

Subgroups of fractional dimension in nilpotent or solvable Lie groups

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Abstract: We construct dense Borel measurable subgroups of Lie groups of intermediate Hausdorff dimension. In particular, we generalize the Erdős-Volkmann construction [6], showing that any nilpotent σ -compact Lie group N admits dense Borel subgroups of arbitrary dimension between 0 and $\dim N$. In algebraic groups defined over a finite extension of the rationals, using diophantine properties of algebraic numbers, we are also able to construct dense subgroups of arbitrary dimension, but the general case remains open. In particular, we raise the following question: does there exist a measurable proper subgroup of \mathbb{R} of positive Hausdorff dimension which is stable under multiplication by a transcendental number? Subgroups of nilpotent p -adic analytic groups are also discussed.

1 Introduction

In 1966, Erdős and Volkmann [6] constructed measurable additive subgroups of \mathbb{R} of arbitrary dimension between 0 and 1 (see also [7], example 12.4). This particular problem can be put in a more general setting: given a metric group G , what can be the Hausdorff dimension of a measurable subgroup of G ? In this paper, we investigate the case of nilpotent Lie groups, endowed with a left-invariant Riemannian metric. In particular, we show the following theorem.

Theorem 1.1. *If G is a nilpotent real Lie group, and Γ is a countable subgroup of G , then, for all $\alpha \in [0, \dim G]$, there exists a measurable subgroup of G containing Γ that has Hausdorff dimension α . In particular, if G has at most countably many connected components, then G admits dense measurable subgroups of arbitrary dimension.*

The construction used to prove this theorem is based on induction, and the subgroups we obtain are not very "homogeneous" in the sense that, although they have intermediate dimension, their projection on the quotients of the lower central series might not. Therefore, we investigate an alternative way to construct dense subgroups having intermediate dimension in all quotients. This alternative construction is an adaptation of the Erdős-Volkmann construction, using approximating lattices in G ; as such, it only works for nilpotent Lie groups having rational structure constants. We obtain the following proposition.

Proposition 1.2. *Let N be a connected nilpotent Lie group having rational structure constants. Denote l the nilpotency class of N , and $(C^i)_{0 \leq i \leq l}$ the lower central series of G . Then, for all $a > l$, N admits a dense measurable subgroup F such that, for all $p \in \{0, \dots, l\}$, the Hausdorff dimension of the projection of F on N/C^p is equal to $\frac{1}{a} \sum_{i=1}^p i(\dim C^{i-1}/C^i)$.*

Using elementary diophantine properties of algebraic numbers, we are also able to get a partial result for solvable groups:

Theorem 1.3. *If G is the group of real points of a solvable linear algebraic group defined over a number field, then G admits dense measurable subgroups of arbitrary Hausdorff dimension.*

Theorems 1.1 and 1.3 can be put into contrast with the main result of [11]: if G is a compact simple Lie groups, then measurable subgroups cannot have Hausdorff dimension arbitrarily close to $\dim G$ (see also [10]). This phenomenon is closely related to what happens in additive combinatorics: nilpotent groups admit approximate subgroups whereas simple algebraic groups do not (see for instance [3, 4]).

For p -adic analytic groups, the study of the Hausdorff dimension was initiated by Abercrombie [1] who, among other things, proved the analog of the Erdős-Volkmann theorem for \mathbb{Q}_p . Here, the ideas used in the real setting also apply for p -adic analytic groups, and yield the following result:

Theorem 1.4. *Let N be a p -adic analytic nilpotent group, then N admits measurable subgroups of arbitrary dimension between 0 and $\dim N$ that are dense in a neighborhood of the identity.*

Again, it should be noted that such a theorem is false without the nilpotency assumption, as one can see by studying the case of $SL_d(\mathbb{Z}_p)$ (see [11]).

The organization of the paper is as follows. After a few preliminary statements given in section 2, the first theorem is obtained by a simple induction on the dimension of the group, starting from the Erdős-Volkmann theorem. This is discussed in section 3, together with the other constructions in nilpotent groups. In section 4, we investigate subgroups of solvable groups and prove theorem 1.3. The induction we use there is very similar to the one used in the proof of theorem 1.1, but the starting point requires more effort: we need to construct additive subgroups of the Euclidean space of arbitrary dimension, which are invariant under matrix transformations with entries in a number field. This is where we use some basic diophantine properties of algebraic numbers. Finally, in section 5, we describe the p -adic setting and prove theorem 1.4.

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2 General setting, notations and preliminary lemmas

If A is a subset of a metric space G , for $s \in \mathbb{R}^+$ and $\epsilon > 0$, we define

$$\mathcal{H}_\epsilon^{(s)}(A) = \inf \left\{ \sum_i r_i^s \right\}$$

where the infimum is taken over all countable coverings of A with balls of radius at most ϵ . Then we let $\mathcal{H}^{(s)}(A) = \lim_{\epsilon \rightarrow 0} \mathcal{H}_\epsilon^{(s)}(A)$. The Hausdorff dimension of A is defined to be the nonnegative number

$$\dim_H A = \inf \{s \in \mathbb{R}_+^* \mid \mathcal{H}^{(s)}(A) = 0\}.$$

Here, G will be a real Lie group or an analytic p -adic Lie group. If G is a real Lie group, we endow it with a left-invariant Riemannian metric, whereas if it is a p -adic analytic, we will use any left-invariant metric such that the local charts from G to $\mathbb{Z}_p^{\dim G}$ are bilipschitz maps.

