OPTIMAL DECAY RATES OF THE ENERGY OF A HYPERBOLIC-PARABOLIC SYSTEM COUPLED BY AN INTERFACE

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Abstract. This work is dedicated to the study of a linear model arising in fluid-structure interaction and introduced by Rauch, Zhang and Zuazua. The system is formed of a heat and a wave equation, taking place in two distinct domains, and coupled by transmission conditions at the interface of the domains. Two different transmission conditions are considered.

In both cases, when the interface geometrically controls the wave domain, we show the quick polynomial decay of the energy for solutions with smooth initial data, improving the rate of decay obtained by the previous authors. The polynomial stability is deduced from an optimal observability inequality conjectured in their work. The proof of this estimate mainly relies on a known generalized trace lemma for solutions of partial differential equations and the results of Bardos, Lebeau and Rauch on the control of the wave equation.

Without the geometric condition, we show, using a Carleman inequality of Lebeau and Robbiano and an abstract theorem of N. Burq, a logarithmic decay for solution of the system with one of the two transmission conditions. This result improves the speed of decay obtained by Zhang and Zuazua, and is also optimal in some geometries.

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1. Introduction

1.1. **Presentation of the system.** Let Ω_1 and Ω_2 be two disjoint bounded open sets of $\mathbb{R}^d$ $(d \geq 2)$ with smooth boundaries, and

$$\gamma := \partial\Omega_1 \cap \partial\Omega_2, \quad \Gamma_1 := \partial\Omega_1 \backslash \overline{\gamma}, \quad \Gamma_2 := \partial\Omega_2 \backslash \overline{\gamma}.$$

Assume that γ is not empty, and denote by Ω the interior of $\overline{\Omega}_1 \cup \overline{\Omega}_2$.

In this work, we are interested with the following system, introduced in [22, 27]

$$\begin{aligned} &\partial_t u - \Delta u = 0, \quad x \in \Omega_1, \quad t > 0,\\ &\partial_t^2 v - \Delta v = 0, \quad x \in \Omega_2, \quad t > 0, \end{aligned} \tag{1}$$

with one of the following initial-boundary conditions

$$\begin{aligned} &u_{\restriction \Gamma_1} = 0, \quad v_{\restriction \Gamma_2} = 0\\ &u_{\restriction \gamma} = v_{\restriction \gamma}, \quad \partial_{n_1} u_{\restriction \gamma} = -\partial_{n_2} v_{\restriction \gamma} \end{aligned} \tag{BCa}$$

$$u_{\restriction t=0} = u_0 \in H^1(\Omega_1), \quad v_{\restriction t=0} = v_0 \in H^1(\Omega_2), \quad \partial_t v_{\restriction t=0} = v_1 \in L^2(\Omega_2), \tag{ICa}$$

(model with “naive” transmission conditions);

$$\begin{aligned} &u_{\restriction \Gamma_1} = 0, \qquad v_{\restriction \Gamma_2} = 0\\ &u_{\restriction \gamma} = \partial_t v_{\restriction \gamma}, \quad \partial_{n_1} u_{\restriction \gamma} = -\partial_{n_2} v_{\restriction \gamma} \end{aligned} \tag{BCb}$$

$$u_{\restriction t=0} = u_0 \in L^2(\Omega_1), \quad v_{\restriction t=0} = v_0 \in H^1(\Omega_2), \quad \partial_t v_{\restriction t=0} = v_1 \in L^2(\Omega_2), \tag{ICb}$$

(model with “natural” transmission conditions).

Here and in the sequel ∂_{n_j} is the outer normal derivative on $\partial\Omega_j$. We will refer to System (1) with initial-boundary conditions (BCa) and (ICa) as (1a), and to the same equation, but with initial-boundary conditions (BCb) and (ICb) as (1b). These systems are rough linear models for fluid-structure interaction, the heat part modeling the fluid and the wave part the solid structure. To get more precise models, one could replace the heat equation by a Navier-Stokes system and the wave by an elastic equation. In this context, the transmission condition of System (1b), $u_{\restriction \gamma} = \partial_t v_{\restriction \gamma}$ is more natural, u being the speed of the fluid and v the displacement of the solid. A model similar to (1b), but including the pressure of the fluid, is mentionned in the classical book of Dautray and Lions [7, XVIII, §7.5].

This work is concerned with the stability of the solutions of the two systems, that is the speed of decay to 0 of the energy. One of our goals is to contribute, through a simple example, to a better understanding of general transmission problems for partial differential equations.

Another motivation is to give a first example of precise control-theoretic results on fluid-structure interaction models, whereas most theoretical works on the subject concern the issue of existence and uniqueness of solutions for more refined models than (1a) and (1b), including non-linearities and most of the time a free boundary (see [2, 24, 8, 9, 6, 25]).

We now state the previous stability results for systems (1a) and (1b). Denote by $U(t) := \big(u(t), v(t), \partial_t v(t)\big)$ the solution of (1a) or (1b). If X is a Banach space, we will denote by $\|\cdot\|_X$ the norm on X, and (when X is an Hilbert space), by $\big(\cdot\,\big|\,\cdot\big)_X$ the scalar product on X. For two positive functions f and g, we shall write $f \lesssim g$ when there is a constant C depending only on Ω_1 and Ω_2, such that $f \leq Cg$.

Both systems (1a) and (1b) have a natural energy, that we shall denote respectively by E_a and E_b

$$E_a(U(t)) := \frac{1}{2}\left(\int_{\Omega_1} |\nabla u(t)|^2 dx + \int_{\Omega_2} |\partial_t v(t)|^2 dx + \int_{\Omega_2} |\nabla v(t)|^2 dx\right), \tag{2}$$

$$E_b(U(t)) := \frac{1}{2}\left(\int_{\Omega_1} |u(t)|^2 dx + \int_{\Omega_2} |\partial_t v(t)|^2 dx + \int_{\Omega_2} |\nabla v(t)|^2 dx\right). \tag{3}$$

Both energies are decreasing, the dissipation coming from the heat component u of the system (see Section 2 for details).

1.2. **The case of naive transmission conditions.** In [22] J. Rauch, X. Zhang and E. Zuazua have proven that the energy of the solutions of (1a) tends to 0 as t goes to infinity. Using Gaussian beam solutions of the two equations, they have shown that this decay is not uniform with respect to the initial energy. Furthermore, under the condition that the interface γ geometrically controls Ω_2 (see subsection 1.4), they have shown polynomial decay for smooth initial conditions

$$E_a(U(t)) \lesssim \frac{1}{t}\|U_0\|^2_{D(\mathcal{A}_a)}. \tag{4}$$

where $\|U_0\|^2_{D(\mathcal{A}_a)}$ is the first order energy at time $t = 0$ (the sum of the energies of $U_{\restriction t=0}$ and $\partial_t U_{\restriction t=0}$). This decay is deduced from an observability inequality with loss of one time derivative on (1a) which implies (4). The author also conjectured that the optimal decay was not (4), but rather a decay in $\frac{1}{t^2}$ for solutions of finite first order energy, which corresponds to the same observability inequality, but with the loss of only half a derivative. As they noticed, according to their Gaussian beam construction, this observability inequality would be optimal.

In [27], Zhang and Zuazua have shown, without the geometric control condition, the following logarithmic decay result

$$\sqrt{E_a}(U(t)) \lesssim \frac{1}{\log^s(1+t)}\|U_0\|_{D(\mathcal{A}_a)} \tag{5}$$

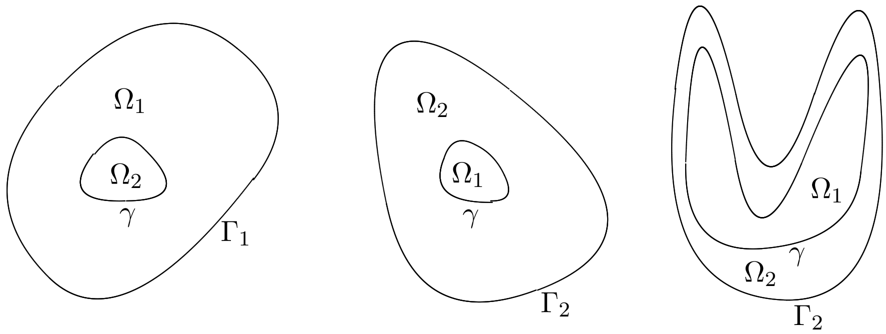
where s is any number strictly smaller than 1/8. Once again, this result does not seem optimal, as the power s does not usually appear in this context (Carleman inequalities usually yield exponential loss, which gives the same decay with $s = 1$).

1.3. **The case of natural transmission conditions.** System (1b) was studied in the recent work by Zhang and Zuazua [27]. Note that according to the definition (3) of E_b, the two equations are on a different energy scale, L^2 for u and H^1 for v, and that $\sqrt{E_b}$ is a norm on the energy space only when Γ_2 is not empty. Otherwise, there exists nontrivial stationnary solutions for the system, of energy 0, such that v is constant and $u = 0$. The stability should be understood as the convergence for infinitely large time to this one-dimensional space of constant solutions.

Zhang and Zuazua have shown that all solutions tend to 0 in the energy space, but without uniformity with respect to the initial energy. Under the geometric control condition of Ω_2 by γ, they have also proven that the energy of finite first-order energy solutions decays at the rate $1/t^{1/3}$, which corresponds to the loss of 3 time-derivatives in the observability inequality. As in the previous case, they conjectured a loss of time-derivative of only 1/2, which should be optimal according to the ray construction.

The study of System (1b) seems more difficult, due in particular to the unusual fact that the embedding of the domain $D(\mathcal{A}_b)$ of the dissipative operator associated to the

FIGURE 1



a. Heat outside GCC satisfied | b. Heat inside GCC not satisfied | c. Heat inside GCC satisfied

system in the energy space is not compact. Indeed, as observed in [27] this is the main obstacle to show the logarithmic decay for solution of (1b) in the absence of geometric control condition.

1.4. **Main results.** In this work, we study a model case (see figure 1), where Ω_1 and Ω_2 are smooth domains, and the C^∞ manifolds γ, Γ_1 and Γ_2 have empty intersections so that

$$\partial\Omega_1 = \gamma \cup \Gamma_1, \quad \partial\Omega_2 = \gamma \cup \Gamma_2.$$

We show, under the geometric control condition, the optimal observability and a polynomial decay at the speed $1/t^{2-\varepsilon}$ for solution of finite first order energy. Furthermore, for solutions of (1a), we show the optimal logarithmic decay (that is, with $s = 1$ in (5)) without further condition.

