Dynamics of semilinear wave equations
2024-2025
Final exam. Correction

Problem 1

1.1) We have $\frac{1}{a}+\frac{3}{b}=\frac{28}{56}=\frac{1}{2}$, and $\frac{1}{14}+\frac{3}{7}=\frac{7}{14}=\frac{1}{2}$

According to Theorem of the course, this gives the desired inequality.

1.2) By definition, $E=\frac{1}{2}\|\vec{u}(t)\|^2_{\dot{\mathcal{H}}^1}+\frac{1}{6}\|u(t)\|^6_{L^6}\geq\frac{1}{2}\|\vec{u}(t)\|^2_{\dot{\mathcal{H}}^1}$

As proved in the course, it is independent of t.

Using 1.1) we have

$$\|u\|_{L^a([T_0,T);L^b)}\leq C\|\vec{u}(T_0)\|_{\dot{\mathcal{H}}^1}+C\|u^5\|_{L^1([T_0,T);L^2)}$$

By Hölder inequality, $\|u^5\|_{L^1([T_0,T);L^2)}\leq\|u\|_{L^{14}([T_0,T);L^7)}\|u^4\|_{(*)}$

$$(*)=L^{\frac{14}{13}}([T_0,T);L^{\frac{14}{5}})$$

This gives the desired inequality (taking a larger C) since $\|\vec{u}(T_0)\|_{\dot{\mathcal{H}}^1}\leq\sqrt{2E}$

1.3) We use a bootstrap argument. Assume that

$\|u\|_{L^a([T_0,T);L^b)}\leq 2C\sqrt{E}$ for some $T\in(T_0,T_+)$

Then by 1.2),

$$\|u\|_{L^a([T_0,T);L^b)} \leq C\sqrt{E} + C\varepsilon\,(2C\sqrt{E})^4$$

Fixing $\varepsilon = \frac{1}{(2C\sqrt{E})^4} \times \frac{\sqrt{E}}{2}$, we deduce $\|u\|_{L^a([T_0,T);L^b)} \leq \frac{3}{2}C\sqrt{E}$

Since $F(T) = \|u\|_{L^a([T_0,T);L^b)}$ defines a continuous function of T such that $F(T_0)=0$, the intermediate value theorem implies:

$\forall T \in [T_0, T_+)$, $\|u\|_{L^a([T_0,T);L^b)} \leq \frac{3}{2}C\sqrt{E}$, and thus

$\|u\|_{L^a([T_0,T_+);L^b)} \leq \frac{3}{2}C\sqrt{E}$. In particular,

$u \in L^a([0,T_+);L^b)$. (we have also used that by 1.1, and the def. of a solution, $u \in L^a([0,T_0];L^b)$)

1.4) We have $\frac{1}{5} = \frac{\theta}{a} + \frac{1-\theta}{14}$ (where $\theta = \frac{4}{5}$)

$$\frac{1}{10} = \frac{\theta}{b} + \frac{1-\theta}{7}$$

Thus $\|u\|_{L^5([0,T_+);L^{10})} \leq \|u\|^{\frac{4}{5}}_{L^a([0,T_+);L^b)} \cdot \|u\|^{\frac{1}{5}}_{L^{14}L^7}$

i.e. $(L^{14}([0,T_+);L^7))$

By 1.3 and the assumption $u \in L^{14}([0,T_+);L^7)$, we obtain $u \in L^5([0,T_+);L^{10})$. Thus by the blow-up criterion $T_+ = +\infty$, and by the scattering criterion, there exist $(v_0,v_1) \in \dot{\mathcal{H}}^1$ such that

$$\lim_{t\to\infty} \|\vec{u}(t) - \vec{S}_L(t)(v_0,v_1)\|_{\dot{\mathcal{H}}^1} = 0$$

By (*), (**) and the Riesz-Thorin interpolation theorem:

$$\|e^{it|D|} f\|_{L^p} \lesssim \left(\frac{2^{2j}}{|t|}\right)^{\theta} \|f\|_{L^q}; \quad \theta \in]0,1[$$

$$\frac{1}{p} = \frac{1-\theta}{2} \qquad \frac{1}{q} = \theta + \frac{1-\theta}{2} = \frac{1+\theta}{2}$$

Since θ is arbitrary in $]0,1[$, that gives an inequality for arbitrary $\frac{1}{p} \in]0, \frac{1}{2}[$, that is for arbitrary $p \in]2,\infty[$, with $\theta = 1 - \frac{2}{p}$,

$\frac{1}{q} = \frac{1}{2} + \frac{\theta}{2} = \frac{1}{2} + \frac{1}{2} - \frac{1}{p} = 1 - \frac{1}{p}$ i.e. $q = p'$. Hence

$$\|e^{it|D|} f\|_{L^p} \lesssim \left(\frac{2^{2j}}{|t|}\right)^{1-\frac{2}{p}} \|f\|_{L^{p'}}$$