We will make use of the Vinogradov notation: if x and y are two quantities, we write $x \ll y$ if there exists an absolute constant C such that $x \leq Cy$, and $x \simeq y$ if $x \ll y$ if $y \ll x$.

We now recall two basic lemmas which we will use in the course of the paper. The first one will be used for the inductive constructions in nilpotent Lie groups, or Lie algebras:

Lemma 2.1. *Denote $\mathbb{K} = \mathbb{R}$ or \mathbb{Z}_p . If A is a measurable subset of \mathbb{K}^k , then*

$$\dim_H(A \times \mathbb{K}^l) = l + \dim_H A.$$

Proof. This follows from formulae 7.2 and 7.5 of [7], because the Hausdorff dimension of \mathbb{K}^l is equal to its upper Minkowski dimension, which is equal to l . \square

The second one will enable us to go easily from nilpotent Lie algebras to connected nilpotent Lie groups using the fact that the exponential map is a covering map. For completeness, we include a proof.

Lemma 2.2. *Let $p : X \rightarrow Y$ be a covering of smooth manifolds, and $A \subset X$. Then, $\dim_H A = \dim_H p(A)$.*

Proof. First, p is locally Lipschitz, so

$$\dim_H p(A) \leq \dim_H A.$$

For the converse inequality, as p is a covering map, we may choose a countable open covering $(X_i)_{i \in \mathbb{N}}$ of X such that for all i , p induce a diffeomorphism from X_i onto $p(X_i)$. Then,

$$\dim_H A = \max_i \dim_H(A \cap X_i) = \max_i \dim_H p(A \cap X_i) \geq \dim_H p(A).$$

\square

3 Nilpotent Lie groups

3.1 The Abelian case

Let us start by recalling the Erdős-Volkmann result about subgroups of \mathbb{R}^d :

Theorem 3.1 (Erdős-Volkmann, 1966). *There exist measurable additive subgroups of \mathbb{R}^d of arbitrary Hausdorff dimension in $[0, d]$.*

Using lemma 2.2, one can reformulate the Erdős-Volkmann theorem in the slightly more general setting of connected Abelian Lie groups:

Theorem 3.2. *If G is a connected Abelian Lie group of dimension d , then G admits measurable subgroups of arbitrary dimension in $[0, d]$.*

Now in order to generalize this theorem to nilpotent Lie groups having at most countably many connected components, it is convenient to start in the Lie algebra setting.

Definition 3.3. *If \mathfrak{g} is a Lie algebra, we define a \mathbb{Q} -subalgebra of \mathfrak{g} to be a \mathbb{Q} -vector subspace of \mathfrak{g} that is stable under the bracket operation.*

Definition 3.4. *Let V be a vector space. A subgroup of $GL(V)$ will be called unipotent if there exists a basis of V in which all the elements of G are upper triangular with all diagonal entries equal to 1.*

With these definitions, we have the following:

Proposition 3.5. *Let \mathfrak{n} be a nilpotent Lie algebra of dimension d , and D be a countable unipotent subgroup of $\text{Aut}(\mathfrak{n})$. For any $\alpha \in [0, \dim \mathfrak{n}]$, there exists a dense measurable \mathbb{Q} -subalgebra of \mathfrak{n} of Hausdorff dimension α that is stable under the elements of D .*

Proof. We use induction on the dimension of \mathfrak{n} .

If $d = 1$, the only unipotent element is the identity, so the result follows immediately from the Erdős-Volkmann theorem, noting that if H_α is a subgroup of dimension α , then $\bigcup_{k \in \mathbb{N}^*} \frac{1}{k} H_\alpha$ is a \mathbb{Q} -vector subspace of same dimension.

Now let \mathfrak{n} be a nilpotent Lie algebra, and suppose we know the result for nilpotent Lie algebras of dimension less than that of \mathfrak{n} . As D is unipotent, there exists a one-dimensional subspace $V \subset \mathfrak{n}$ which is fixed under D . Moreover, as the center of \mathfrak{n} is stable under the elements of D , we may assume that V is central in \mathfrak{n} .

First case: $\alpha \leq 1$.

Then, by the $d = 1$ case, we can choose a measurable dense \mathbb{Q} -subspace V_α of V of Hausdorff dimension α . Choose any dense countable \mathbb{Q} -subalgebra C_0 of \mathfrak{n} stable under D (this is easily seen to exist) and let $\mathfrak{n}_\alpha = C_0 + V_\alpha$. Then \mathfrak{n}_α is a countable union of copies of V_α , so $\dim_H \mathfrak{n}_\alpha = \alpha$; \mathfrak{n}_α is a dense measurable \mathbb{Q} -subalgebra of \mathfrak{n} ; and \mathfrak{n}_α is stable under D , because C_0 and V_α are.

Second case: $\alpha > 1$

In this case, we choose a subspace V' such that $\mathfrak{n} = V \oplus V'$. Identifying V' with \mathfrak{n}/V , and using the induction hypothesis, we can find a dense measurable \mathbb{Q} -subalgebra $V'_{\alpha-1}$ of V' of Hausdorff dimension $\alpha - 1$ which is stable under D . Let $\mathfrak{n}_\alpha = V + V'_{\alpha-1}$. This is a dense measurable \mathbb{Q} -subalgebra of \mathfrak{n} ; by lemma 2.1, it has Hausdorff dimension α ; and it is stable under D because V is stable, $V'_{\alpha-1}$ is stable modulo V , and $V \subset \mathfrak{n}_\alpha$. \square

The following observation is the tool we need to pass from nilpotent Lie algebras to nilpotent Lie groups.

