfrak{g}$ is perfect, we must have

$$\mathfrak{r} = [\mathfrak{g}, \mathfrak{r}],$$

and $\mathfrak{r}$ is spanned by all subspaces $[H, \mathfrak{r}]$, $H \in \mathfrak{g}$. Now for any $H \in \mathfrak{g}$, the Lie algebra $\mathbb{C}H \oplus \mathfrak{r}$ is solvable and therefore, by Lie's theorem again, any element in $[H, \mathfrak{r}]$ is ad-nilpotent. This is what we wanted. $\square$

Proof of Proposition 2.2. By the above lemma, the Lie algebra $\mathfrak{g}$ is the semi-direct product of a nilpotent ideal $\mathfrak{u}$ with a semisimple algebra $\mathfrak{s}$. Let $\mathcal{I}$ be the ideal generated by X. Since the projection of X on any simple factor is non-zero, we must have $\mathfrak{g} = \mathcal{I} + \mathfrak{u}$. Let $\mathcal{J} = \mathcal{I} \cap \mathfrak{u}$. Since $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, we have

$$\mathfrak{u} = [\mathcal{I}, \mathfrak{u}] + [\mathfrak{u}, \mathfrak{u}] = \mathcal{J} + [\mathfrak{u}, \mathfrak{u}].$$

By induction on the nilpotency step of $\mathfrak{u}$, this readily implies that $\mathcal{J} = \mathfrak{u}$, and thus, $\mathcal{I} = \mathfrak{g}$. $\square$

With Proposition 2.2 at hand, the rest of the proof of Theorem 2.1 will follow the lines of [4, Section 5].

Proof of Theorem 2.1. Let G be a connected perfect real Lie group. By Proposition 2.2, we may fix a compact torus H in G with Lie algebra $\mathfrak{h}$ such that the smallest ideal containing $\mathfrak{h}$ is equal to $\mathfrak{g}$. For some integer n, consider the map

$$\begin{array}{cccc} \pi^{(n)}: & (G\times H\times H)^n & \longrightarrow & G \\ & (g_1,t_1,s_1,\dots,g_n,t_n,s_n) & \longmapsto & \prod_{i=1}^n t_ig_it_i^{-1}s_ig_i^{-1}s_i^{-1} \end{array}$$

By the above assumption on $\mathfrak{h}$, we may find an integer n and elements $g_1,\dots,g_n$ in G such that $\mathfrak{g}=\bigoplus_{i=1}^n(\mathrm{Ad}\,g_i)\mathfrak{h}$. Fixing such a choice of g_i, the tangent map at the identity of $(t_1,s_1,\dots,t_n,s_n)\mapsto\pi^{(n)}(g_1,t_1,s_1,\dots,g_n,t_n,s_n)$ is onto. This implies that there exists a point at which the tangent map of $\pi^{(n)}$ has maximal rank $\dim G$, and since $\pi^{(n)}$ is analytic, this must hold in fact for almost every point in $(G\times H\times H)^n$. We can then reproduce the proof of [4, $(*)$, page 35] and find that for some integer k ($k=8d$) and some $\tau>0$, if B is a fixed compact subset of G, then, for all $F_1,F_2\in L^2(B)$ and all $\delta>0$,

$$\int_G|\langle F_1,T_gF_2\rangle|^2\,\mathrm{d}g\ll\|F_1\|_2^2(\delta^\tau\|F_2\|_2^2+\|F_2\|_2^{2-\frac{1}{k}}\|P_\delta*F_2\|_2^{\frac{1}{k}})$$

where the implied constant in the $\ll$ notation depends only on B. In particular, if $\|P_{\sqrt{\delta}}*F_2\|_2\ll\sqrt{\delta}\|F_2\|_2$, then the above applied at scale $\sqrt{\delta}$ yields

$$\int_G|\langle F_1,T_gF_2\rangle|^2\,\mathrm{d}g\ll\delta^{\frac{\tau}{2}}\|F_1\|_2^2\|F_2\|_2^2 \tag{1}$$