2.2) By definition, for $f \in \mathcal{C}_0^\infty(\mathbb{R}^3)$, $g \in \mathcal{C}_0^\infty(\mathbb{R}^4)$, we have

$$\iint_{\mathbb{R}\,\mathbb{R}^3} T_j f \,\bar{g}\, dx\,dt = \int_{\mathbb{R}^3} f\, \overline{T_j^* g}\, dx$$

$$\iint_{\mathbb{R}\,\mathbb{R}^3} e^{it|D|} \Delta_j f\, \bar{g}\, dx\,dt = \int_{\mathbb{R}} \int_{\mathbb{R}^3} f\, \overline{e^{-it|D|} \Delta_j g}\, dx\,dt$$

$$= \int_{\mathbb{R}^3} f\, \overline{\int_{\mathbb{R}} e^{-it|D|} \Delta_j g\, dt}\, dx$$

Thus $T_j^* g = \int_{\mathbb{R}} e^{-it|D|} \Delta_j g\, dt$

2.3)

1.5) Assume u satisfies $T_+(u) = +\infty$ and

$\exists (v_0, v_1) \in \dot{H}^1$ such that

$$\lim_{t \to +\infty} \| \vec{u}(t) - \vec{S}_L(t)(v_0, v_1) \|_{\dot{H}^1} = 0$$

By the scattering criterion of the course, $u \in L^5([0,+\infty[; L^{10})$. Since

$$\|u\|_{L^{14}([0,+\infty[; L^7)} \leq \|u\|_{L^\infty([0,+\infty[; L^6)}^{\frac{9}{14}} \|u\|_{L^5([0,+\infty[; L^{10})}^{\frac{5}{14}}$$

by Hölder, we obtain $u \in L^{14}([0,+\infty[; L^7)$

Problem 2

2.1) We have (*) $\|e^{it|D|} \Delta_j f\|_{L^2} = \|\Delta_j f\|_{L^2}$ by Plancherel equality. By the dispersion inequality for the wave equation $\left\| \frac{\sin(t|D|)}{|D|} \Delta_j f \right\|_{L^\infty} \lesssim \frac{1}{|t|} \|\Delta_j f\|_{\dot{W}^{1,1}}$

and thus $\|\sin(t|D|) \Delta_j f\|_{L^\infty} \lesssim \frac{2^{2j}}{|t|} \|\Delta_j f\|_{L^1} \lesssim \frac{2^{2j}}{|t|} \|f\|_{L^1}$

Also $\|\cos(t|D|) \Delta_j f\|_{L^\infty} \lesssim \frac{1}{|t|} \|\Delta_j f\|_{\dot{W}^{2,1}} \lesssim \frac{2^{2j}}{|t|} \|f\|_{L^1}$

(the notations $\dot{W}^{1,1}$ and $\dot{W}^{2,1}$ are those of chapter 1 of the course)

Writing $e^{it|D|} = \cos(t|D|) + i\sin(t|D|)$, we obtain

(**) $$\|e^{it|D|} \Delta_j f\|_{L^\infty} \lesssim \frac{2^{2j}}{|t|} \|f\|_{L^1}$$

By density, (4) is valid if and only if,

$$\forall f \in \mathcal{C}_0^\infty(\mathbb{R}^3); \quad \|T_j f\|_{L^q L^r} \leq \bar{C}\|f\|_{L^2}$$

By duality, this is equivalent to

$$\forall f \in \mathcal{C}_0^\infty(\mathbb{R}^3); \; \forall g \in \mathcal{C}_0^\infty(\mathbb{R}^4)$$

$$\left|\iint T_j f \, \bar{g} \, dx dt\right| \leq \bar{C}\|f\|_{L^2}\|g\|_{L^{q'}L^{r'}}$$

i.e

$$\forall f \in \mathcal{C}_0^\infty(\mathbb{R}^3); \; \forall g \in \mathcal{C}_0^\infty(\mathbb{R}^4)$$

$$\left|\int_{\mathbb{R}^3} f \, \overline{T_j^* g} \, dx\right| \leq \bar{C}\|f\|_{L^2}\|g\|_{L^{q'}L^{r'}}$$

That is: $\|T_j^* g\|_{L^2} \leq \bar{C}\|g\|_{L^{q'}L^{r'}} ; \forall g \in \mathcal{C}_0^\infty(\mathbb{R}^4)$

By direct computation and the definition of T_j^*

$$\|T_j^* g\|_{L^2}^2 = \int T_j^* g \, \overline{T_j^* g} \, dx$$

$$= \iint T_j T_j^* g \, \bar{g} \, dx dt$$

We are thus reduced to prove:

$$\forall g \in \mathcal{C}_0^\infty(\mathbb{R}^4), \quad \left|\iint T_j T_j^* g \, \bar{g} \, dx dt\right| \leq \bar{C}^2 \|g\|_{L^{q'}L^{r'}}^2$$

For this, by Hölder's inequality, it is sufficient to prove:

$$\forall g \in \mathcal{C}_0^\infty(\mathbb{R}^4) \quad \|T_j T_j^* g\|_{L^q L^r} \leq \bar{C}^2 \|g\|_{L^{q'}L^{r'}}$$

2.4.) We have

$$\left(T_j T_j^* g\right)(t) = \int_{\mathbb{R}} e^{i(t-s)|D|} \Delta_j g(s)\, ds$$