Lemma 3.6. *Let N be a nilpotent Lie group and \mathfrak{n} its Lie algebra. If \mathfrak{n}_α is a \mathbb{Q} -subalgebra of \mathfrak{n} , then $N_\alpha = \exp \mathfrak{n}_\alpha = \{\exp x \mid x \in \mathfrak{n}_\alpha\}$ is a subgroup of N of Hausdorff dimension α . Moreover, if \mathfrak{n}_α is stable under a family D of automorphisms of \mathfrak{n} , then N_α is stable under the automorphisms of N whose tangent map at the identity lie in D .*

Remark: In the preceding lemma, if \mathfrak{n}_α is dense and N is connected, then N_α is dense in N .

Proof. The fact that $\exp \mathfrak{n}_\alpha$ is a subgroup follows from the Campbell-Hausdorff formula (see [5], chapter 1):

$$\exp X \exp Y = \exp(X + Y + P(X, Y))$$

where $P(X, Y)$ is a finite (because N is nilpotent) \mathbb{Q} -linear combination of multiple brackets of X and Y . And it must have dimension α , by lemma 2.2, because the exponential map is a covering map from \mathfrak{n} to the identity component of N . Finally, if φ is an automorphism of N whose tangent map $T\varphi$ lies in D , then we have, for $X \in \mathfrak{n}_\alpha$,

$$\varphi(\exp X) = \exp(T\varphi(X)) \in N_\alpha,$$

because \mathfrak{n}_α is stable under $T\varphi$. □

We can finally state and prove our theorem about measurable subgroups of nilpotent Lie groups:

Theorem 3.7. *If N is a nilpotent Lie group and D is a countable subgroup of N , then, for all $\alpha \in [0, \dim N]$, there exists a measurable subgroup of N containing D which has Hausdorff dimension α .*

Proof. Denote N^0 the identity component of N and \mathfrak{n} the Lie algebra of N . Then N acts on \mathfrak{n} by the adjoint action. If $a \in N$, we want to show that $\text{Ad } a$ is a unipotent transformation on \mathfrak{n} . For this, we show (with a slight abuse of notation) that $\text{Ad } a$ is unipotent on each $c^p(\mathfrak{n})/c^{p+1}(\mathfrak{n})$, where $c^p(\mathfrak{n})$ is the lower central series of \mathfrak{n} (defined by $c^0(\mathfrak{n}) = \mathfrak{n}$ and, for $p \geq 0$, $c^{p+1}(\mathfrak{n}) = [\mathfrak{n}, c^p(\mathfrak{n})]$). Using that $c^p(\mathfrak{n})/c^{p+1}(\mathfrak{n})$ is Abelian, we find, for $X \in c^p(\mathfrak{n})/c^{p+1}(\mathfrak{n})$,

$$\exp((\text{Ad } a - 1)X) = a \exp X a^{-1} \exp(-X) = (a, \exp X),$$

where $(x, y) = xyx^{-1}y^{-1}$ is the commutator map, and the exponential is from $c^p(\mathfrak{n})/c^{p+1}(\mathfrak{n})$ to $N/\langle \exp c^{p+1}(\mathfrak{n}) \rangle$. Thus, for $n \in \mathbb{N}$ large enough, and independent of X ,

$$\exp((\text{Ad } a - 1)^n X) = (a, (a, \dots, (a, \exp(X)) \dots)) = 1.$$

Therefore, $(\text{Ad } a - 1)^n (c^p(\mathfrak{n})/c^{p+1}(\mathfrak{n}))$ lies in the discrete set $\exp^{-1}(\{1\})$ and, by connectedness, $(\text{Ad } a - 1)^n (c^p(\mathfrak{n})/c^{p+1}(\mathfrak{n})) = \{0\}$, i.e. $\text{Ad } a$ is unipotent on $c^p(\mathfrak{n})/c^{p+1}(\mathfrak{n})$. As $(c^p(\mathfrak{n}))_{p \geq 0}$ terminates at $\{0\}$, we indeed get that $\text{Ad } a$ is unipotent on \mathfrak{n} .

By Kolchin's theorem (see [12], part I, page 5.7), the group $\text{Ad } D$ is unipotent.

As it is also countable, we may apply proposition 3.5 and get that there exists a dense measurable \mathbb{Q} -subalgebra of \mathfrak{n} of dimension α which is stable under $\text{Ad } D$. Denote by N_α its image under the exponential map; by lemma 3.6, this is a measurable subgroup of N^0 of Hausdorff dimension α , which is stable under conjugation by the elements of D . Finally, let M_α be the subgroup generated by N_α and D . Then, M_α is a measurable subgroup of N containing D . Moreover, using that N_α is stable under conjugation by elements of D , we see that any element of M_α can be written $d.n_\alpha$ where $d \in D$ and $n_\alpha \in N_\alpha$. Hence M_α is a countable union of copies of N_α , so it has Hausdorff dimension α . This is what we wanted. \square

Using that any σ -compact Lie group admits a dense countable subgroup, we get,

Corollary 3.8. *If N is a nilpotent Lie group having at most countably many connected components, then it admits dense measurable subgroups of arbitrary dimension.*

Remark: This corollary is false without the σ -compactness assumption. Indeed, let $U = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}; x \in \mathbb{R} \right\}$ endowed with the discrete topology, and consider the semi-direct product $G = U \ltimes \mathbb{R}^2$. If H is a dense subgroup of G , then, as U is discrete, H must contain a representative of each coset of G modulo \mathbb{R}^2 . But then, H contains the orbit of a nonzero vector of \mathbb{R}^2 under the whole action of U , which has dimension 1. So any dense subgroup of G has Hausdorff dimension at least 1.