provided $\tau\leq\frac{1}{k}$. By symmetry, (1) also holds if $\|P_{\sqrt{\delta}}*F_1\|_2\ll\sqrt{\delta}\|F_1\|_2$.
Now write $F_i=P_\delta*F_i+G_i$, $i=1,2$.
It is easy to check that

$$\|P_{\sqrt{\delta}}-P_{\sqrt{\delta}}*P_\delta\|_1\ll\sqrt{\delta}, \tag{2}$$

so that for each i, $\|P_{\sqrt{\delta}}*G_i\|_2\ll\sqrt{\delta}\|F_i\|_2$. Moreover,

$$\begin{aligned}|\langle F_1,T_gF_2\rangle|^2\ll{}&|\langle G_1,T_gG_2\rangle|^2+|\langle P_\delta*F_1,T_gG_2\rangle|^2+|\langle G_1,T_gP_\delta*F_2\rangle|^2\\&+|\langle P_\delta*F_1,T_gP_\delta*F_2\rangle|^2\end{aligned}$$

Applying (1) to the three first terms, and bounding trivially the last one, we obtain what we wanted to show:

$$\int_G|\langle F_1,T_gF_2\rangle|^2\,\mathrm{d}g\ll\|P_\delta*F_1\|_2^2\|P_\delta*F_2\|_2^2+\delta^\kappa\|F_1\|_2^2\|F_2\|_2^2,$$

with $\kappa=\frac{\tau}{2}$. □

2.2 Application to Frostman measures

It was observed by Bourgain in [1] that bounds of the type of Theorem 2.1 above imply a convolution inequality for Frostman measures on G. Recall that given $\alpha\in[0,d]$, a Borel measure μ on G is called *α-Frostman* if there exists C such that for all $r>0$ and all x in G, $\mu(B(x,r))\leq Cr^\alpha$. In our context, we obtain the following, which is a refined version of the mixing inequality proved in [4, Proposition 5.6].

Corollary 2.4 (Convolution inequality)**.** *Let G be a connected perfect Lie group. There exist a neighborhood U of the identity in G and $\kappa' > 0$ such that if μ is any $(d-\kappa')$-Frostman probability measure supported on V and $F \in L^2(U)$, then, for all $\delta > 0$,*

$$\|\mu * F\|_2 \ll \|P_\delta * F\|_2 + \delta^{\kappa'}\|F\|_2,$$

where the implied constant in $\ll$ depends only on μ.

Proof. Throughout the proof, C will denote a large constant, depending only on μ, and whose value may increase from one line to the other.

Let $\tilde{\mu} \in \mathcal{P}(G)$ be the pushforward of the measure $\mu \otimes \mu$ on $G \times G$ under the map $(x, y) \mapsto xy^{-1}$. Of course $\tilde{\mu}$ is $(d-\kappa')$-Frostman. Without loss of generality, we may assume $\|F\|_2 = 1$. We use the Paley-Littlewood decomposition of F, as introduced in [2], $F = \sum_{i\geq 0} \Delta_i F$, where

$$\Delta_0 F = P_{\frac{1}{2}} * F \text{ and } \forall i \geq 1,\ \Delta_i F = P_{2^{-i-1}} * F - P_{2^{-i}} * F.$$

We refer to [4, Theorem 5.1] for a nice exposition of the basic properties of this decomposition. In addition, we note that there exists an absolute constant C such that for all integer i, the function $\Delta_i F$ is $C2^i$-Lipschitz.

We assume for simplicity that $\delta = 2^{-i_0}$ for some $i_0 \geq 1$. We have

$$\mu * F = \mu * P_\delta * F + \sum_{i\geq i_0} \mu * \Delta_i F$$

Since μ is a probability measure, the first term is bounded in L^2-norm by

$$\|\mu * P_\delta * F\|_2 \leq \|P_\delta * F\|_2.$$

On the other hand, for $i \geq i_0$, we write

$$\begin{aligned}\|\mu * \Delta_i F\|_2^2 &= \int_G \langle T_g \Delta_i F, \Delta_i F\rangle \,\mathrm{d}\tilde{\mu}(g)\\ &\leq \int_G |\langle T_g \Delta_i F, \Delta_i F\rangle| \,\mathrm{d}\tilde{\mu}(g)\end{aligned}$$