Consider the following wave equation on Ω_2 with (inhomogeneous) Dirichlet boundary conditions

$$\begin{aligned} &\partial_t^2 w - \Delta w = 0, \quad w_{\restriction\gamma} = g, \quad w_{\restriction\Gamma_2} = 0 \\ &w_{\restriction t=0} = w_0 \in H_0^1(\Omega_2), \quad \partial_t w_{\restriction t=0} = w_1 \in L^2(\Omega_2). \end{aligned} \tag{6}$$

We say that γ controls geometrically Ω_2 in a time T when all the rays of geometric optic in Ω_2 of length T hit γ at least one time at a point which is not strictly diffractive [1]. According to the classical work [1] by Bardos, Lebeau and Rauch, this condition is equivalent to the exact control, in time T, of the wave equation (6) by a function $g \in H^1((0,T) \times \gamma)$. The typical situation where this Geometric Control Condition (GCC) is fullfilled in our case is when Γ_2 is empty and γ is the entire boundary $\partial\Omega_2$ (see figure 1, a). In this case, Ω_2 is inside Ω_1. Nevertheless, even under our strong smoothness assumptions, there exist geometries such that the heat domain Ω_1 is inside the wave domain Ω_2, but γ controls Ω_2 (see figure 1, c). We now state our main results.

Theorem 1. *Assume that γ controls Ω_2 in a time $T' > 0$ and that $\partial\Omega_2$ does not have any contact of infinite order with its tangents. Let $T > T'$. Then the following observability inequality (with loss of one half time-derivative) holds for solutions of* (1a) *with initial conditions in $D(\mathcal{A}_a)$*

$$E_a(U(T)) \lesssim \|\partial_t u\|^2_{H^{1/2}(0,T;L^2(\Omega_1))}. \tag{7}$$

[1] A strictly diffractive point ρ is one where the optic ray is tangent to γ and such that γ is, near ρ, contained in the interior of Ω_2

Assume furthermore that Γ_1 is not empty. Then the observability inequality with loss of one half time-derivative also holds for solutions of (1b) *with initial conditions in $D(\mathcal{A}_b)$*

$$E_b(U(T)) \lesssim \|\nabla u\|^2_{H^{1/2}(0,T;L^2(\Omega_1))}. \tag{8}$$

Remark 1.1. The assumption that $\partial\Omega_2$ does not have any contact of infinite order with its tangents, that ensures the existence and uniqueness of the optic flow in Ω_2, is standard in this context. It is fullfilled, for example, when Ω_2 is analytic.

Remark 1.2. The observability inequalities (7) and (8) are the one conjectured by Rauch, Zhang and Zuazua in [22, 27]. In these works, weaker form of these estimates are proven, namely with H^1 instead of $H^{1/2}$ in (7) and with H^3 instead of $H^{1/2}$ in (8).

Theorem 2. *Under the assumptions of Theorem 1, the following holds for solutions of* (1a) *or* (1b)

$$\forall s < 2,\ \exists C_s > 0,\ \forall U_0 \in D(\mathcal{A}), \quad E(U(t)) \lesssim \frac{C_s}{t^s}\|U_0\|^2_{D(\mathcal{A})}. \tag{9}$$

Here $D(\mathcal{A})$ is the space of initial conditions with finite first order energy (see Section 2) for the operator $\mathcal{A}_a$ or $\mathcal{A}_b$, and E is the energy E_a or E_b of the equation.

Indeed it would be natural to expect a decay in $1/t^2$, but for technical reasons (due to the fact that the loss of derivative is fractional), we were not able to show better than Theorem 2. Apart from this technical difficulty, the deduction of polynomial decay from observability inequality with loss is standard.

The proof of Theorem 1 has two steps. The first one consists in applying a generalized trace lemma (from Hörmander [13]), which allows to bound some traces of u (and thus, by the transmission condition, of v), by the observation. The loss of one-half derivative in the observability inequalities comes from the same loss in the trace theorems. In the second step of the proof we apply the control results of [1] to observe, with the traces, all the interior energy of the equation.

Remark 1.3. To weaken the smoothness assumptions in Theorems 1 and 2 one should use the results of N. Burq in [3] generalizing the work of Bardos, Lebeau and Rauch to open sets of class C^2. This would yield a lot of technicalities (in the use of the control results of [3] but also in the trace theorems of Subsection 3.1), so that we prefered to restrict ourself to this model case to show precisely the influence of the geometry on the decay of the solutions.

Remark 1.4. In the case of System (1b), when Γ_2 is empty, Theorem 2 asserts that U converges (at the speed given by (9)) in the natural energy space H_b defined in Subsection 2.2, to the one-dimensional linear subspace of stationnary solutions of (1b).

The logarithmic decay result reads as follows:

Theorem 3. *Assume that the boundary of each connex component of Ω_2 has a non-empty intersection with γ. Then the energy of a smooth solution of* (1a) *decays at logarithmic speed:*

$$\sqrt{E_a}(U(t)) \leq \frac{C}{\log(t+1)}\|u\|_{D(\mathcal{A}_a)}.$$

Remark 1.5. When Ω is a ball and $\partial\Omega_2$ is the circle $\partial\Omega$, the decay of Theorem 3 is optimal, due to the so-called *whispering gallery eigenfunctions*, which form a sequence of eigenfunctions of the Dirichlet Laplace operator on Ω which concentrate exponentially on $\partial\Omega$ (see for example the remark in 2.3 of [12]).

The proof of Theorem 3 uses two known results: an abstract theorem of N. Burq (see [4]) which links, for dissipative operators, logarithmic decay to resolvent estimates with exponential loss, and some high frequency Carleman inequalities of G. Lebeau and L. Robbiano [16]. Apart from these two deep results, the argument is elementary.

It is a general property of dissipative systems that in Theorem 2 and 3, the speed of decay is related to the smoothness of the initial data. More precisely, for initial conditions in $D(\mathcal{A}^{r_0})$, $r_0 > 0$, $\sqrt{E}$ decays at least as fast as C_r/t^r, for all $r < r_0$, in the case of Theorem 2 and as $C/\log^{r_0}(t+1)$ in the case of Theorem 3, which may be shown by elementary interpolation and iteration arguments.

It is interesting to compare System (1) with a wave equation in Ω_2, with Dirichlet boundary condition on Γ_2 and strict dissipation on γ. According to the work [1] of Bardos, Lebeau and Rauch, under the geometric control condition, the stability is uniform (and thus exponential). When this condition does not hold, Lebeau and Robbiano [17] have shown the logarithmic decay of the energy as in Theorem 3, which is again optimal in some geometries. According to the gaussian beams constructions of [22] and [27], and Theorems 1 and 2 above, the dissipation on γ is half a time-derivative weaker for System (1) than for the damped wave equation. Naturally, this difference of polynomial order is not seen in Theorem 3, where exponential losses are allowed. From this point of view, one may also hope Theorem 3 to hold for solutions of (1b), but it is not clear whether the difficulties posed by the lack of compactness of $\mathcal{A}_b^{-1}$ are of technical nature, or if they are really an obstacle to the decay.

Let us mention other linear systems coupling two, or more, equations by an interface. The behaviour of the solutions of the classical transmission problem for Schrödinger and wave equations is by now well understood (see the complete study of L. Miller [20] in a microlocal setting). In the recent work of Koch and Zuazua [15], a system coupling 3 waves equation is introduced. The mathematical novelty brought by the study of System (1) is a better understanding of the coupling of two equations of different nature and of the transfer, through γ, of the dissipative properties of the parabolic equation to the hyperbolic one.

Our work is also related to the stability of system coupling an elastic wave equation with a dissipative equation, both taking place in the same domain. In this context, the speed of decay of the energy has also been linked, through microlocal tools, to geometric optical-type conditions on the domain (see [18, 5, 10, 26]). In these systems, the transmission problem is at the boundary of the domain, where the two components (transversal and longitudinal) of the elastic wave are coupled by a Dirichlet boundary condition. For related works using multipliers techniques see for example [14, 21].

The structure of the paper is the following. In Section 2, we recall the semi-group formulation of (1a) and (1b) which ensures the well-posedness of the equations. In Section 3, we show the observability inequality of Theorem 1. In Section 5 we explain how this inequality implies the decay stated in Theorem 2. Section 6 deals with the same questions on System (1b). In Section 7 we prove Theorem 3.

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2. Well-posedness of systems (1a) and (1b)

In this section we quickly recall, from [22, 27], the definition in term of semi-groups and well-posedness of systems (1a) and (1b).

2.1. **System** (1a).

Notation. If u is a distribution on Ω_1, we shall denote by $\mathbb{1}_{\Omega_1}u$ the distribution on Ω obtained by extending u by 0 on Ω_2. Such an extension is well defined if u is "not too rough" with respect to the distance z to γ (for example when u is locally in one of the spaces $H^{0,-N}$ defined in Subsection 3.1). Similarly, we will use the notation $\mathbb{1}_{\Omega_2}v$, where v is a distribution on Ω_2.

If u is a distribution on Ω_1, v a distribution on Ω_2, and E a space of distributions over Ω, we write:

$$(u,v) \in E$$

whenever $\mathbb{1}_{\Omega_1}u + \mathbb{1}_{\Omega_2}v$ defines a distribution on Ω which belongs to E. For example

$$(u,v) \in H^2(\Omega) \iff u \in H^2(\Omega_1),\ v \in H^2(\Omega_2),\ u_{\restriction\gamma} = v_{\restriction\gamma},\ \text{and } \partial_{n_1}u_{\restriction\gamma} = -\partial_{n_2}v_{\restriction\gamma}.$$

Define the energy space H_a and the operator $\mathcal{A}_a$ on H_a, of domain $D(\mathcal{A}_a)$ by

$$H_a := \left\{U_0 = (u_0,v_0,v_1) \in H^1(\Omega_1) \times H^1(\Omega_2) \times L^2(\Omega_2),\ (u_0,v_0) \in H^1_0(\Omega)\right\},$$
$$\mathcal{A}_aU_0 := (\Delta u_0, v_1, \Delta v_0)$$
$$D(\mathcal{A}_a) := \left\{U_0 \in H_a,\ (u_0,v_0) \in H^2(\Omega),\ v_1 \in H^1(\Omega_2),\ (\Delta u_0, v_1) \in H^1_0(\Omega)\right\}.$$

Note that the condition $(u_0,v_0) \in H^1_0(\Omega) \cap H^2(\Omega)$ for functions in the domain of $\mathcal{A}_a$ implies the boundary and transmission conditions (BCa). System (1a) may thus be rewritten in the abstract form

$$\partial_t U = \mathcal{A}_a U, \quad U(t) := (u(t), v(t), \partial_t v(t)). \tag{10}$$

The following proposition (see [22]) ensures the well-posedness of System (1a).

Proposition 2.1. *The domain $D(\mathcal{A}_a)$ is dense in H_a. The operator $\mathcal{A}_a$ is the generator of a contractive semi-group. The energy E_a of a solution of* (1a)*, defined by* (2)*, is decreasing and:*

$$\forall T > 0, \quad E_a(U(0)) - E_a(U(T)) = \int_0^T \int_{\Omega_1} |\partial_t u(t,x)|^2 dxdt. \tag{11}$$

Furthermore, 0 *is in the resolvent set of $\mathcal{A}_a$, A_a^{-1} is compact as an operator in H_a and for every $U_0 \in H_a$, the solution U of* (1a) *satisfies*

$$U(t) \underset{t\to\infty}{\longrightarrow} 0 \text{ in } H_a.$$

2.2. **System** (1b).