3.2 Approximating lattices

The subgroups constructed in the proof of theorem 3.7, although they have an intermediate dimension in G , may not have an intermediate dimension in all the quotients of the lower central series. In fact, it is easily seen that they have intermediate dimension in at most one of the quotients.

Recall the *lower central series* of a group N is defined inductively by $C^0(N) = N$ and, for $p \geq 0$, $C^{p+1}(N) = (N, C^p(N))$, the subgroup generated by the commutators $(x, y) = xyx^{-1}y^{-1}$ with $x \in N$ and $y \in C^p(N)$. If N is nilpotent, the series terminates to $\{1\}$, and the smallest integer l such that $C^l(N) = \{1\}$ is called the *nilpotency class* of N .

It is natural to ask whether it is possible to construct more "homogeneous" intermediate dimensional subgroups. It turns out that in the case where N admits approximating lattices, we are able to mimic the Erdős-Volkmann construction and so to prove the following.

Proposition 3.9. *Let N be a connected simply connected nilpotent Lie group having rational structure constants. Denote l the nilpotency class of N . Then, for all $a > l$, N admits a dense measurable subgroup F satisfying:
For all $p \in \{0, \dots, l-1\}$, the Hausdorff dimension of the projection of F on $N/C^p(N)$ is equal to $\frac{1}{a} \sum_{i=1}^p i(\dim C^{i-1}(N)/C^i(N))$.*

Recall that if N is a connected nilpotent Lie group, N admits a lattice if and only if it has rational structure constants (see [9], theorem 2.12) this is the reason why we have to restrict attention to these groups when trying to adapt the Erdős-Volkmann proof.

Proof of proposition 3.9. Let N be a connected simply connected nilpotent Lie group having rational structure constants. Let \mathcal{B} be a basis of \mathfrak{n} in which the structure constants are rational. Let $(C^p(\mathfrak{n}))_{0 \leq p \leq l}$ be the lower central series of \mathfrak{n} , defined by $C^0(\mathfrak{n}) = \mathfrak{n}$, and, for $p \geq 0$, $C^{p+1}(\mathfrak{n}) = [\mathfrak{n}, C^p(\mathfrak{n})]$. Starting from \mathcal{B} , and using successive brackets of elements of \mathcal{B} , one easily obtains subspaces m_p such that, for each p ,

$$C^{p-1}(\mathfrak{n}) = C^p(\mathfrak{n}) \oplus m_p$$

together with a basis (e_i) of \mathfrak{n} adapted to the decomposition $\mathfrak{n} = \bigoplus m_p$ in which the structure constants of \mathfrak{n} are rational. The Campbell-Hausdorff formula (cf. [5], chapter 1) then describes the product law in $N = \mathbb{R}^d = \bigoplus \mathbb{R}e_i$:

For $x \in \mathfrak{n}$ we write $x = x_1 e_1 + \dots + x_d e_d$, and, if $\alpha \in \mathbb{N}^d$, $x^\alpha = \prod_{i=1}^d x_i^{\alpha_i}$. Denote $\deg(e_i)$ the largest integer j such that $e_i \in C^{j-1}(\mathfrak{n})$. Also denote, for $\alpha \in \mathbb{N}^d$, $d_\alpha = \sum \deg(e_i) \alpha_i$. Then, one has

$$(xy)_i = x_i + y_i + \sum C_{\alpha,\beta} x^\alpha y^\beta$$

where the $C_{\alpha,\beta}$ are rational constants, and the sum is on the indices α and β such that $d_\alpha + d_\beta \leq \deg(e_i)$, $d_\alpha \geq 1$ and $d_\beta \geq 1$. We may replace the basis (e_i) by a proportional basis $(e'_i) = (\lambda e_i)$; then, the Campbell-Hausdorff formula becomes

$$(xy)'_i = x'_i + y'_i + \sum \lambda^{|\alpha|+|\beta|-1} C_{\alpha,\beta} x'^\alpha y'^\beta$$

so that, without loss of generality, we may assume that the $C_{\alpha,\beta}$ are integers. We then have, for $n \in \mathbb{N}^*$,

$$\Gamma_n = \bigoplus \mathbb{Z} \frac{e_i}{n^{\deg(e_i)}} \text{ is a lattice of } N.$$

Now we define, for (n_k) an increasing sequence of integers:

$$F_r = \{x \in N : \forall k, d(x, \Gamma_{n_k}) \leq r n_k^{-a}\}$$

and

$$F = \bigcup_{r \in \mathbb{N}^*} F_r.$$

Of course F is a dense measurable subgroup of N . Let us compute its Hausdorff dimension. For conveniency, denote $A = \sum_p p \dim m_p$ and $B = \dim N = \sum_p \dim m_p$. First, for any ball U , $F \cap U$ is covered by at most $C n_k^{-A}$ balls of radius $r n_k^{-a}$, and therefore $\dim_H F_r \leq \frac{A}{a}$. Using that F is a countable union of F_r , we get

$$\dim_H F \leq \frac{A}{a}.$$

To prove the reverse inequality, we show $\dim_H F_1 \cap U \geq \frac{A}{a}$ for some open ball U containing 1. For this we use the mass distribution principle as described in Falconer [7], chapter 4:

Starting from a total mass 1 (for U), we equidistribute it at each step on the balls of lower scale which appear in the construction of F_1 . This gives us a sequence of measures μ_n . Then we choose a converging subsequence of (μ_n) to obtain a Borel measure μ supported on $F_1 \cap U$ and we check that the measure μ is s -Hölder for $s \geq \frac{A}{a}$.