Since $\Delta_i F$ is $C2^i$-Lipschitz, so is the map $g \mapsto |\langle T_g \Delta_i F, \Delta_i F\rangle|$ – adjusting possibly the value of the constant C – so we have, for all $g \in G$,

$$|\langle T_g \Delta_i F, \Delta_i F\rangle| \leq C2^{-i} + \frac{1}{|B(g, 2^{-2i})|}\int_{B(g,2^{-2i})} |\langle T_g \Delta_i F, \Delta_i F\rangle| \,\mathrm{d}g.$$

Therefore

$$\int_G |\langle T_g \Delta_i F, \Delta_i F\rangle| \,\mathrm{d}\tilde{\mu}(g) \leq C2^{-i} + \int_G |\langle T_g \Delta_i F, \Delta_i F\rangle| \,\mathrm{d}(\tilde{\mu} * P_{2^{-2i}})(g)$$

Since $\tilde{\mu}$ is $(d-\kappa')$-Frostman the function $\tilde{\mu} * P_{2^{-2i}}$ is bounded in L^∞-norm by $C2^{2\kappa' i}$, so that

$$\int_G |\langle T_g \Delta_i F, \Delta_i F\rangle| \,\mathrm{d}\tilde{\mu}(g) \ll 2^{-i} + 2^{2\kappa' i}\int_G |\langle T_g \Delta_i F, \Delta_i F\rangle| \,\mathrm{d}g$$

Thus, by Theorem 2.1 applied to $\Delta_i F$ at scale $2^{-i/2}$,

$$\|\mu * \Delta_i F\|_2^2 \ll 2^{-i} + 2^{2\kappa' i}(\|P_{2^{-i/2}} * \Delta_i F\|_2^2 + 2^{-\frac{i\kappa}{2}}).$$

To conclude, note that (2) implies that $\|P_{2^{-i/2}} * \Delta_i F\|_2 \leq C2^{-i/4}$, so that

$$\|\mu * \Delta_i F\|_2^2 \ll 2^{-i} + 2^{(2\kappa' - \frac{1}{2})i} + 2^{(2\kappa' - \frac{\kappa}{2})i}.$$

Assuming $\kappa' < \min(\frac{1}{8}, \frac{\kappa}{8})$, we can sum over i to get

$$\sum_{2^i \geq \delta} \|\mu * \Delta_i F\|_2 \ll \delta^{\kappa'}$$

and in turn,

$$\|\mu * F\|_2 \ll \|P_\delta * F\|_2 + \delta^{\kappa'}.$$

□

3 Local action on the torus and Fourier series

Let G be a Lie group and $\mathbb{T}^d$ be the d-dimensional torus. In this section, we discuss local actions of G on a small open set V of $\mathbb{T}^d$. The reader may think of V as a small open set in an arbitrary manifold that we embedded in $\mathbb{T}^d$ via a local chart, in order to be able to use Fourier series to study the action of G on elements of $L^2(V)$.

Definition 3.1. Given neighborhoods U and V of the identity in G and $\mathbb{T}^d$, respectively, we define a *local action* of G on $\mathbb{T}^d$ as a smooth map

$$\begin{array}{ccc} U \times V & \to & \mathbb{T}^d \\ (g, x) & \mapsto & g \cdot x \end{array}$$

satisfying, for all $g, h \in U$ and $x \in V$, $g(g^{-1}x) = x$ if $g^{-1}x \in V$ and $g(hx) = (gh)x$ if $hx \in V$ and $gh \in U$.

Suppose now that we are given a local action of a Lie group G on $\mathbb{T}^d$ defined on $U \times V$. If f is a real-valued function supported on V that is integrable for the Haar measure $\mathrm{d}x$ on $\mathbb{T}^d$ and μ any Borel probability measure supported on U, we define the convolution product $\mu * f$ to be the integrable function on $\mathbb{T}^d$ defined by the formula, for almost every $x \in \mathbb{T}^d$,

$$\mu * f(x) = \int_G f(g^{-1}x)\, \mathrm{d}\mu(g). \tag{3}$$

3.1 Convolution and high-frequency harmonics

As before, suppose we have a local action $U \times V \to \mathbb{T}^d$ of a Lie group G on the torus $\mathbb{T}^d$. Given $f \in L^2(V)$, we can decompose it into its Fourier series

$$f = \sum_{n \in \mathbb{Z}^d} \hat{f}(n) e_n$$

where $\hat{f}(n) = \int_{\mathbb{T}^d} f(x) e^{-2i\pi n \cdot x} dx$ and $e_n(x) = e^{2i\pi n \cdot x}$ for $x \in \mathbb{T}^d$. If μ is a probability measure supported on U, we want to control the Fourier expansion of $\mu * f$ in terms of that of f.