Define the energy space H_b by

$$H_b := \left\{U_0 = (u_0,v_0,v_1) \in L^2(\Omega_1) \times H^1_{\Gamma_2}(\Omega_2) \times L^2(\Omega_2)\right\},$$

when $H^1_{\Gamma_2}(\Omega_2)$ is defined as the space

$$H^1_{\Gamma_2}(\Omega_2) := \left\{v_0 \in H^1(\Omega_2),\ v_{0\restriction\Gamma_2} = 0\right\},$$

endowed with the usual H^1 norm. Note that when Γ_2 is not empty, the H^1 norm of v_0 is equivalent, on $H^1_{\Gamma_2}(\Omega_2)$, to $\|\nabla v_0\|_{L^2(\Omega_2)}$, so that the energy defined by (3) is the square of a norm on H_b. However, this is not the case when Γ_2 is empty (the elements of H_b: $(u_0 = 0, v_0 = c, v_1 = 0)$, where c is an arbitrary constant, have 0 energy). Consider the operator $\mathcal{A}_b$ on H_b, of domain $D(\mathcal{A}_b)$

$$\mathcal{A}_bU_0 := (\Delta u_0, v_1, \Delta v_0)$$
$$D(\mathcal{A}_b) := \Big\{(u_0,v_0,v_1) \in H_b,\ u_0 \in H^1(\Omega_1),\ \Delta u_0 \in L^2(\Omega_1),$$
$$v_1 \in H^1_{\Gamma_2}(\Omega_2),\ \Delta v_0 \in L^2(\Omega_2),\ u_{0\restriction\gamma} = v_{1\restriction\gamma},\ \partial_{n_1}u_{0\restriction\gamma} = -\partial_{n_2}v_{0\restriction\gamma}\Big\}.$$

Proposition 2.2. *The domain $D(\mathcal{A}_b)$ is dense in H_b. The operator $\mathcal{A}_b$ is the generator of a strongly continuous semi-group. Furthermore, for every $U_0 \in H_b$, the solution U of* (1b) *satisfies*

$$E_b(U(t)) \underset{t\to\infty}{\longrightarrow} 0.$$

The energy $E_b(U(t))$ of a solution of (1b) *decays. More precisely*

$$\forall T > 0, \quad E_b(U(0)) - E_b(U(T)) = \int_0^T \int_{\Omega_1} |\nabla u(t,x)|^2 dx dt. \tag{12}$$

Remark 2.3. When the energy E_b is not the square of a norm on H_b, i.e. in the case $\Gamma_2 = \emptyset$, we will also need to study the natural norm on H_b. Note that

$$\|U(t)\|^2_{H_b} = 2E_b(U(t)) + \|v(t)\|^2_{L^2(\Omega_2)}. \tag{13}$$

Differentiating (13) and using Gronwall's inequality, it is easy to show

$$\forall t \geq t' \geq 0, \quad \|U(t)\|^2_{H_b} \leq e^{t-t'} \|U(t')\|^2_{H_b}. \tag{14}$$

Remark 2.4. The embedding of $D(\mathcal{A}_b)$ into H_b is not compact, due to the lack of gain of regularity of v_0 near γ (that is, if $U_0 = (u_0, v_0, v_1)$ is in $D(\mathcal{A}_b)$, v_0 has a local H^2 regularity in the interior of Ω_2, but not better, in general, than a global H^1 regularity on Ω_2).

3. High-frequency observability inequality for (1a)

In this section (first step of the proof of Theorem 1), we show

Proposition 3.1. *Under the assumptions of Theorem 1, consider a smooth cut-off function ψ on $(0,T)$, equal to 1 in the neighborhood of an interval (α,β) of length greater than T'. Then for all solution U of* (1a) *with initial condition in $D(\mathcal{A}_a)$*

$$E(U, t = T) \lesssim \underbrace{\|\psi(t)\partial_t u\|^2_{H^{1/2}(\mathbb{R},L^2(\Omega_1))} + \|\psi(t)u\|^2_{L^2(\mathbb{R}\times\Omega_1)} + \|\psi(t)v\|^2_{L^2(\mathbb{R}\times\Omega_2)}}_{\mathcal{Q}_T(U)}. \tag{15}$$

Note that if $U(t)$ is a solution of (1a) with initial condition in $D(\mathcal{A}_a)$, both quantities

$$\|\partial_t u\|^2_{L^2(0,T;L^2(\Omega_1))}, \quad \|\partial_t^2 u\|^2_{L^2(0,T;L^2(\Omega_1))}$$

are well defined and finite according to the energy decay law (11), so that (15) makes sense for such a solution.

Proposition 3.1 says that the observability inequality (7) is valid, up to terms that are compact with respect to the energy space H_a. These terms will be removed in the next section using a standard uniqueness-compactness argument.

The proof takes two steps. We first show (paragraph 3.2) that $\mathcal{Q}_T(U)$ controls the traces of $\partial_t^2 v$ on γ, in suitable Sobolev spaces. For this we need a generalized trace lemma for solutions of partial differential equations that are transverse to the boundary. This result which we deduce from two lemmas of Hörmander, is stated in paragraph 3.1.

The second step (paragraphs 3.3 and 3.4) of the proof of Proposition 3.1 consists in using the geometric control condition to bound v in the energy space by its traces on γ. The direct application of the geometric control condition, in paragraph 3.3, will only yield a bound on $\|\partial_t^2 v\|^2_{H^{-1}}$. Nevertheless, it is easy to deduce from this bound the complete high-frequency observability estimate (15) (paragraph 3.4). The "compact" terms in $\mathcal{Q}_T(U)$ appears in this last step.

3.1. **Asymmetric Sobolev spaces and trace lemma.**

3.1.1. *Asymmetric Sobolev spaces.* In this paragraph we work in the Euclidean space $\mathbb{R}^n$, where $n \geq 2$. We will denote a point of $\mathbb{R}^n$ by $x = (y, z)$, with y in $\mathbb{R}^{n-1}$, and z in $\mathbb{R}$. Let $\mathbb{R}^n_+$ be the open half-space, formed by all points such that $z > 0$, $\overline{\mathbb{R}}^n_+$ the closed half-space, formed by all points such that $z \geq 0$, and $\partial\mathbb{R}^n_+$ the boundary $\{z = 0\}$. It is natural, in the context of boundary-value problems, to introduce Sobolev spaces that are asymmetric with respect to the tangential variable y and the normal variable z. Let r and s be two real numbers and

$$H^{r,s}(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n),\ \|f\|^2_{r,s} := \int |\hat{f}(\xi)|^2 \left(1 + |\xi|^2\right)^r \left(1 + |\eta|^2\right)^s d\xi < +\infty \right\},$$

where $\hat{f}$ is the Fourier transform of f, and the Fourier variable ξ is decomposed as (η, ζ), with η in $\mathbb{R}^{n-1}$ and ζ in $\mathbb{R}$. Note that

$$H^{r,0}(\mathbb{R}^n) = H^r(\mathbb{R}^n), \quad H^{0,s}(\mathbb{R}^n) = L^2\big(\mathbb{R}_z, H^s(\mathbb{R}^{n-1}_y)\big).$$

The spaces $H^{r,s}(\mathbb{R}^n)$ restrict to spaces $H^{r,s}(\mathbb{R}^n_+)$, that may be defined as quotient spaces with respect to the equivalence relation

$$f \sim g \iff f_{\restriction z>0} = g_{\restriction z>0}.$$

Each space $H^{r,s}(\mathbb{R}^n_+)$ may be identified with a space of distribution over $\mathbb{R}^n_+$. This identification gives in particular

$$H^{r,0}(\mathbb{R}^n_+) = H^r(\mathbb{R}^n_+), \quad H^{0,s}(\mathbb{R}^n_+) = L^2\big((0, +\infty)_z, H^s(\mathbb{R}^{n-1}_y)\big).$$

Furthermore, we have

$$f \in H^{r,s}(\mathbb{R}^n_+) \text{ and } \chi \in C^\infty_0(\mathbb{R}^n) \Longrightarrow \chi f \in H^{r,s}(\mathbb{R}^n_+).$$

Thus the local spaces

$$H^{r,s}_{\text{loc}}(\mathbb{R}^n_+) = \left\{ f \in \mathcal{D}'(\mathbb{R}^n_+) \,\middle|\, \forall \chi \in C^\infty_0(\mathbb{R}^n),\ \chi f \in H^{r,s}(\mathbb{R}^n_+) \right\}$$

are well defined.

The properties of the spaces $H^{r,s}$ are similar to those of standard Sobolev spaces. We will use in particular that a differential operator of degree m with C^∞ coefficients on $\overline{\mathbb{R}}^n_+$ maps $H^{r+m,s}_{\text{loc}}(\mathbb{R}^n_+)$ to $H^{r,s}_{\text{loc}}(\mathbb{R}^n_+)$.

Denote by $C^\infty_0(\overline{\mathbb{R}}^n_+)$ the set of smooth functions with compact support in $\{z \geq 0\}$ (thus this functions do not need to be 0 on $\{z = 0\}$). This space is dense in each of the Sobolev space $H^{r,s}(\mathbb{R}^n_+)$. We now state two lemmas of Hörmander concerning these asymmetric spaces. The first one (which is Theorem B.2.7 in [13]) is a trace theorem.

Lemma 3.2. *If j is a natural number and r a real number such that $r > j + 1/2$, then the mapping*

$$C^\infty_0(\overline{\mathbb{R}}^n_+) \ni f \longmapsto \partial^j_z f(.,0)$$

can be extended continuously to a mapping from $H^{r,s}_{\text{loc}}(\mathbb{R}^n_+)$ to $H^{r+s-j-1/2}_{\text{loc}}(\{z = 0\})$.

The proof of Lemma 3.2 is a staightforward generalization of that of the classical trace theorem in Sobolev spaces H^r, $r > j + 1/2$.

The second lemma (see [13, Thm B.2.9]) shows the equivalence of the tangential variable y and global variable $x = (y, z)$ for solutions of partial differential equations transverse to the boundary.