The μ -measure of a n_k^{-a} -ball B_k appearing at stage k is bounded above by the inverse of the number of such balls, which is

$$\frac{n_1^A}{\text{Vol } U} \cdot \frac{n_2^A}{n_1^{aB}} \cdots \frac{n_k^A}{n_{k-1}^{aB}} = \frac{n_k^A}{\text{Vol } U} \prod_{i=1}^{k-1} n_i^{A-aB}.$$

Hence,

$$\mu(B_k) \leq \frac{\text{Vol } U}{n_k^A} \prod_{i=1}^{k-1} n_i^{aB-A}.$$

Now, let B_ρ a ball of radius $\rho > 0$. Choose k such that $n_k \leq \rho < n_{k-1}$. Then B_ρ meets at most $(\rho^B n_k^A)$ many balls of scale k , so

$$\mu(B_\rho) \leq (\rho^B n_k^A) \frac{\text{Vol } U}{n_k^A} \prod_{i=1}^{k-1} n_i^{aB-A} = \rho^B \text{Vol } U \prod_{i=1}^{k-1} n_i^{aB-A}.$$

At the same time, B_ρ meets at most one ball of scale $k-1$, hence

$$\mu(B_\rho) \leq \frac{\text{Vol } U}{n_{k-1}^A} \prod_{i=1}^{k-2} n_i^{aB-A}.$$

Taking the geometric mean of ratio $s \in (0, 1)$, we find,

$$\begin{aligned} \mu(B_\rho) &\leq \left(\rho^B \text{Vol } U \prod_{i=1}^{k-1} n_i^{aB-A} \right)^s \cdot \left(\frac{\text{Vol } U}{n_{k-1}^A} \prod_{i=1}^{k-2} n_i^{aB-A} \right)^{1-s} \\ &= \rho^{sB} n_{k-1}^{A-saB} \prod_{i=1}^{k-2} n_i^{A-aB}. \end{aligned}$$

Therefore, if we take (n_k) to increase sufficiently rapidly, we have, for $s > \frac{A}{aB}$,

$$\mu(B_\rho) \ll \rho^{sB}.$$

The measure μ has support in F and is (sB) -Hölder for all $s > \frac{A}{a}$, thus,

$$\dim_H F \geq \frac{A}{a},$$

which is exactly the content of the proposition, in case $p = l$. Exactly the same computation can be made for any p in $\{0, \dots, l\}$ in order to prove the proposition, we leave the details to the reader. \square

4 Solvable Lie groups

4.1 Additive subgroups of \mathbb{R}^d stable under matrices with coefficients in a number field

We explain here how the diophantine properties of algebraic numbers can be used to construct subgroups of \mathbb{R}^d of arbitrary dimension that are stable under the action of matrices with entries in a number field. The fundamental observation is the following lemma, which should be classical, but for which we could not find a suitable reference.

Lemma 4.1. *If α is a real algebraic number of degree p , then there exists a constant $c = c(\alpha) > 0$ such that for all $n \in \mathbb{N}^*$,*

$$\min\{|a_0 + a_1\alpha + \cdots + a_{p-1}\alpha^{p-1}|; |a_i| \in \{-n, \dots, n\}\} \geq \frac{c}{n^{p-1}}.$$

Proof. Without loss of generality, we may assume that α is an algebraic integer. Otherwise, we replace α by $k\alpha$ where $k \in \mathbb{N}^*$ is appropriately chosen. Denote $\beta = a_0 + \cdots + a_{p-1}\alpha^{p-1}$. Then β is an algebraic integer, hence its norm $N(\beta)$ is a nonzero integer; in particular, $|N(\beta)| \geq 1$. Now using the fact that $N(\beta)$ is the product of all Galois conjugates of β , we get

$$\begin{aligned} |N(\beta)| &= \prod_{i=1}^p |\sigma_i(\beta)| \\ &= \prod_{i=1}^p |a_0 + a_1\sigma_i(\alpha) + \cdots + a_{p-1}\sigma_i(\alpha)^{p-1}| \\ &= |\beta| \prod_{i=2}^p |a_0 + a_1\sigma_i(\alpha) + \cdots + a_{p-1}\sigma_i(\alpha)^{p-1}| \\ &\leq C_\alpha |\beta| n^{p-1}, \end{aligned}$$

where $C_\alpha = \max\{1, |\sigma_2(\alpha)|^p, \dots, |\sigma_p(\alpha)|^p\}$. So that

$$|\beta| \geq \frac{1}{C_\alpha} \frac{1}{n^{p-1}}.$$

□

In order to construct intermediate dimensional subgroups that are stable under multiplication by some algebraic numbers, it is convenient to have the following lemma, which generalizes the Erdős-Volkmann construction.