Notation. Given a positive integer N, and a continuous function f on $\mathbb{T}^d$, we write $\operatorname{Supp}\hat{f} \cap N = \emptyset$ if $\hat{f}(n) = 0$ whenever $|n| \leq N$.

Lemma 3.2. *Let G be a Lie group and $U \times V \to \mathbb{T}^d$ a local action of G on $\mathbb{T}^d$. Fix $\theta \in C_c^\infty(V)$. Given positive constants τ, A, there exists $C = C_{\tau,A}$ such that the following holds.*
Suppose μ is a Borel probability measure with $\operatorname{Supp}\mu \subset U$ and $f \in L^2(\mathbb{T}^d)$ is such that $\operatorname{Supp}\hat{f} \cap N = \emptyset$. Then, one can decompose

$$\mu * (\theta f) = f_0 + f_1 \quad \textit{with} \begin{cases} \|f_0\|_2 \leq C_{\tau,A} N^{-A} \|f\|_2 \\ \operatorname{Supp}\hat{f}_1 \cap N^{1-\tau} = \emptyset \end{cases}$$

Proof. Let $\psi = \mu * (\theta f)$. The Fourier coefficient $\widehat{\psi}(n)$ is given by

$$\begin{aligned}\widehat{\psi}(n) &= \int_G \int_{\mathbb{T}^d} e^{-2i\pi n\cdot x} \theta(g^{-1}x) f(g^{-1}x) dx \, \mathrm{d}\mu(g) \\ &= \int_G \int_{\mathbb{T}^d} e^{-2i\pi n\cdot (gx)} \theta(x) f(x) |g'(x)| dx \, \mathrm{d}\mu(g)\end{aligned}$$

where $|g'(x)|$ denotes the Jacobian of the diffeomorphism $x \mapsto g(x)$, defined for $x \in V$ and $g \in U$. Fixing $g \in U$, let $\eta_{n,g}(x) = e^{-2i\pi n\cdot(gx)}|g'(x)|\theta(x)$, so that the inner integral above rewrites

$$\int_{\mathbb{T}^d} e^{-2i\pi n\cdot (gx)} \theta(x) f(x) |g'(x)| dx = \int_{\mathbb{T}^d} \eta_{n,g}(x) f(x) dx$$

If Δ denotes the usual Laplacian on $\mathbb{T}^d$, we can bound, for p any positive integer, if $n \neq 0$,

$$\|\Delta^p \eta_{n,g}\|_\infty \leq C_p |n|^{2p},$$

where the constant C_p depends only on p. On the other hand, if $h_p \in L^2(\mathbb{T}^d)$ is such that $\Delta^p h_p = f$ and h_p has zero average, we have

$$\|h_p\|_2 \leq N^{-2p} \|f\|_2$$

because $\hat{f}(n) = 0$ whenever $|n| \leq N$. Therefore, integrating by parts p times, we find

$$\begin{aligned}\int_{\mathbb{T}^d} \eta_{n,g}(x) f(x) dx &= \int_{\mathbb{T}^d} \Delta^p \eta_{n,g}(x) h_p(x) dx \\ &\leq C_p \left(\frac{|n|}{N}\right)^{2p} \|f\|_2\end{aligned}$$