Lemma 3.3. *Let $\mathbf{P}$ be a differential operator of order $N \geq 1$ with C^∞ coefficients on $\overline{\mathbb{R}}^n_+$*

$$\mathbf{P} = \sum_{0 \leq \alpha + |\beta| \leq N} a_{\alpha,\beta}(x) \partial^\alpha_z \partial^\beta_y,$$

with $a_{N,0} := 1$. *Let* r_1, r_2, s_1, s_2 *be some real numbers. Consider* f *such that*

$$f \in H^{r_1,s_1}_{\mathrm{loc}}(\mathbb{R}^n_+) \text{ and } \mathbf{P}f \in H^{r_2-N,s_2}_{\mathrm{loc}}(\mathbb{R}^n_+).$$

Let r, s *be real numbers such that*

$$r+s \leq r_j+s_j,\ j=1,2 \text{ and } r\leq r_2.$$

Then f *is in* $H^{r,s}_{\mathrm{loc}}(\mathbb{R}^n_+)$ *and satisfies the estimate*

$$\|\varphi f\|_{r,s} \lesssim \|\psi f\|_{r_1,s_1} + \|\psi \mathbf{P} f\|_{r_2-N,s_2},$$

where $\varphi \in C_0^\infty(\overline{\mathbb{R}}^n_+)$ *and* $\psi \in C_0^\infty(\overline{\mathbb{R}}^n_+)$ *is* 1 *on the support of* φ.

3.1.2. *A trace lemma for solutions of partial differential equations.*

Lemma 3.4. *Let* ω *be an open set of* $\mathbb{R}^n$ *with smooth boundary* $\partial\omega$. *Let* P *be a second-order differential operator with* C^∞ *coefficients in* $\overline{\omega}$, *which takes the following form near* $\partial\omega$

$$P = \partial_n^2 + Q_1\partial_n + Q_2, \tag{16}$$

where the Q_j*'s are differential operators of order* j, *tangential to* $\partial\omega$, *and* ∂_n *is the normal derivative with respect to* $\partial\omega$. *Let* q *be any real number,* $f \in H^q(\omega)$ *such that* $Pf \in H^k(\omega)$, *for some* $k > -1/2$. *Then the traces* $f_{\restriction\partial\omega}$ *and* $\partial_n f_{\restriction\partial\omega}$ *are well defined and*

$$f_{\restriction\partial\omega} \in H^{q-1/2}_{\mathrm{loc}}(\partial\omega), \quad \partial_n f_{\restriction\partial\omega} \in H^{q-3/2}_{\mathrm{loc}}(\partial\omega),$$

with the estimate

$$\|\chi f_{\restriction\partial\omega}\|_{H^{q-1/2}(\partial\omega)} + \|\chi\partial_n f_{\restriction\partial\omega}\|_{H^{q-3/2}(\partial\omega)} \leq C\left(\|f\|_{H^q(\omega)} + \|Pf\|_{H^k(\omega)}\right),$$

where χ *is any function in* $C_0^\infty(\partial\omega)$.

Proof. It is just a simple application of Lemma 3.2 and Lemma 3.3. Note that the result is an immediate consequence of the usual trace lemma when $q > 3/2$. We may thus assume that $q \leq 3/2$. Furthermore, it is sufficient to show that for all φ in $C_0^\infty(\overline{\omega})$ such that $\operatorname{supp}\varphi$ is small and close to the boundary, the traces of the function φf exist and satisfy the desired estimates. Using local geodesic normal coordinates, one may identify a small open subset $\mathcal{V}$ of $\overline{\omega}$ near the boundary with an open subset of $\overline{\mathbb{R}}^n_+$ located near $\{z=0\}$. On $\mathcal{V}$, the operator P takes, according to (16), the form

$$P = \partial_z^2 - Q_1\partial_z + Q_2.$$

We will show by an induction argument that for all real m such that $q+m \leq k+2$

$$\forall\varphi \in C_0^\infty(\mathcal{V}), \quad \varphi f \in H^{q+m,-m}(\overline{\mathbb{R}}^n_+) \text{ and}$$
$$\|\varphi f\|_{q+m,-m} \leq C_\varphi\left(\|f\|_{H^q(\omega)} + \|Pf\|_{H^k(\omega)}\right). \tag{17}$$

Note that if (17) is true for some m, it is also true for all smaller m. The assumptions of the lemma says that (17) is true for $m=0$. Assume that (17) is true for some $m \geq 0$. Let $m' \leq m+1$ such that $q+m' \leq k+2$. We have

$$[P,\varphi]f \in H^{q+m-1,-m}(\mathbb{R}^n_+), \quad \varphi P f \in H^{k,0}(\mathbb{R}^n_+),$$

and their norms are bounded by the norm of f in $H^q(\omega)$ and of Pf in $H^k(\omega)$. Furthermore

$$k \geq q-2, \quad k \geq q+m'-2, \quad q+m-1 \geq q+m'-2.$$

By Lemma 3.3 with $s_1 = -m$, $r_1 = q+m$, $r_2 = k+2$, $s_2 = 0$, $r = q+m'$ and $s = -m'$, we get (17) with m' instead of m. Iterating this argument, we get (17) for all m such that $q+m \leq k+2$.

Taking $m = k + 2 - q$, we get
$$\varphi f \in H^{k+2,q-k-2}(\mathbb{R}^n_+).$$
By assumption, $k + 2 > 3/2$. Lemma 3.2 yields the desired result. □

3.2. **Observation of the traces.** In this paragraph, we show that the solutions of (1a) satisfy

$$\|\varphi(t)\partial_t^2 v_{\restriction\gamma}\|^2_{H^{-1}((0,T)\times\gamma)} \lesssim \mathcal{Q}_T(U), \tag{18}$$

$$\|\varphi(t)\partial_t^2 \partial_n v_{\restriction\gamma}\|^2_{H^{-2}((0,T)\times\gamma)} \lesssim \mathcal{Q}_T(U), \tag{19}$$

where $\mathcal{Q}_T(U)$ is defined in (15), φ is a smooth cut-off function with support in $\{\psi = 1\}$, and which takes value 1 in a neighborhood of (α, β).

The function $\partial_t^2 u$ is solution of the heat equation in $(0,T) \times \Omega_1$, and the heat operator $-\partial_t + \Delta$ takes the form (16) near the boundary. Consider a smooth cut-off function ψ_1 on $\mathbb{R}$ such that

$$\operatorname{supp}\varphi \subset \{t \in \mathbb{R},\ \psi_1(t) = 1\}, \quad \operatorname{supp}\psi_1 \subset \{t \in \mathbb{R},\ \psi(t) = 1\}. \tag{20}$$

Thus, using Lemma 3.4, with $\omega := (0,T)\times\Omega_1$ (or rather, to satisfy exactly the smoothness assumption in this lemma, a smooth open neighborhood of $\operatorname{supp}\psi \times \Omega_1$ in $(0,T) \times \Omega_1$), we get

$$\begin{aligned}\|\varphi(t)\partial_t^2 u_{\restriction\gamma}\|_{H^{-1}((0,T)\times\gamma)} + \|\varphi(t)\partial_t^2\partial_n u_{\restriction\gamma}\|_{H^{-2}((0,T)\times\gamma)} &\lesssim \|\psi_1(t)\partial_t^2 u\|_{H^{-1/2}(\mathbb{R}\times\Omega_1)}\\ &\lesssim \|\psi_1(t)\partial_t^2 u\|_{H^{-1/2}(\mathbb{R},L^2(\Omega_1))}\\ &\lesssim \mathcal{Q}_T(U).\end{aligned}$$

Using the transmission condition in (BCa), we get (18) and (19).

3.3. **Observation of $\partial_t^2 v$.** The next step of the proof of Proposition 3.1 is to show that under the geometric control condition

$$\|\varphi(t)\partial_t^2 v\|^2_{H^{-1}((0,T)\times\Omega_2)} \lesssim \mathcal{Q}_T(U). \tag{21}$$

We first recall a theorem of Bardos, Lebeau and Rauch [1]:

Theorem 3.5 (Observability from the boundary)**.** *Let $s \in \mathbb{R}$. Assume that (γ, T') controls geometrically Ω_2. Consider w a solution of the wave equation in Ω_2*

$$\partial_t^2 w - \Delta w = 0, \quad (t,x) \in (0,T') \times \Omega_2 \tag{22}$$

$$w \in H^s((0,T') \times \Omega_2) \tag{23}$$

$$w_{\restriction\partial\Omega_2} = 0. \tag{24}$$

Then there exists a constant C, independent of w, such that
$$\|w\|_{H^s((0,T')\times\Omega_2)} \leq C\|\partial_n w_{\restriction\gamma}\|_{H^{s-1}((0,T')\times\gamma)}.$$

Indeed this is Corollary 3.7 in [1]. Note that in our case, the set of invisible solutions (solutions such that $\partial_n w_{\restriction\gamma}$ is 0 on $(0,T')$) is reduced to $\{0\}$. We now deduce from Theorem 3.5 an observability result when the Dirichlet boundary condition (24) is non-homogeneous.

Corollary 3.6. *Assume that (γ, T') controls geometrically Ω_2. Consider a solution w of the wave equation in Ω_2*

$$\partial_t^2 w - \Delta w = 0, \quad (t,x) \in (0,T') \times \Omega_2 \tag{25}$$

$$w \in H^{-1}((0,T') \times \Omega_2) \tag{26}$$

$$w_{\restriction\partial\Omega_2} = f \in H^{-1}((0,T') \times \partial\Omega_2), \quad \partial_{n_2} w_{\restriction\gamma} \in H^{-2}((0,T') \times \gamma). \tag{27}$$

Then there exists a constant C, independent of w, such that

$$\|\varphi(t)w\|_{H^{-1}((0,T')\times\Omega_2)} \leq C\left\{\|w_{\restriction\partial\Omega_2}\|_{H^{-1}((0,T')\times\partial\Omega_2)} + \|\partial_n w_{\restriction\gamma}\|_{H^{-2}((0,T')\times\gamma)}\right\}.$$

Note that the norm of $\partial_n w$ in the right-hand side is only taken over γ, whereas the norm of w has to be taken over all $\partial\Omega_2$.