Definition 4.2. *If S is a subset of a metric space U , we will say that S is δ -dense in U if U can be covered by balls of radius δ centered at points of S .*

Lemma 4.3. *Denote Γ a finitely generated subgroup of \mathbb{R}^d . Let $B_\Gamma(n)$ be the ball of radius n centered at 1 in Γ , for the word metric associated to some finite generating set of Γ . Assume that for some $a > 1$, for all ball U centered at 0 in \mathbb{R}^d , there exists $\delta > 0$ arbitrarily small and $n_\delta \in \mathbb{N}^*$ such that*

$$B_\Gamma(n_\delta) \text{ is } \delta\text{-dense in } U \tag{1}$$

and

$$N(B_\Gamma(n_\delta) \cap U, \delta^a) \simeq \delta^{-d} \cdot \text{Vol } U, \tag{2}$$

where $N(A, \delta)$ denotes the minimal cardinality of a covering of A with ball of radius δ . Choose a sequence (δ_k) of such δ and, denoting $n_k = n_{\delta_k}$, let

$$F = \bigcup_{r,s \geq 1} F_{r,s}$$

with

$$F_{r,s} = \bigcap (B_\Gamma(sn_k))_{r\delta_k^a},$$

where $A_{r\delta_k^a}$ denotes the $r\delta_k^a$ neighborhood of the set A .

Then F is a measurable subgroup of \mathbb{R}^d of Hausdorff dimension $\frac{d}{a}$, provided (δ_k) decreases sufficiently fast to zero.

Proof. First, it is clear that F is measurable, and that it is a subgroup of N . Besides, (2) shows that $F_{r,s}$ can be covered by at most δ_k^{-d} balls of radius $r\delta_k^a$, and therefore that $\dim_H F_{r,s} \leq \frac{d}{a}$. Using that F is a countable union of $F_{r,s}$, we get

$$\dim_H F \leq \frac{d}{a}.$$

Now we have to prove the reverse inequality. For this we use the mass distribution principle (see Falconer [7], chapter 4). Choose a maximal δ_k -separated set S_k in each $B_\Gamma(n_k) \cap U$, and, starting from a total mass 1 (for U), equidistribute it at each step on the balls of lower scale. Finally choose a converging subsequence to obtain a Borel measure μ supported on $F_{1,1} \cap U$. The μ -measure of a δ_k^a -ball B_k at stage k is bounded above by

$$\frac{1}{N(B_\Gamma(n_k) \cap U, \delta_1)} \cdots \frac{1}{\min_{x \in S_{k-1}} \{N(B_\Gamma(n_k) \cap B(x, \delta_{k-1}^a), \delta_k)\}}.$$

Now, using assumption (1), we have, for all $x \in S_{i-1}$,

$$N(B_\Gamma(n_k) \cap B(x, \delta_{i-1}^a), \delta_i) \gg \delta_{i-1}^{ad} \delta_i^{-d}$$

and so

$$\mu(B_k) \ll \prod_{i=1}^k \delta_{i-1}^{-ad} \delta_i^d.$$

Now we claim that μ is Hölder. To see this, let B_ρ a ball of radius $\rho > 0$. Choose k such that $\delta_k \leq \rho < \delta_{k-1}$. Then B_ρ meets at most $(\frac{\rho}{\delta_k})^d$ many balls of radius δ_k^{-a} , so

$$\mu(B_\rho) \ll \left(\frac{\rho}{\delta_k}\right)^d \prod_{i=1}^k \delta_{i-1}^{-ad} \delta_i^d.$$

And at the same time, B_ρ meets at most one ball of radius δ_{k-1}^a centered at an element of S_{k-1} , hence

$$\mu(B_\rho) \ll \left(\frac{\delta_{k-1}^a}{\delta_k}\right)^d \prod_{i=1}^k \delta_{i-1}^{-ad} \delta_i^d.$$

Therefore,

$$\mu(B_\rho) \ll \left(\frac{\min\{\rho, \delta_{k-1}^a\}}{\delta_k}\right)^d \prod_{i=1}^k \delta_{i-1}^{-ad} \delta_i^d.$$

Now, for $\beta \in]0, 1[$, $\min\{\rho, \delta_{k-1}^a\} \leq \rho^\beta \delta_{k-1}^{a(1-\beta)}$ so

$$\begin{aligned} \mu(B_\rho) &\ll \rho^{d\beta} \delta_{k-1}^{ad(1-\beta)-ad+d} \delta_{k-2}^{d-ad} \delta_1^{d-ad} \\ &\ll \rho^{d\beta} \delta_{k-1}^{d(1-a\beta)} \delta_{k-2}^{d-ad} \cdots \delta_1^{d-ad} \\ &\ll \rho^{d\beta} \end{aligned}$$

as long as $\beta > \frac{1}{a}$ and (δ_k) decrease sufficiently fast to 0.

The measure μ has support in F , and is (βd) -Hölder for all $\beta > \frac{1}{a}$, therefore,

$$\dim_H F \geq \frac{d}{a}.$$

□

Remark: Suppose that Γ is generated by p elements, and that there exists $c = c(\Gamma)$ such that, for all $n \in \mathbb{N}$, $B_\Gamma(n)$ is $c.n^{-\frac{2}{a}+1}$ -separated. Then Γ satisfies conditions (1) and (2) for all $a > 1$.