Integrating over $\mathrm{d}\mu(g)$, we find, provided $|n| \leq N^{1-\tau}$,

$$|\widehat{\psi}(n)| \leq C_p N^{-2p\tau} \|f\|_2.$$

Let $f_0 = \sum_{|n| \leq N^{1-\tau}} \widehat{\psi}(n) e_n$ and $f_1 = \sum_{|n| > N^{1-\tau}} \widehat{\psi}(n) e_n$. One has of course $\operatorname{Supp}\hat{f}_1 \cap N^{1-\tau} = \emptyset$, and by the above estimates, provided p has been chosen so that $2p\tau - d > A$, one also has

$$\|f_0\|_2 \leq C_{\tau,A} N^{-A} \|f\|_2.$$

This proves the lemma. □

3.2 The regular local action

We define now the *regular local action* of a d-dimensional Lie group G on the torus $\mathbb{T}^d$. It corresponds to the action of G on itself by left-multiplication, in a neighborhood of the identity. More precisely, let U be a neighborhood of the identity in G such that $\iota = \exp_{\mathbb{T}^d} \circ \exp_G^{-1}$ induces a diffeomorphism from U to a neighborhood V of the identity in $\mathbb{T}^d$. We define the regular local action of G on $\mathbb{T}^d$ by

$$\forall g \in U,\ x \in V,\ g \cdot x = \iota(g\iota^{-1}(x)).$$

Recall that P_δ denotes the family of $C_c^\infty(G)$ approximations to the identity defined at the beginning of Section 2. The lemma below relates the Fourier expansion of a function f on $\mathbb{T}^d$ with the size of $P_\delta * f$, which is the regularization of f at scale δ. Here the convolution product refers to the local action of G on $\mathbb{T}^d$, as defined in (3).

Lemma 3.3. *Let G be a d-dimensional Lie group and consider the regular local action $\cdot : U \times V \to \mathbb{T}^d$ as defined above. Fix $\theta \in C_c^\infty(V)$. For all positive integer k, there exists a constant C_k such that for all $N \geq 1$, for all $\delta > \frac{1}{N}$, for all $f \in L^2(\mathbb{T}^d)$ with* $\operatorname{Supp} \hat{f} \cap N = \emptyset$,

$$\|P_\delta * (\theta f)\|_2 \leq C_k \delta^{-d} \left(\frac{1}{\delta N}\right)^{2k} \|f\|_2$$

Proof. The proof is similar to that of Lemma 3.2 and will follow from integrating by parts. We have

$$P_\delta * (\theta f)(x) = \int_G \theta(g^{-1}x) f(g^{-1}x) P_\delta(g)\, \mathrm{d}g$$

Use change of variables $u = gx$, and let $J_x(u)$ be the Jacobian of $u \mapsto xu^{-1}$, to write

$$\begin{aligned} P_\delta * (\theta f)(x) &= \int_{\mathbb{T}^d} f(u)\theta(u) P_\delta(g(u)) J_x(u)\, \mathrm{d}u \\ &= \int_{\mathbb{T}^d} f(u)\psi(u)\, \mathrm{d}u, \end{aligned}$$

where $\psi(u) = \theta(u) P_\delta(g(u)) J_x(u)$. For any positive integer k, we can bound

$$\Delta^k \psi(u) \leq C_k \delta^{-2k-d}$$

and on the other side, if η_k is a mean-zero function on $\mathbb{T}^d$ such that $\Delta^k \eta_k = f$,

$$\|\eta_k\|_2 \leq N^{-2k} \|f\|_2$$

because $\operatorname{Supp} \hat{f} \cap N = \emptyset$. Therefore,

$$\left| \int_{\mathbb{T}^d} f(u)\psi(u) du \right| \leq C_k \delta^{-2k-d} N^{-2k} \|f\|_2.$$

$\square$

3.3 Iterating the convolution inequality

Given a Frostman measure μ on a connected Lie group G, the goal of the rest of this section is to apply Corollary 2.4 inductively. More precisely, we prove Proposition 3.4 below, which will allow us to show in Section 4 that large powers of Frostman measures become arbitrarily smooth.