Proof. Let ψ_1 be a smooth cut-off function on $\mathbb{R}$ satisfying (20). Denote by I the interval $\{\psi_1 = 1\}$, which contains the support of φ. Consider the solution $\tilde{w}$ of the wave equation with non-homogeneous Dirichlet boundary conditions

$$\begin{aligned} &\partial_t^2\tilde{w} - \Delta\tilde{w} = 0 \text{ on } (0,T')\times\Omega_2 \\ &\tilde{w}_{\restriction(0,T')\times\partial\Omega_2} = \psi_1(t)f, \quad \tilde{w}_{\restriction t=0} = 0, \quad \partial_t\tilde{w}_{\restriction t=0} = 0. \end{aligned} \tag{28}$$

By standard mixed problems theory, such a solution exists and satisfies the inequality

$$\|\tilde{w}\|_{H^{-1}((0,T')\times\Omega_2)} + \|\partial_n\tilde{w}_{\restriction\partial\Omega_2}\|_{H^{-2}((0,T')\times\partial\Omega_2)} \lesssim \|\psi_1(t)f\|_{H^{-1}((0,T')\times\partial\Omega_2)} \tag{29}$$

Let $W = w - \tilde{w}$ which is solution of the wave equation on $I\times\Omega_2$ with homogeneous Dirichlet boundary condition

$$\partial_t^2 W - \Delta W = 0 \text{ on } I\times\Omega_2, \quad W_{\restriction I\times\partial\Omega_2} = 0 \tag{30}$$

By Theorem 3.5

$$\|W\|_{H^{-1}(I\times\Omega_2)} \lesssim \|\partial_n W_{\restriction\gamma}\|_{H^{-2}(I\times\gamma)}. \tag{31}$$

By (29) and (31), one gets

$$\begin{aligned} \|\varphi(t)w\|_{H^{-1}((0,T')\times\Omega_2)} &\lesssim \|W\|_{H^{-1}(I\times\Omega_2)} + \|\tilde{w}\|_{H^{-1}(I\times\Omega_2)} \\ &\lesssim \|\partial_n W_{\restriction\gamma}\|_{H^{-2}((0,T')\times\gamma)} + \|f\|_{H^{-1}((0,T')\times\partial\Omega_2)} \\ &\lesssim \|\partial_n\tilde{w}_{\restriction\gamma}\|_{H^{-2}((0,T')\times\gamma)} + \|\partial_n w_{\restriction\gamma}\|_{H^{-2}((0,T')\times\gamma)} + \|f\|_{H^{-1}((0,T')\times\partial\Omega_2)} \\ &\lesssim \|\partial_n w_{\restriction\gamma}\|_{H^{-2}((0,T')\times\gamma)} + \|f\|_{H^{-1}((0,T')\times\partial\Omega_2)}, \end{aligned}$$

which shows the corollary. □

Using Corollary 3.6 together with (18) and (19), one gets (21).

3.4. End of the proof of Proposition 3.1.

Let $f := 1\!\!1_{\Omega_1}u + 1\!\!1_{\Omega_2}v$, which is a function defined on $(0,T)\times\Omega$ and satisfying homogeneous Dirichlet boundary conditions on Ω. By the transmission conditions (BCa), and the equations satisfied by u and v, we have

$$\partial_t^2 f + \Delta f = \underbrace{1\!\!1_{\Omega_1}(\partial_t u + \partial_t^2 u) + 2 1\!\!1_{\Omega_2}\partial_t^2 v}_{g}.$$

To get a bound on f in H^1, using this elliptic equation, we need to bound g in H^{-1}. Note that the bound on $\partial_t^2 v$ in $H^{-1}((0,T)\times\Omega_2)$ given by (21) does not immediately yield a bound on $1\!\!1_{\Omega_2}\partial_t^2 v$ in $H^{-1}((0,T)\times\Omega)$. In the next lemma, we use the equation satisfied by v to obtain such a bound. In this lemma, we take the liberty to shrink the support of φ, still keeping a function with support in $\{\psi = 1\}$ which takes value 1 in a neighborhood of (α,β).

Lemma 3.7. *The function $\varphi(t)g$ is in $H^{-1}(\mathbb{R}\times\Omega)$, and its norm in this space is bounded by $C\mathcal{Q}_T(U)$ for a certain constant C independent of the solution.*

Proof. Indeed we shall show that each of the three terms in $\varphi(t)g$ satisfies this property. First note that for solutions with initial condition in $D(\mathcal{A}_a)$, both $\varphi(t)\partial_t u$ and $\varphi(t)\partial_t^2 u$ are in $H^{-1/2}(\mathbb{R}, L^2(\Omega_1))$ and satisfy the elementary bound

$$\begin{aligned} \|\varphi(t)\partial_t u\|^2_{H^{-1/2}(\mathbb{R},L^2(\Omega_1))} + \left\|\varphi(t)\partial_t^2 u\right\|^2_{H^{-1/2}(\mathbb{R},L^2(\Omega_1))} &\lesssim \|\psi(t)\partial_t u\|^2_{H^{1/2}(\mathbb{R},L^2(\Omega_1))} \\ &\lesssim \mathcal{Q}_T(U). \end{aligned}$$

Thus

$$\|\varphi(t)\mathbb{1}_{\Omega_1}\partial_t u\|^2_{H^{-1/2}(\mathbb{R},L^2(\Omega))} + \left\|\varphi(t)\mathbb{1}_{\Omega_1}\partial_t^2 u\right\|^2_{H^{-1/2}(\mathbb{R},L^2(\Omega))} \lesssim \mathcal{Q}_T(U). \tag{32}$$

Obviously $H^{-1/2}(\mathbb{R}, L^2(\Omega))$ is included in $H^{-1}(\mathbb{R}\times\Omega)$, which yields the desired bound on both $\partial_t u$ and $\partial_t^2 u$.

The case of $\partial_t^2 v$ is a little more delicate. Note that if χ is a smooth function on Ω_2 with support away from γ, then, according to (21),

$$\left\|\mathbb{1}_{\Omega_2}\varphi(t)\chi\partial_t^2 v\right\|^2_{H^{-1}(\mathbb{R}\times\Omega)} \lesssim \mathcal{Q}_T(U). \tag{33}$$

It remains to show (33) when χ has small support near γ. In a neighborhood of the support of such a χ, we use local geodesic normal coordinates (y, z), where $y = (t, y')$ in $\mathbb{R}^d$ is the tangential space-time variable, and z is the distance to γ, and introduce the asymmetric Sobolev spaces of paragraph 3.1 (with $n = d + 1$). The distribution $\partial_t^2 v$ is solution of an equation $Pf = 0$, with P taking the form (16). According to (21)

$$\chi(x)\varphi(t)\partial_t^2 v \in H^{-1,0}\big(\mathbb{R}^{d+1}_+\big), \quad \left\|\chi(x)\varphi(t)\partial_t^2 v\right\|^2_{-1,0} \lesssim \mathcal{Q}_T(U).$$

In view of Lemma 3.3 it is easy to show that

$$\chi(x)\varphi(t)\partial_t^2 v \in H^{0,-1}\big(\mathbb{R}^{d+1}_+\big) \quad \left\|\chi(x)\varphi(t)\partial_t^2 v\right\|^2_{0,-1} \lesssim \mathcal{Q}_T(U).$$

This is an L^2 space in the normal variable z. Hence

$$\mathbb{1}_{\{z>0\}}\chi(x)\varphi(t)\partial_t^2 v \in L^2(\mathbb{R}, H^{-1}(\mathbb{R}^d)),$$

with the usual bound of its norm by $\mathcal{Q}_T(U)$. Noticing that the space $H^{-1}(\mathbb{R}^{d+1})$ contains $L^2(\mathbb{R}, H^{-1}(\mathbb{R}^d))$, and going back to the original system of coordinates, we also get (33) when χ is supported near γ, which completes the proof of the lemma. $\square$

Now we have

$$(\partial_t^2 + \Delta)(\varphi(t)f) = \underbrace{\varphi''(t)f + 2\varphi'(t)\partial_t f + \varphi(t)g}_{h} \tag{34}$$

Noticing that f is trivially bounded in L^2 in a neighborhood of $(\alpha, \beta)\times\Omega$ by $Q_T(U)$ and using the preceding lemma, we get

$$\|h\|^2_{H^{-1}(\mathbb{R}\times\Omega)} \lesssim Q_T(U).$$

Equation (34) is elliptic, with homogeneous Dirichlet boundary on any open set of $\mathbb{R}\times\Omega$ which contains $(0, T)\times\Omega$. By standard elliptic theory, we get in particular

$$\|f\|_{H^1((\alpha,\beta)\times\Omega)} \lesssim Q_T(U).$$

This yields the observability inequality

$$\int_\alpha^\beta E(U, t)dt \lesssim Q_T(U).$$

The function $E(U, t)$ being time-decreasing, we obtain (15). The proof of Proposition 3.1 is complete.

4. From high frequency observability to observability

In this part, we show that (15) implies (7), that is, for $T > T'$

$$E(U, t = T) \lesssim \|\partial_t u\|^2_{H^{1/2}(0,T;L^2(\Omega_1))}, \tag{7}$$

which concludes the proof of Theorem 1. We will use a classical uniqueness/compactness argument (see e.g. [19]).

Note that according to (11), for every solution of (1a) with initial condition in H_a, we have

$$\int_0^T \|\partial_t u\|^2_{L^2(\Omega_1)} dt \leq \|U_0\|^2_{H_a}.$$

Thus the mapping

$$S_a : U_0 \longmapsto (u(t), v(t)),$$

where (u, v) is the solution of (1a) with initial condition $U_0 = (u_0, v_0, v_1)$, is continuous from H_a to

$$X_T := \big\{(u,v), \quad u \in C^0(0,T;H^1(\Omega_1)),\ \partial_t u \in L^2((0,T)\times\Omega_1),\\ v \in C^0(0,T;H^1(\Omega_2)) \cap C^1(0,T;L^2(\Omega_2))\big\}.$$

As a consequence, S_a is a **compact** map from H_a to

$$Y_T := L^2\big((0,T)\times\Omega_1\big) \ \times\ L^2\big((0,T)\times\Omega_2\big).$$

Assume that (7) does not hold. Then there is a sequence (U_0^k) in $D(\mathcal{A}_a)$ such that

$$1 = \|U^k(T)\|_{H_a} > k\|\partial_t u^k\|_{H^{1/2}(0,T;L^2(\Omega_1))}. \tag{35}$$

(where as usual $U^k(t) = \big(u^k(t), v^k(t), \partial_t v^k(t)\big)$ is the solution of (1a) with initial condition U_0^k). According to (35) and the energy dissipation law (11), U_0^k is bounded in H_a. Thus, up to the extraction of a subsequence, one may assume that there is a $\widetilde{U}_0$ in H_a such that

$$U^k \underset{k\to\infty}{\rightharpoonup} \widetilde{U}_0 \text{ in } H_a. \tag{36}$$

By continuity of S_a

$$(u^k, v^k) \underset{k\to\infty}{\rightharpoonup} (\tilde{u}, \tilde{v}) \text{ in } X_T$$

where $(\tilde{u}, \tilde{v}) = S_a(\widetilde{U}_0)$. Using (35), this shows that $(\tilde{u}, \tilde{v})$ is in the vector space G_T of solutions of (1a) such that $\partial_t u = 0$ on $(0,T)\times\Omega_1$. Let us show that G_T is reduced to $\{0\}$. Let (u, v) be an element of G_T. Then

$$\partial_t u = 0 \text{ in } (0,T)\times\Omega_1. \tag{37}$$

Thus by (BCa), one has

$$\partial_t v_{\restriction(0,T)\times\partial\Omega_2} = 0, \quad \partial_{n_2}\partial_t v_{\restriction(0,T)\times\gamma} = 0.$$

The function $\partial_t v$ is a (weak) solution of the wave equation on $(0,T)\times\Omega_2$ with Dirichlet boundary condition, whose traces vanish on γ. It is well known (see [1]), that the geometric control of Ω_2 by $(0,T')\times\gamma$ implies

$$\partial_t v = 0 \text{ in } (0,T)\times\Omega_2. \tag{38}$$

Let $f := \mathbb{1}_{\Omega_1} u + \mathbb{1}_{\Omega_2} v$. According to (37), (38) and (BCa)

$$\Delta f = 0, \text{ in } (0,T)\times\Omega$$
$$f = 0, \text{ in } (0,T)\times\partial\Omega.$$

Thus $f = 0$. This shows as announced that $G_T = \{0\}$.