Proposition 4.4. *Let K be a real number field. For any $s \in [0, d]$, there exists a dense measurable subgroup of \mathbb{R}^d of Hausdorff dimension s which is stable under multiplication by the matrices with entries in K .*

Proof. From the primitive element theorem, we may write $K = \mathbb{Q}[\alpha]$ where α is an algebraic integer. Let $\Gamma = \mathbb{Z}^d + \mathbb{Z}^d\alpha + \dots + \mathbb{Z}^d\alpha^{p-1}$ where p is the degree of α . Endow Γ with the word metric for the set of generators $e_1, \alpha e_1, \dots, \alpha^{p-1}e_1, \dots, e_d, \dots, \alpha^{p-1}e_d$, where the (e_i) is the usual basis of \mathbb{R}^d . From lemma 4.1, for all $n \in \mathbb{N}^*$, the ball $B_\Gamma(n)$ is $\frac{c}{n^{p-1}}$ -separated. Hence, using lemma 4.3 and the following remark, we can construct a dense measurable subgroup F of dimension $\frac{1}{a}$, which is easily checked to be invariant under multiplication by matrices with entries in $\mathbb{Z}[\alpha]$. Letting

$$F' = \bigcup_{n \in \mathbb{N}^*} \frac{1}{n}F,$$

one gets a dense measurable subgroup of \mathbb{R}^d of dimension $\frac{d}{a}$ which is stable under multiplication by matrices with entries in $K = \mathbb{Q}[\alpha]$. \square

The previous proposition leads to the following natural questions: For what real numbers α does there exist a measurable subgroup of \mathbb{R} having arbitrary (or just positive) Hausdorff dimension which is invariant under multiplication by α ? Does there exist a measurable subgroup of \mathbb{R} of positive Hausdorff dimension which is invariant under multiplication by a transcendental real number? etc.

4.2 Solvable algebraic groups defined over a number field

Starting from proposition 4.4 and using induction, we will prove the following lemma, which we will need for our study of solvable algebraic groups:

Lemma 4.5. *Let \mathfrak{n} be a nilpotent Lie algebra, and let D be a family of automorphisms of \mathfrak{n} such that, in an appropriate basis of \mathfrak{n} , all the elements of \mathfrak{n} have entries in K , where K is a real number field. Then for any $\alpha \in [0, \dim \mathfrak{n}]$, there exists a dense measurable \mathbb{Q} -subalgebra of \mathfrak{n} of Hausdorff dimension α that is invariant under D .*

Proof. We use induction on the nilpotency class of \mathfrak{n} .

If \mathfrak{n} is Abelian, then the result follows from proposition 4.4.

Now let \mathfrak{n} be a nilpotent Lie algebra, and suppose we know the result for nilpotent Lie algebras of nilpotency class less than that of \mathfrak{n} . As the elements of D are automorphisms of the Lie algebra \mathfrak{n} , the center \mathfrak{z} of \mathfrak{n} is invariant under D .

First case: $\alpha \leq \dim \mathfrak{z}$.

Then, by the Abelian case, we can choose a measurable dense \mathbb{Q} -subspace \mathfrak{z}_α of \mathfrak{z} of Hausdorff dimension α . Choose any dense countable \mathbb{Q} -subalgebra C_0 of \mathfrak{n} stable under D (this is easily seen to exist) and let $\mathfrak{n}_\alpha = C_0 + \mathfrak{z}_\alpha$. Then \mathfrak{n}_α is a countable union of copies of \mathfrak{z}_α so $\dim_H \mathfrak{n}_\alpha = \alpha$; \mathfrak{n}_α is a dense measurable \mathbb{Q} -subalgebra of \mathfrak{n} ; and \mathfrak{n}_α is invariant under D .

Second case: $\alpha > \dim \mathfrak{z}$

In this case, we choose a subspace V' such that $\mathfrak{n} = \mathfrak{z} \oplus V'$. Identifying V' with $\mathfrak{n}/\mathfrak{z}$, and using the induction hypothesis, we can find a dense measurable subgroup $V'_{\alpha-1}$ of V' of Hausdorff dimension $\alpha - \dim \mathfrak{z}$ which is stable under D . Let $\mathfrak{n}_\alpha = \mathfrak{z} + V'_{\alpha-1}$; this is a dense measurable \mathbb{Q} -subalgebra of \mathfrak{n} ; by lemma 2.1 it has Hausdorff dimension α ; and it is invariant under D because V is stable, $V'_{\alpha-1}$ is stable modulo \mathfrak{z} , and $\mathfrak{z} \subset \mathfrak{n}_\alpha$. \square

We are now ready to state and prove our theorem about measurable subgroups of connected solvable algebraic groups defined over a number field.

Theorem 4.6. *If $G = \mathbf{G}(\mathbb{R})$ is the group of real points of a connected solvable algebraic group \mathbf{G} defined over a number field K , then G admits dense measurable subgroups of arbitrary dimension.*

Proof. By the structure theorem for connected solvable algebraic groups (see Borel [2], theorem 10.6), G can be written as a semidirect product $G = T.G_u$ where T is a maximal torus and G_u is the connected unipotent group of unipotents elements of G . Denote \mathfrak{g}_u the Lie algebra of G_u . We distinguish two cases.

First case: $\alpha \leq \dim G_u$

From theorem 7.7 of [8], the group $D = \mathbf{G}(\mathbb{K})$ is a dense subgroup of G . Moreover, it acts (by the adjoint action) on \mathfrak{g}_u with matrices with entries in K . Therefore, by lemma 4.5 we can choose a dense measurable \mathbb{Q} -subalgebra of \mathfrak{g}_u of Hausdorff dimension α which is invariant under $\text{Ad } D$. By lemma 3.6 and the remark following it, its image under the exponential map is then a dense measurable subgroup $G_{u,\alpha}$ of G_u of dimension α which is stable under conjugation by the elements of D . We define G_α to be the subgroup generated by D and $G_{u,\alpha}$. One easily checks that G_α has the required properties.