Proposition 3.4. *Given a connected perfect Lie group G, we consider the regular local action $U \times V \to \mathbb{T}^d$, as defined in the previous paragraph. There exists $\kappa > 0$ such that for all $A > 0$, there exist $U' \subset U$, $V' \subset V$ and $p \in \mathbb{N}^*$ such that for $\theta \in C_c^\infty(V')$, if $\mu_1, ..., \mu_p$ are $(d-\kappa)$-Frostman probability measures supported on U', then there exists $C > 0$ such that, for all $f \in L^2(\mathbb{T}^d)$ with $\operatorname{Supp} \hat{f} \cap N = \emptyset$,*

$$\|\mu_p * \cdots * \mu_1 * (\theta f)\|_2 \leq CN^{-A}\|f\|_2.$$

We start by a lemma which will be an easy consequence of Corollary 2.4 and Lemma 3.3.

Lemma 3.5. *Let G be a perfect Lie group and $U \times V \to \mathbb{T}^d$ the regular local action. There exists $\kappa > 0$ such that if $\theta \in C_c^\infty(V)$, for all $(d-\kappa)$-Frostman probability measure μ, there exists $C \geq 0$ such that for all $f \in L^2(\mathbb{T}^d)$ such that $\operatorname{Supp} \hat{f} \cap N = \emptyset$,*

$$\|\mu * (\theta f)\|_2 \leq CN^{-\kappa}\|f\|_2.$$

Proof. Since $\iota = \exp_{\mathbb{T}^d} \circ \exp_G^{-1}$ induces a diffeomorphism from U to V, any function ψ on V corresponds to a function $F = \psi \circ \iota$ on U. Moreover, for some constant C depending only on the ambiant group G,

$$\frac{1}{C}\|\psi\|_2 \leq \|F\|_2 \leq C\|\psi\|_2.$$

We apply this observation to $\psi = \theta f$, and denote $F = (\theta f) \circ \iota$. Thus, by Corollary 2.4, for all $\delta > 0$,

$$\|\mu * (\theta f)\|_2 \ll \|\mu * F\|_2 \ll \|P_\delta * F\|_2 + \delta^{\kappa'}\|F\|_2.$$

Using Lemma 3.3, we can bound the first term on the right

$$\|P_\delta * F\|_2 \leq C_k \delta^{-d} \left(\frac{1}{\delta N}\right)^{2k} \|f\|_2$$

so that choosing $\delta = N^{-\frac{1}{2}}$ and $k \geq \frac{d}{2} + \kappa'$ we obtain

$$\|\mu * (\theta f)\|_2 \ll N^{-\kappa'}\|f\|_2.$$

This proves the lemma, with $\kappa = \kappa'$ given by Corollary 2.4. □

We are now ready to prove Proposition 3.4.

Proof of Proposition 3.4. Again, we assume without loss of generality that $\|f\|_2 = 1$. Let $\kappa > 0$ be as in Lemma 3.5. Choose $\tau > 0$ and $p \in \mathbb{N}^*$ such that

$$\kappa + (1-\tau)\kappa + \cdots + (1-\tau)^{p-1}\kappa > A.$$

Choose the neighborhoods U', V' so that $U'^p * V' \subset \frac{1}{2}V$, and for each $i = 1, ..., p$, let $\theta_i \in C_c^\infty(U)$ be such that

$$\theta_i(x) = \begin{cases} 1 & \text{if } x \in U'^i * V' \\ 0 & \text{if } x \notin U'^{i+1} * V' \end{cases}$$

By Lemma 3.2 we can write

$$\mu_1 * (\theta f) = f_0 + f_1 \quad \text{with} \begin{cases} \|f_0\|_2 \ll N^{-A} \\ \operatorname{Supp} \hat{f}_1 \cap N^{1-\tau} = \emptyset \end{cases}$$

By Lemma 3.5, $\|f_1\|_2 \ll N^{-\tau}$. Note that since $\mu_1 * (\theta f)$ is supported on $U' * V'$, we have $\mu_1 * (\theta f) = \theta_1(\mu_1 * (\theta f)) = \theta_1 f_0 + \theta_1 f_1$. So we apply Lemma 3.2 again, this time to $\theta_1 f_1$, at scale $N^{1-\tau}$ to write

$$\mu_2 * (\theta_1 f_1) = f_{10} + f_{11} \quad \text{with} \begin{cases} \|f_{10}\|_2 \ll N^{-A} \\ \operatorname{Supp} \widehat{f_{11}} \cap N^{(1-\tau)^2} = \emptyset \end{cases}$$

This time, by Lemma 3.5, $\|f_{11}\|_2 \leq N^{-\kappa-\kappa(1-\tau)}$. We apply this procedure p times to obtain in the end

$$\|\mu_p * \cdots * \mu_1 * f\|_2 \ll N^{-A} + N^{-\kappa-\kappa(1-\tau)-\cdots-\kappa(1-\tau)^{p-1}} \ll N^{-A}.$$

□

4 Applications

4.1 Differentiability of convolution powers

We now prove the main result of this note.