Consequently, $(\tilde{u},\tilde{v})$ is identically zero. The compactness of S_a from H_a to Y_T yields together with the weak convergence (36)

$$(u^k,v^k)\underset{k\to\infty}{\longrightarrow} 0 \text{ in } Y_T.$$

Using high-frequency inequality (15) together with (35), we obtain that $U^k(T)$ goes to 0 in H_a as k goes to ∞, which contradicts the equality in (35). The proof of Theorem 1 is complete.

5. From observability inequalities to polynomial decay

In this section we show that Theorem 1 implies Theorem 2. The strategy of the proof is very classical and goes back to the work of Russell [23]. To get the expected decay rate at the speed $1/t^2$, we would need the following observability estimate (omitting the subscripts a, as we shall do in all the section)

$$\|U(T)\|_H^2 \lesssim \|U(0)\|^2_{D(\mathcal{A}^{1/2})} - \|U(T)\|^2_{D(\mathcal{A}^{1/2})}. \tag{39}$$

Estimate (7) is very similar, but we were not able to prove that it implies exactly (39). Note that when the loss of time-derivatives is an integer, the analogues of estimates (7) and (39) are trivially the same. In this section we use a trick to avoid the technical obstacle caused by fractional time-derivatives, and show the decay of the energy at the speed $1/t^s$, $s<2$ when the initial condition is in $D(\mathcal{A})$.

Assume that the hypothesis of Theorem 1 are fulfilled and consider a solution U of (1a) with regular initial condition. By Theorem 1, (and the fact that the energy decreases with time) we have, for T large enough and some well-chosen φ in $C_0^\infty((0,T))$

$$E(U,T) \lesssim \|\varphi(t)\partial_t u\|^2_{H_T^{1/2}L^2(\Omega_1)}. \tag{40}$$

The interpolation inequalities

$$\|f\|_{H^{s_0}(\mathbb{R})} \leq \|f\|^{\alpha_1}_{H^{s_1}(\mathbb{R})}\|f\|^{\alpha_2}_{H^{s_2}(\mathbb{R})},$$
$$\alpha_1 s_1+\alpha_2 s_2 = s_0,\ \alpha_1+\alpha_2=1,\ \alpha_1,\alpha_2\in(0,1),$$

hold for Hilbert-valued functions by Fourier transform. Let $N\in\mathbb{N}^*$ and $\beta := 1-\frac{1}{2N}$. In the sequel we will write

$$H_T^s B = H^s(0,T;B)$$

where B is a Hilbert space, T and s are real nonnegative numbers. Estimate (40) yields

$$E(U,T) \lesssim \|\partial_t u\|^{2(1-\beta)}_{H_T^N L^2(\Omega_1)}\|\partial_t u\|^{2\beta}_{L_T^2 L^2(\Omega_1)}. \tag{41}$$

Consider the natural norm on $D(\mathcal{A}^N)$

$$\|F\|^2_{D(\mathcal{A}^N)} := \|F\|_H^2 + \|\mathcal{A}F\|_H^2 + \ldots + \left\|\mathcal{A}^N F\right\|_H^2.$$

It is easy to check that (11) implies

$$\|\partial_t u\|^2_{H_T^N L^2(\Omega_1)} = \|U(0)\|^2_{D(\mathcal{A}^N)} - \|U(T)\|^2_{D(\mathcal{A}^N)} \leq \|U(0)\|^2_{D(\mathcal{A}^N)}.$$

From (41), we get

$$\|U(T)\|_H^2 \lesssim \|U(0)\|^{2(1-\beta)}_{D(\mathcal{A}^N)}\left(\|U(0)\|_H^2 - \|U(T)\|_H^2\right)^\beta. \tag{42}$$

Let $a_k := \|U(kT)\|_H^2$. The inequality (42) with initial time kT yields, taking into account the decay of U in $D(\mathcal{A}^N)$

$$a_{k+1}^{1/\beta} \leq M(a_k - a_{k+1}) \tag{43}$$

where the constant $M>0$ depends on $\|U(0)\|_{D(\mathcal{A}^N)}$. The following lemma is classical in this setting (see [23]):

Lemma 5.1. *Let (a_k) be a sequence of positive number satisfying (43) with $\beta\in(0,1)$. Then there is a constant C such that*

$$a_k \leq Ck^{-\alpha}, \quad \alpha := \frac{\beta}{1-\beta}.$$

Thus, we obtain, when U_0 is in $D(\mathcal{A}^N)$

$$\|U(t)\|_H^2 \leq \frac{C(U_0)}{t^\alpha}$$

An easy application of the closed graph theorem yields

$$\|U(t)\|_H^2 \leq \frac{C}{t^\alpha}\|U_0\|^2_{D(\mathcal{A}^N)},$$

where the constant C does not depend on U. Interpolating with the decay bound

$$\|U(t)\|_H^2 \leq \|U(0)\|_H^2,$$

we get

$$\|U(t)\|_H^2 \leq \frac{C}{t^{\frac{\alpha}{N}}}\|U_0\|^2_{D(\mathcal{A})}. \tag{44}$$

Furthermore $\frac{\alpha}{N}=2-\frac{1}{N}$. Thus it may be chosen as close to 2 as desired. The proof of Theorem 2 is complete.

6. The case of natural transmission conditions

In this section, we follow the steps of the two previous ones, showing first a high frequency observation inequality, then Theorems 1 and 2 for System (1b). We will keep exactly the same notations as before but $U=(u,v,\partial_t v)$ will now always denote a solution of (1b). We will assume as announced in Theorems 1 and 2 that Γ_1 is not empty, so that

$$\forall u_0\in H^1(\Omega_1) \text{ s.t. } u_{0\restriction\Gamma_1}=0, \quad \|u_0\|_{H^1(\Omega_1)}\lesssim \|\nabla u_0\|_{L^2(\Omega_1)}.$$

6.1. **High frequency observability inequality.** The next proposition is the analogue to Proposition 3.1.

Proposition 6.1. *Under the assumptions of Theorem 1, consider ψ in $C_0^\infty(0,T)$ which takes value 1 on an interval (α,β) of length more than T'. Then*

$$\|U(T)\|_{H_b}^2 \lesssim \underbrace{\|\nabla u\|^2_{H_T^{1/2}L^2(\Omega_1)} + \|\psi(t)u\|^2_{H^{-1}((0,T)\times\Omega_1)} + \|\psi(t)v\|^2_{L^2((0,T)\times\Omega_2)}}_{\mathcal{Q}_T(U)}. \tag{45}$$

The strategy of the proof of (45) is exactly the same than the one of (15): to use Lemma 3.4 to control some traces of v on γ (Paragraph 6.1.1), and then to use the geometric condition to control v in the energy space (Paragraph 6.1.2). As before, we will rather control first $\partial_t^2 v$ in H^{-1}. The compact terms in the right-hand side will appear when converting this estimate on $\partial_t^2 v$ into the estimate on the energy of v.

6.1.1. *Control of the traces.* Here we show the following

$$\|\varphi(t)\partial_t^2 v_{\restriction\gamma}\|^2_{H^{-1}((0,T)\times\gamma)} \lesssim \mathcal{Q}_T(U), \tag{46}$$

$$\|\varphi(t)\partial_n\partial_t^2 v_{\restriction\gamma}\|^2_{H^{-2}((0,T)\times\gamma)} \lesssim \mathcal{Q}_T(U), \tag{47}$$

$$\|\varphi(t)\partial_n v_{\restriction\gamma}\|^2_{H^{-1/2}((0,T)\times\gamma)} \lesssim Q_T(U), \tag{48}$$

where φ is as before a function with compact support in $(0,T)$ and taking value 1 near (α,β). Consider the global system of space coordinate $x=(x_1,x_2,..,x_n)$ in which $\Delta=\sum\partial^2_{x_j}$. Using Lemma 3.4 as in paragraph 3.2, we get, for any j

$$\|\varphi(t)\partial_{x_j}\partial_t u_{\restriction\gamma}\|^2_{H^{-1}((0,T)\times\gamma)} \lesssim \mathcal{Q}_T(U). \tag{49}$$

The normal derivative ∂_{n_1} may be extended, near a point x' of γ, to a vector field in the interior of Ω_1, of the form

$$\partial_{n_1}=\sum_j a_j(x)\partial_{x_j},$$

where $a_1,..,a_n$ are C^∞ in a neighborhood of x'. Thus (49) implies, together with the transmission condition $\partial_{n_1}u_{\restriction\gamma}=-\partial_{n_2}v_{\restriction\gamma}$, estimate (47).

Under the assumption that Γ_1 is not empty, and using the fact that u vanishes on Γ_1, there is a constant C such that

$$\forall t\geq 0,\quad \|u(t)\|_{L^2(\Omega_1)}\leq C\|\nabla u(t)\|_{L^2(\Omega_1)}.$$

Hence

$$\|\psi(t)u\|^2_{H^{1/2}(\mathbb{R},L^2(\Omega_1))}\lesssim \mathcal{Q}_T(U).$$

By Lemma 3.4, we get

$$\|\varphi(t)\partial_t u\|^2_{H^{-1}((0,T)\times\gamma)}\lesssim \mathcal{Q}_T(U),$$

which yields (46) with the transmission condition $u_{\restriction\gamma}=\partial_t v_{\restriction\gamma}$. The estimate (48) is easily deduced from Lemma 3.4 and the bound

$$\|\nabla u\|^2_{L^2((0,T)\times\Omega_1)}\lesssim \mathcal{Q}_T(U).$$

6.1.2. *Observation in Ω_1 and Ω_2.* Corollary 3.6 yields, together with (46) and (47)

$$\|\varphi(t)\partial_t^2 v\|^2_{H^{-1}((0,T)\times\Omega_2)}\lesssim Q_T(U). \tag{50}$$

We now show that (48), (50) and the Dirichlet boundary condition on Γ_2 suffice to observe all the energy of v with $\mathcal{Q}_T(U)$.