Second case: $\alpha > \dim G_u$

Then, as T is a connected Abelian Lie group, it contains a dense subgroup $T_{\alpha - \dim G_u}$ of dimension $(\alpha - \dim G_u)$. So we let $G_\alpha = T_{\alpha - \dim G_u}.G_u$. The group G_α is a dense measurable subgroup of G , and it has Hausdorff dimension α , because of lemma 2.1. \square

Example: The more general question which asks whether any solvable Lie group admits dense measurable subgroups of arbitrary dimension remains open. For instance, let G be the semidirect product $\mathbb{R}_+^* \rtimes_\theta \mathbb{R}^2$ where

$$\begin{aligned} \theta: \mathbb{R}_+^* &\rightarrow \text{Aut } \mathbb{R}^2 \\ t &\mapsto ((x, y) \mapsto (tx, t^\lambda y)), \end{aligned}$$

with $\lambda \in \mathbb{R} - \mathbb{Q}$. If $\alpha \geq 2$, one easily construct a dense subgroup of G of dimension α of the form $H_\alpha \times \mathbb{R}^2$, where H_α is a subgroup of \mathbb{R}_+^* of dimension $\alpha - 2$. If $\alpha \in [1, 2]$, it is still possible to construct a dense subgroup of dimension α of the form $\mathbb{Q}_+^* \times (H_\alpha \times \mathbb{R})$, with H_α a dense subgroup of \mathbb{R} of dimension $\alpha - 1$ which is invariant under multiplication under the rationals. However, if $\alpha < 1$, then the existence of a dense measurable subgroup of dimension α is unclear.

5 The p -adic setting

5.1 Nilpotent p -adic analytic groups

The proof of theorem 1.4 follows the same lines as in the real case: we start with the Abelian case, then use induction to study nilpotent Lie algebras, and finally

lift the result to nilpotent Lie groups, using the Campbell-Hausdorff formula instead of the exponential map.

In the p -adic setting, dense subgroups of arbitrary dimension of \mathbb{Z}_p were constructed by Abercrombie [1].

Theorem 5.1 (Abercrombie). *There are dense measurable subgroups of \mathbb{Z}_p of arbitrary dimension between 0 and 1.*

In fact, one can prove this theorem using almost the same construction as Erdős and Volkmann: define

$$F_r = \{x \in \mathbb{Z}_p \mid \forall k, \exists x' \in \{0, \dots, rp^{n_k} - 1\} : |x - x'| \leq p^{-an_k}\},$$

$$F = \bigcup_{r \in \mathbb{N}} F_r,$$

and check that F is a measurable subgroup of \mathbb{Z}_p of Hausdorff dimension $\frac{1}{a}$.

The analog of proposition 3.5 is the following.

Proposition 5.2. *Let \mathfrak{n} be a nilpotent Lie algebra over \mathbb{Q}_p of dimension d , and D be a countable unipotent subgroup of $\text{Aut}(\mathfrak{n})$. For any $\alpha \in [0, \dim \mathfrak{n}]$, there exists a measurable \mathbb{Q} -subalgebra of \mathfrak{n} of Hausdorff dimension α which is stable under the elements of D .*

The proof is exactly the same as that of proposition 3.5, so we omit it. Now, in order to obtain results about p -adic nilpotent Lie groups, we start by the following consequence of the proposition:

Corollary 5.3. *Let \mathfrak{n} be a p -adic nilpotent Lie algebra. Define N_0 to be the group whose elements are the elements of \mathfrak{n} , and whose law is defined using the Campbell-Hausdorff formula (see [12], part II, page 5.18, remark 1). Let Γ be a countable subgroup of N_0 . Then, for all $\alpha \in [0, \dim N_0]$, there exists a measurable subgroup of N_0 of Hausdorff dimension α containing Γ . In particular, N_0 contains a dense subgroup of Hausdorff dimension α .*

Proof. The group $D = \text{Ad } \Gamma$ is a countable unipotent subgroup of $GL(\mathfrak{n})$, so, by the previous proposition, there exists a dense \mathbb{Q} -subalgebra \mathfrak{n}_α of \mathfrak{n} of Hausdorff dimension α which is stable under D . Then, as in the proof of proposition 3.6, one checks that $N_\alpha = \exp(\mathfrak{n}_\alpha)$ is a subgroup of N_0 of Hausdorff dimension α which is stable under conjugation by the elements of Γ . This implies that the subgroup M_α generated by N_α and Γ is a countable union of copies of N_α and hence has Hausdorff dimension α . Of course, M_α is measurable and contains Γ so we have what we wanted. \square

We can finally state and prove our theorem on nilpotent p -adic analytic groups:

Theorem 5.4. *If N is a nilpotent p -adic analytic group, then, for all $\alpha \in [0, \dim N]$, there exists a measurable subgroup of N of Hausdorff dimension α which is dense in a neighborhood of the identity.*

Proof. From [12], page 5.34, corollary 1, N contains an open subgroup which is isomorphic to an open subgroup U of the group N_0 above. But, by the previous proposition, U_0 contains a dense subgroup of Hausdorff dimension α , so we are done. \square

Remark: If N contains a normal open subgroup which is isomorphic to a subgroup of N_0 , then, reasoning in the same way as we did in the real setting, it is possible to show that N has a dense subgroup of Hausdorff dimension α . So we ask the following question: given a nilpotent p -adic analytic group N does there exist an open normal subgroup of N which is isomorphic to a subgroup of N_0 , where N_0 is defined from the Lie algebra of N , using the Campbell-Hausdorff formula?

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