Theorem 4.1 (Convolution of Frostman measures)**.** *Let G be a connected perfect Lie group. Given a positive integer k, there exists $\kappa > 0$ and $p \in \mathbb{N}^*$ such that if $\mu_1, \ldots, \mu_p$ are compactly supported $(d-\kappa)$-Frostman measure on G, then the k-fold convolution $\mu_p * \cdots * \mu_1$ is absolutely continuous with respect to Haar measure, with density k times differentiable.*

Proof. Let $\kappa > 0$ and $p \in \mathbb{N}^*$ be such that Proposition 3.4 holds for $A > 2k + d$. Assume first that each μ_i is supported in the neighborhood U' given by Proposition 3.4. We identify the measure $\nu = \mu_p * \cdots * \mu_1$ with its pushforward under the diffeomorphism $\iota = \exp_{\mathbb{T}^d} \circ \exp_G^{-1}$. Recall that Δ denotes the usual Laplacian on $\mathbb{T}^d$. It suffices to show that the distribution $\Delta^k \nu$ lies in $L^2(\mathbb{T}^d)$. Since $\nu * \Delta^k P_\delta = \Delta^k \nu * P_\delta$ converges to $\Delta^k \nu$ when δ goes to zero, we just need to bound the L^2-norm $\|\nu * \Delta^k P_\delta\|_2$ independently of δ. Fix $\theta \in C_c(V')$ equal to 1 on a neighborhood of 0 and, for $\delta > 0$ small enough, decompose $P_\delta = \theta P_\delta$ into its Fourier series:

$$P_\delta = \sum_{n \in \mathbb{Z}^d} \hat{P}_\delta(n) e_n = \sum_{n \in \mathbb{Z}^d} \hat{P}_\delta(n) \theta e_n,$$

and

$$\Delta^k P_\delta = \sum_{n \in \mathbb{Z}^d} |n|^{2k} \hat{P}_\delta(n) \theta e_n.$$

By Proposition 3.4 applied to e_n, we get

$$\|\nu * (\theta e_n)\|_2 \ll |n|^{-A}$$

and therefore, since $|\hat{P}_\delta(n)| \leq 1$,

$$\|\nu * \Delta^k P_\delta\|_2 \ll \sum_{n \in \mathbb{Z}^d} |n|^{-A+2k} \leq C_A < \infty.$$

This shows that $\Delta^k \nu$ lies in L^2 and therefore that ν is k times differentiable.

The general case reduces to the case where the μ_i are supported in U'. Indeed, if B is a large symmetric compact set containing $\bigcup_{i=1}^p \operatorname{Supp} \mu_i$, cover B by a finite union of translates $g_j U''$, where U'' is chosen so that for all g in the product set B^p, $gU''g^{-1} \subset U'$. Taking a partition of unity associated to this cover, we see that we may assume that for each i, μ_i is supported on some $g_i U''$. Let

$$\nu_i = \delta_{(g_i \dots g_1)^{-1}} * \mu_i * \delta_{g_{i-1} \dots g_1},$$

so that each ν_i is supported on U' and

$$\mu_p * \dots * \mu_1 = \delta_{g_p \dots g_1} * \nu_p * \dots * \nu_1.$$

Since $\nu_p * \dots * \nu_1$ is k-times differentiable by the first part of the proof, so is $\mu_p * \dots * \mu_1$, and we are done. $\square$

As a consequence of the above theorem, we have the following.