Let $\tilde\varphi$ be a C^∞, cut-off function with support in $\varphi=1$. Then we have

$$\big(\partial_t^2+\Delta_x\big)\big(\tilde\varphi(t)v\big)=2\tilde\varphi(t)\partial_t^2 v+2\tilde\varphi'(t)\partial_t v+\tilde\varphi''(t)v. \tag{51}$$

Using (50) to bound, in $H^{-1}((0,T)\times\Omega_2)$, the first term of the right member of (51), and the norm of $\psi(t)v$ in $L^2((0,T)\times\Omega_1)$ to bound the two other terms, we get

$$\begin{gathered}\big(\partial_t^2+\Delta_x\big)\big(\tilde\varphi(t)v\big)=f,\quad \|f\|^2_{H^{-1}((0,T)\times\Omega_2)}\lesssim \mathcal{Q}_T(U)\\ \tilde\varphi(t)v_{\restriction\mathbb{R}\times\Gamma_2}=0,\quad \tilde\varphi(t)\partial_n v_{\restriction\mathbb{R}\times\gamma}=g,\quad \|g\|^2_{H^{-1/2}(\mathbb{R}\times\gamma)}\lesssim \mathcal{Q}_T(U).\end{gathered} \tag{52}$$

Consider a bounded, smooth, open subset of $\mathbb{R}\times\Omega_2$ which contains $(0,T)\times\Omega_2$. Equation (52) is elliptic, with homogeneous Dirichlet boundary conditions on $\mathbb{R}\times\Gamma_2$ and Neumann inhomogeneous boundary condition in $\mathbb{R}\times\gamma$. It is well known that it implies

$$\|\tilde\varphi(t)v\|^2_{H^1((0,T)\times\Omega_2)}\lesssim \mathcal{Q}_T(U).$$

Using that

$$\|u\|_{L^2((0,T)\times\Omega_1)}\lesssim\|\nabla u\|_{L^2((0,T)\times\Omega_1)},$$

we get

$$\int_\alpha^\beta \|U(t)\|_{H_b}^2 dt \lesssim Q_T(U),$$

which yields, with (14), the high frequency observability inequality (45).

6.2. **From high-frequency observability to observability.** In this section we show, adapting the uniqueness-compactness argument of Section 4, the observation inequality

$$E_b(U(T)) \lesssim \|\nabla u\|^2_{H^{1/2}(0,T;L^2(\Omega_1))}, \tag{8}$$

for solutions of System (1b). This will conclude the proof of Theorem 1 in the case of (1b).

The only new difficulty is when $\Gamma_2 = \emptyset$ (so that $\sqrt{E_b}$ is not a norm on H_b), which we shall assume in the sequel. The proof of the case $\Gamma_2 \neq \emptyset$ is exactly the same as in Section 4 and therefore we omit it.

The map

$$S_b : U_0 = (u_0, v_0, v_1) \mapsto (u(t), v(t)),$$

where $(u(t), v(t))$ is the solution of (1b) with initial conditions (u_0, v_0, v_1), is continuous from H_b to

$$C^0\big([0,T]; L^2(\Omega_1)\big) \times \Big\{C^1\big([0,T]; L^2(\Omega_2)\big) \cap C^0\big([0,T]; H^1(\Omega_2)\big)\Big\}$$

thus compact from H_b to

$$Y_T := H^{-1}\big((0,T)\times\Omega_1\big) \times L^2\big((0,T)\times\Omega_2\big).$$

Assume that (8) does not hold. Then there is a sequence $\big(U_0^k\big)_k$ of elements of H_b such that

$$1 = E_b(U^k(T)) > k\big\|\nabla u^k\big\|^2_{H^{1/2}(0,T;L^2(\Omega_1))}. \tag{53}$$

where $U^k(t) = \big(u^k(t), v^k(t), \partial_t v^k(t)\big)$ is the solution of (1b) with initial condition U_0^k. Replacing, if necessary, $(u^k(t,x), v^k(t,x))$ by

$$\Big(u^k(t,x), v^k(t,x) - \frac{1}{\mu(\Omega_2)}\int_{\Omega_2} v^k(0,x)dx\Big), \quad \mu(\Omega_2) = \int_{\Omega_2} dx$$

which is also a solution of (1b), we may assume

$$\int_{\Omega_2} v^k(0,x)dx = 0.$$

Furthermore

$$\begin{aligned}\left|\partial_t \int_{\Omega_2} v^k(t,x)dx\right| &= \left|\int_{\Omega_2} \partial_t v^k(t,x)dx\right| \\ &\leq \Big\{2\mu(\Omega_2)E_b(U^k(t))\Big\}^{1/2}.\end{aligned}$$

According to the decay law (12) and to (53), $E_b(U^k)$ is bounded independently of $t \in (0,T)$ and $k \in \mathbb{N}$. Thus

$$\exists C > 0, \quad \forall t \in [0,T], \quad \forall k \in \mathbb{N}, \quad \left|\int v^k(t,x)dx\right| \leq C. \tag{54}$$

By (53) and (54)

$$\exists C > 0, \quad \forall t \in [0,T], \quad \forall k \in \mathbb{N}, \quad \big\|v^k(t)\big\|_{H^1(\Omega_2)} \leq C.$$

Thus in view of (53), $\left(U_0^k\right)$ is bounded in H_b. Up to the extraction of a subsequence, there is a $\tilde{U}_0=(\tilde{u}_0,\tilde{v}_0,\tilde{v}_1)$ in H_b such that

$$U_0^k \underset{k\to\infty}{\rightharpoonup} \tilde{U}_0 \text{ weakly in } H_b.$$

The map S_b being compact from H_b to Y_T, (u^k,v^k) converges strongly in Y_T to the solution $(\tilde{u},\tilde{v})$ of (1b) with initial condition $\tilde{U}_0$. According to (53)

$$\nabla u^k \underset{k\to\infty}{\longrightarrow} 0 \text{ in } H^{1/2}(0,T;L^2(\Omega_1)).$$

Thus $\nabla\tilde{u}$ is identically 0 on $(0,T)\times\Omega_1$. By similar arguments as in Section 4, one may deduce from this fact

$$\exists C_0>0 \text{ s.t. } \tilde{u}=0,\quad \tilde{v}=C_0,\quad (t,x)\in(0,T)\times\Omega_2. \tag{55}$$

Furthermore,

$$\int_{\Omega_2} v_0^k dx=(v_0^k,1)_{L^2(\Omega_2)},$$

so that by weak convergence of v_0^k in H_b,

$$0=\int_{\Omega_2} v_0^k dx \underset{k\to\infty}{\longrightarrow} \int_{\Omega_2}\tilde{v}_0 dx.$$

Thus $C_0=0$, and $\tilde{U}=0$. This shows by the high frequency inequality (45) and the strong convergence of (u^k,v^k) to $(\tilde{u},\tilde{v})$ in Y_T that

$$U^k(T)\underset{k\to\infty}{\longrightarrow} 0 \text{ in } H_b,$$

which contradicts (53). Hence (8) holds, which concludes the proof of Theorem 1.

6.3. **Polynomial decay.** The proof of the polynomial decay for System (1b) from observability inequality (8) is exactly the same as in the case of System (1a). We therefore omit the proof of Theorem 2 for System (1b).

7. Logarithmic decay

In this section we prove Theorem 3. We start by recalling, from [4] that it suffices to show some high-frequency estimates with exponential loss on the resolvent. We will then deduce this estimates from known Carleman inequalities.

7.1. **Preliminaries.** Treating separately each connex component of Ω, we may assume that Ω is connex, which we shall do in the sequel.

The proof of the logarithmic decay relies on an abstract result of N. Burq, which links it to estimates on the resolvent of $\mathcal{A}_a$. Let $\mathcal{A}$ be any maximal dissipative operator, with dense domain $D(\mathcal{A})$, in a Hilbert space H. Denote by $\mathcal{R}(\mathcal{A})$ the resolvent set of $\mathcal{A}$ and, for complex numbers λ in $\mathcal{R}(\mathcal{A})$, by R_λ the resolvent $(\mathcal{A}-\lambda)^{-1}$. Then the following holds (see [4, Thm 3])

Theorem 7.1. *Let $D>0$, and*

$$\mathcal{O}_D:=\left\{\lambda\in\mathbb{C},\ |Re\lambda|<D^{-1}e^{-D|\mathrm{Im}\,\lambda|}\right\}.$$

Assume that for some $D>0$, $\mathcal{O}_D$ is included in $\mathcal{R}(\mathcal{A})$, and that in $\mathcal{O}_D$ there is a positive constant C such that

$$\|R_\lambda\|_{\mathcal{L}(H)}\leq Ce^{C|\mathrm{Im}\,\lambda|}. \tag{56}$$

Then for all k there exists C_k such that

$$\forall u_0 \in D(\mathcal{A}^k),\ \|e^{t\mathcal{A}}u_0\|_H \leq \frac{C_k}{(\log(t+2))^k}\|u_0\|_{D(\mathcal{A}^k)}.$$

Theorem 3 of [4] is written in a more general setting. To check the exact assumptions of this result, recall that the resolvent R_λ is analytic in the resolvent set, as a function with values in $\mathcal{L}(\mathcal{H})$.

The main difficulty is the high frequency problem, which we will solve using some Carleman inequalities due to G. Lebeau and L. Robbiano (see [16]) and in this exact form to N. Burq in [4]. This is the object of the following proposition:

Proposition 7.2. *Assume that Ω is connex and Ω_1 non-empty. Then there exists $C_0 > 0$ such that for every $\mu \in \mathbb{R}$ with $|\mu| \geq C_0$,*

$$\|(\mathcal{A}_a - i\mu)^{-1}\|_{\mathcal{L}(H_a)} \leq C_0 e^{C_0|\mu|}. \tag{57}$$

In all the sequel we shall omit the subscripts a, writing H and $\mathcal{A}$ for H_a and $\mathcal{A}_a$. We shall prove Proposition 7.2 in Subsection 7.2. Let us first check that it implies the assumptions of Theorem 7.1. All the imaginary axis is included in $\mathcal{R}(\mathcal{A})$, which is an open set. Thus, for small enough ε

$$Q_\varepsilon := \{\lambda \in \mathbb{C},\ |\operatorname{Re}\lambda| \leq \varepsilon,\ |\operatorname{Im}\lambda| \leq C_0\} \subset \mathcal{R}(\mathcal{A}),$$

where C_0 is given by Proposition 7.2. Note that by continuity, R_λ is bounded in Q_ε.

Let us now consider $\nu \in \mathbb{C}$, $\mu := \operatorname{Im}\nu$, such that

$$|\mu| \geq C_0, \quad |\operatorname{Re}\nu| \leq \frac{1}{2C_0}e^{-C_0|\mu|}. \tag{58}$$

Then, denoting as before $R_{i\mu} := (\mathcal{A} - i\mu)^{-1}$,

$$R_{i\mu}(\mathcal{A} - \nu) = 1 + (i\mu - \nu)R_{i\mu}.$$

Furthermore

$$\begin{aligned}\|(i\mu - \nu)R_{i\mu}\|_{\mathcal{L}(H)} =& |\operatorname{Re}\nu|\ \|R_{i\mu}\|_{\mathcal{L}(H)}\\ \leq& |\operatorname{Re}\nu| C_0 e^{C_0|\mu|} \leq \frac{1}{2}\end{aligned}$$

by (57) and (58). By the Neumann expansion's theorem in the Banach algebra $\mathcal{L}(H)$, $R_{i\mu}(\mathcal{A} - \nu)$ is invertible and the norm of its inverse $R_\nu(A - i\mu)$ satisfies

$$\|R_\nu(\mathcal{A} - i\mu)\|_{\mathcal{L}(H)} \leq 2.$$

Using again (57), we get

$$\|R_\nu\|_{\mathcal{L}(H)} \leq 2C_0 e^{C_0|\mu|}. \tag{59}$$

Take $D > 0$ such that D^{-1} is less than ε and $\frac{1}{2C_0}$. The argument above shows that $\mathcal{O}_D$ is included in $\mathcal{R}(\mathcal{A})$. Inequality (59) and the boundedness of R_λ on Q_ε yield the estimate (56). The assumptions of Theorem 7.1 are fulfilled. It remains to show Proposition 7.2 to conclude the proof of Theorem 3.

7.2. **Proof of Proposition 7.2.**

Proposition 7.2 is a consequence of the following known result:

Proposition 7.3 (High frequency Carleman inequalities)**.** *Let ω be a non-empty, open subset of Ω. Then there is a constant $C > 0$ such that for every large positive number μ, and for every w in $H^2(\Omega)$, with $w_{\restriction\partial\Omega} = 0$*

$$\|(\Delta + \mu^2)w\|^2_{L^2(\Omega)} + \|w\|^2_{H^1(\omega)} \geq e^{-C\mu}\|w\|^2_{H^1(\Omega)}. \tag{60}$$

Sketch of the proof. To prove the proposition we will need to recall the Carleman inequality of N. Burq, G. Lebeau and L. Robbiano. Consider a bounded smooth open set $\mathcal{U}$ of $\mathbb{R}^n$, φ a function in $C^\infty(\overline{\mathcal{U}})$, and

$$p_\varphi(x,\xi) := (\xi + i\nabla\varphi(x)).(\xi + i\nabla\varphi(x)) - 1,$$

(where $X.Y$ is the scalar product $\sum X_i Y_i$). Let $h>0$ be a small parameter. The function p_φ, defined on $\overline{\mathcal{U}}\times\mathbb{R}^n$, is the semi-classical principal symbol of the semi-classical operator $-e^{\varphi/h}\circ(h^2\Delta+1)\circ e^{-\varphi/h}$. For two functions f and g on $\overline{\mathcal{U}}\times\mathbb{R}^n$, define the Poisson bracket of f and g by $\{f,g\} := \sum_j \partial_{\xi_j} f\partial_{x_j} g - \partial_{x_j} f\partial_{\xi_j} g$. Then we have the following (see [4, prop 1.2]):

Proposition 7.4. *Let Γ be an non-empty union of connex components of $\partial\mathcal{U}$. Assume that φ satisfies*

$$\forall (x,\xi)\in\overline{\mathcal{U}}\times\mathbb{R}^d,\ \nabla\varphi\neq 0, \tag{61}$$

$$\forall x\in\partial\mathcal{U},\ \partial_n\varphi(x)\neq 0, \tag{62}$$

$$\forall x\in\Gamma,\ \partial_n\varphi(x)<0, \tag{63}$$

$$\exists c>0,\ \forall (x,\xi)\in\overline{\mathcal{U}}\times\mathbb{R}^n,\ p_\varphi(x,\xi)=0\Rightarrow\{\mathrm{Re}\,p_\varphi,\mathrm{Im}\,p_\varphi\}(x,\xi)\geq c. \tag{64}$$

Then there exists $C>0$ such that for all $u\in H^2(\mathcal{U})$ with $u_{\restriction\Gamma}=0$ and for all small, positive h

$$\int_U e^{2\varphi/h}|h^2\Delta u+u|^2dx + h\int_{\partial\mathcal{U}\backslash\Gamma} e^{2\varphi/h}\left(|u_{\restriction\partial\mathcal{U}}|^2+|h\nabla u_{\restriction\partial\mathcal{U}}|^2\right)dx \geq C^{-1}h\int_{\mathcal{U}} e^{2\varphi/h}(|u|^2+|h\nabla u|^2)dx. \tag{65}$$

Remark 7.5. If u is supported away from $\partial\mathcal{U}\backslash\Gamma$, inequality (65) is valid even if φ does not satisfy (62).

Let $\tilde\omega$ be a small ball such that $\overline{\tilde\omega}\subset\omega$, and χ a smooth function such that

$$\chi(x)=1 \text{ if } x\in\Omega\backslash\omega,\quad \chi(x)=0 \text{ if } x\in\tilde\omega.$$

Let $\mathcal{U}:=\Omega\backslash\tilde\omega$, and $\Gamma:=\partial\Omega$. The function χ clearly vanishes on $\partial\mathcal{U}\backslash\Gamma=\partial\tilde\omega$. Assume that there exists a function φ satisfying (61), (63) and (64). Then for any function $w\in H^2(\Omega)$ vanishing at $\partial\Omega$, we have, for small h

$$C\int_\Omega e^{2\varphi/h}|h^2\Delta(\chi w)+\chi w|^2dx \geq h\int_\Omega e^{2\varphi/h}(|\chi w|^2+|h\nabla(\chi w)|^2)dx.$$

Let $h=\mu^{-1}$. We may replace φ by its maximum at the left-hand side of the equation, and by its minimum at the right hand side. Dropping the powers of μ (which may be done by taking a greater constant in the exponential), and using that the commutator $[\Delta,\chi]$ is of order 1, supported in $\omega\backslash\tilde\omega$ and that χ is 1 outside of ω, we obtain, for large μ

$$e^{C\mu}\left\{\int_\Omega|\Delta w+\mu^2 w|^2dx+\int_{\omega\backslash\tilde\omega}(|\nabla w|^2+|w|^2)dx\right\}\geq\int_\Omega(|w|^2+|\nabla w|^2)dx$$

which yields (60).

It remains to show that there is a function φ satisfying (61), (63) and (64). It is classical (see [11]) that there exists a C^∞ function $\tilde\varphi$ on $\overline{\Omega}$ such that $\partial_n\tilde\varphi<0$ on $\partial\Omega$, and whose critical points, in finite number, are all in $\tilde\omega$. Thus for such a function, (61) and (63) are satisfied. An explicit calculation (cf [4, p.12]) shows that the function $\varphi:=e^{\beta\tilde\varphi}$ (which still satisfies (61) and (63)) also satisfies (64) when β is large enough. $\square$

We are now ready to prove Proposition 7.2. Let μ be a real number with large absolute value. Take

$$F := (f_0, g_0, g_1) \in H, \quad U := (\mathcal{A} - i\mu)^{-1}F, \quad U = (u_0, v_0, v_1)\,. \tag{66}$$

The idea of the proof is to bound all the energy of U by the energy of the heat component u_0 on a small subset ω of Ω_1. This may be done, with exponential loss in μ, using Proposition 7.3.

Taking the real part of the scalar product of equation (66) in H by U, one gets

$$\|\Delta u_0\|^2_{L^2(\Omega_1)} = -\mathrm{Re}\,(\mathcal{A}U, U)_H \leq \|F\|_H \|U\|_H. \tag{67}$$

Equation (66) yields

$$\begin{cases} (\Delta - i\mu)u_0 = f_0 \\ \Delta v_0 + \mu^2 v_0 = g_1 + i\mu g_0 \\ \qquad\quad v_1 = g_0 + i\mu v_0. \end{cases} \tag{68}$$

Let

$$\begin{aligned}\mathcal{K}_0 :=&\mu^2\|F\|^2_H + \|\Delta u_0\|^2_{L^2(\Omega_1)} + \mu^4\|u_0\|^2_{L^2(\Omega_1)}\\ :=&\mu^2\big\{\|f_0\|^2_{H^1(\Omega_1)} + \|g_0\|^2_{H^1(\Omega_2)} + \|g_1|^2_{L^2(\Omega_2)}\big\} + \|\Delta u_0\|^2_{L^2(\Omega_1)} + \mu^4\|u_0\|^2_{L^2(\Omega_1)}.\end{aligned}$$

By (67) and the first line of (68), we get, for large μ

$$\mathcal{K}_0 \lesssim \mu^2\left(\|U\|_H\|F\|_H + \|F\|^2_H\right). \tag{69}$$

Thus it suffices to bound $\|U\|^2_H$ by the product of $\mathcal{K}_0$ and an exponential in μ. The following function, thanks to the boundary conditions (BCa), is in $H^2(\Omega) \cap H^1_0(\Omega)$

$$w_0 := 1\!\!1_{\Omega_1} u_0 + 1\!\!1_{\Omega_2} v_0.$$

We have, by equations (68)

$$(\Delta + \mu^2)w_0 = \underbrace{1\!\!1_{\Omega_1}(i\mu u_0 + \mu^2 u_0 + f_0) + 1\!\!1_{\Omega_2}(g_1 + i\mu g_0)}_{h_0}.$$

Note that $\|h_0\|^2_{L^2(\Omega)}$ is bounded, up to a constant, by $\mathcal{K}_0$. Let ω be a small, non-empty open subset of Ω_1. By the Carleman inequality of Proposition 7.3, we have, for large $|\mu|$

$$\|w_0\|^2_{H^1(\Omega)} \leq e^{C|\mu|}\left\{\|h_0\|^2_{L^2(\Omega)} + \|u_0\|^2_{H^1(\omega)}\right\}$$

Thus, noting that by equations (68)

$$\|v_1\|^2_{L^2(\Omega_2)} \lesssim \mathcal{K}_0 + \mu^2\|w_0\|^2_{L^2(\Omega)},$$

we get, for large μ, and some constant C

$$\|U\|^2_H \leq e^{C|\mu|}\left\{\mathcal{K}_0 + \|u_0\|^2_{H^1(\omega)}\right\}. \tag{70}$$

It is an easy fact that

$$\|u_0\|^2_{H^1(\omega)} \lesssim \|u_0\|^2_{L^2(\Omega_1)} + \|\Delta u_0\|^2_{L^2(\Omega_1)} \lesssim \mathcal{K}_0.$$

We thus get, for some constant C_1

$$\|U\|^2_H \leq e^{C_1|\mu|}\mathcal{K}_0.$$

By (69), using the inequality $ab \leq \frac{t}{2}a^2 + \frac{1}{2t}b^2$

$$\|U\|^2_H \leq e^{C_1|\mu|}\left\{\frac{1}{2}e^{-C_1|\mu|}\|U\|^2_H + \frac{1}{2}e^{C_1|\mu|}\mu^4\|F\|^2_H + \mu^2\|F\|^2_H\right\}.$$

We finally obtain, for large μ, the following bound, which shows the proposition

$$\|U\|_H \leq Ce^{2C_1|\mu|}\mu^4\|F\|_H.$$

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