Corollary 4.2 (Convolution of continuous functions)**.** *Let G be a connected perfect Lie group. Given a positive integer k, there exists $p \in \mathbb{N}^*$ such that*

$$C_c(G)^{*p} \subset C_c^k(G).$$

Example 1. Note that the assumption that G is perfect is necessary. Indeed, the function on $\mathbb{T}^1$ defined by

$$F(x) = \sum_{n \geq 1} \frac{1}{n^2} e^{2^n i \pi x}$$

is continuous, but for all p, F^{*p} is nowhere differentiable. This example can easily be generalized to any connected abelian Lie group.

Corollary 4.3 (Convolution of L^q-functions)**.** *Let $q > 1$ and denote by $L_c^q(G)$ the space of compactly supported L^q-functions on G. Then, given any positive integer k, there exists $p \in \mathbb{N}^*$ such that*

$$L_c^q(G)^{*p} \subset C_c^k(G).$$

Proof. By the classical Young inequalities, $L^q(G) * L^q(G) \subset L^{\frac{q}{2-q}}(G)$ so that for some integer $p \in \mathbb{N}^*$, we get

$$L^q(G)^{*p} \subset L^r(G), \text{ for some } r \geq 2.$$

This implies in particular that

$$L_c^q(G)^{*p} \subset L_c^2(G),$$

and since $L_c^2(G) * L_c^2(G) \subset C_c(G)$, the result follows from the previous corollary. $\square$

Example 2. The assumption that $q > 1$ is necessary, as is shown by the following example. Let $(B(x_n, 2^{-n}))_{n\geq 1}$ be a collection of disjoint balls included in $B_G(1,2)$, and define

$$f(x) = \begin{cases} n^{-2}2^{n\dim G} & \text{if } x \in B(x_n, 2^{-n}) \text{ for some } n \\ 0 & \text{otherwise} \end{cases}$$

Then $f \in L^1_c(G)$ but for all $p \in \mathbb{N}^*$, f^{*p} is not in $L^2(G)$. In particular, no convolution power of f can be continuous.

4.2 Products sets and Hausdorff dimension

Recall Frostman's lemma, which relates sets and Hausdorff dimension to Frostman measures.

Lemma 4.4. *Let G be a Lie group with a Riemannian metric and $A \subset G$ a measurable set. If $\dim_H A > \alpha$, then there exists an α-Frostman measure μ with compact support included in A.*

For subsets $A_1, \dots, A_p$ of a group G, we denote by $A_1A_2\cdots A_p$ the product set of all elements $g \in G$ that can be written $g = a_1a_2\dots a_p$, with each $a_i \in A_i$. The above lemma allows us to deduce the following from the results about Frostman measures of the preceding paragraph.

Theorem 4.5. *Let G be a connected perfect Lie group of dimension d. There exists $\alpha < d$ and $p \in \mathbb{N}^*$ such that for any measurable sets $A_1, \dots, A_p$ with $\dim_H A_i > \alpha$, the product set $A_1A_2\cdots A_p$ contains a non-empty open set.*

Proof. Let $\kappa > 0$ and $p \in \mathbb{N}^*$ be as in Theorem 4.1, with $k = 0$, and set $\alpha = d - \kappa$. By Frostman's lemma, we may choose for each i an α-Frostman probability measure μ_i with support included in A_i. By Theorem 4.1, the convolution product $\mu_1 * \cdots * \mu_p$ is absolutely continuous with respect to Haar measure, with continuous density. In particular, $S = \operatorname{Supp} \mu_1 * \cdots * \mu_p$ contains a non-empty open set. This proves the theorem, because S is included in the product set $A_1\cdots A_p$. □

This has the following consequence about measurable subgroups of a connected perfect group G.

Corollary 4.6. *Let G be a connected perfect Lie group of dimension d. There exists $\alpha < d$ such that G admits no proper measurable subgroup of Hausdorff dimension larger than α.*

It was proven in Saxcé [6] that a connected simple subgroup G has no dense measurable subgroup of intermediate dimension $\alpha \in (0, \dim G)$. The new information contained in the above corollary is that the only measurable subgroup of maximal Hausdorff dimension is G, so that we find:

Theorem 4.7. *Let G be a connected simple Lie group. Any proper dense measurable subgroup of G has Hausdorff dimension 0.*

We recall that such a simple Lie group G always contains many uncountable subgroups of Hausdorff dimension zero.

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