By the inequality of 2.1

$$\|T_j T_j^* g(t)\|_{L^r} \leq C \int_{\mathbb{R}} \frac{\left(2^{2j}\right)^{1-\frac{2}{r}}}{|t-s|^{1-\frac{2}{r}}} \|g(s)\|_{L^{r'}}\, ds$$

We the use Hardy-Littlewood Sobolev inequality, with $\theta = 1 - \frac{2}{r}$ so that

$$\frac{1}{q'} + \theta = 1 + \frac{1}{q} \text{ by the assumption } \frac{1}{q} + \frac{1}{r} = \frac{1}{2}$$

This gives $\|T_j T_j^* g\|_{L^q L^r} \leq \left(2^{2j}\right)^{1-\frac{2}{r}} \|g\|_{L^{q'} L^{r'}}$

This yields the desired inequality with $\bar{C}^2 = \left(2^{2j}\right)^{1-\frac{2}{r}} C^2$, i.e $\bar{C} = C\left(2^j\right)^{1-\frac{2}{r}}$

for some constant $C > 0$ (using 2.3)

2.5) We let $\bar{\Delta}_j = \Delta_{j-1} + \Delta_j + \Delta_{j+1}$ and recall that by the definition of Δ_j, $\bar{\Delta}_j \Delta_j = \Delta_j$.

Using the inequality of 2.4, with $j-1$, j, $j+1$, we obtain

$$\|e^{it|D|} \bar{\Delta}_j f\|_{L^q L^r} \leq 3\bar{C} \|f\|_{L^2}$$

Applying this to $\Delta_j f$ and using $\bar{\Delta}_j \Delta_j f = \Delta_j f$, we have: $\|e^{it|D|} \Delta_j f\|_{L^q L^r} \leq 3\bar{C} \|\Delta_j f\|_{L^2}$

2.6) By 2.5),

$$\star \quad \| e^{it|D|} \Delta_j f \|_{L^q L^r} \leq C\, 2^{j(1-\frac{2}{r})} \| \Delta_j f \|_{L^2}$$

By one of the results of Littlewood-Paley theory, (see Theorem III.4.5 of the course), at fixed $t \in \mathbb{R}$,

$$\| e^{it|D|} f \|^2_{L^r(\mathbb{R}^3)} \lesssim \sum_j \| \Delta_j e^{it|D|} f \|^2_{L^r(\mathbb{R}^3)}$$

$$\| e^{it|D|} f \|^2_{L^q L^r} = \Big\| \| e^{it|D|} f \|^2_{L^r(\mathbb{R}^3)} \Big\|_{L^{q/2}(\mathbb{R})}$$

$$\lesssim \Big\| \sum_{j \in \mathbb{Z}} \| \Delta_j e^{it|D|} f \|^2_{L^r(\mathbb{R}^3)} \Big\|_{L^{q/2}(\mathbb{R})}$$

$$\lesssim \sum_j \Big\| \| \Delta_j e^{it|D|} f \|^2_{L^r(\mathbb{R}^3)} \Big\|_{L^{q/2}(\mathbb{R})}$$

(by Minkowski inequality)

$$\lesssim \sum_j \| \Delta_j e^{it|D|} f \|^2_{L^q L^r}$$

$$\| e^{it|D|} f \|^2_{L^q L^r} \lesssim \sum_j \left(2^{j(1-\frac{2}{r})}\right)^2 \| \Delta_j f \|^2_{L^2} \simeq \| f \|^2_{\dot{H}^{1-\frac{2}{r}}} \text{ by } \star$$

This gives the estimate with $s = 1 - \frac{2}{r}$

When $r=4$ we obtain $q=4$, $s=\frac{1}{2}$ which is one of the estimate proved in the course.

2.7) we have, by 2.6)

$$\| \cos(t|D|) u_0 \|_{L^q L^r} \lesssim \| e^{it|D|} u_0 \|_{L^q L^r} + \| e^{-it|D|} u_0 \|_{L^q L^r}$$

$$\lesssim \| u_0 \|_{\dot{H}^{1-\frac{2}{r}}}$$

and

$$\left\| \frac{\sin(t|D|)}{|D|} u_1 \right\|_{L^q L^r} \lesssim \| |D|^{-1} u_0 \|_{\dot H^{1-\frac{2}{n}}} = \| u_0 \|_{\dot H^{-\frac{2}{n}}}$$

Thus, if u is the solution of $\partial_t^2 u - \Delta u = 0$, with initial data $(u_0, u_1) \in \dot{\mathcal{H}}^{1-\frac{2}{n}} = \dot H^{1-\frac{2}{n}} \times \dot H^{-\frac{2}{n}}$, one has:

$$\|u\|_{L^q L^r} \lesssim \|(u_0, u_1)\|_{\dot{\mathcal{H}}^{1-\frac{2}{n}}}$$

<u>Problem 3</u>

<u>3.1)</u> Let $R>0$, $x_0 \in \mathbb{R}^3$, $f \in \dot H^1(\mathbb{R}^3)$,

$$g(x) = f(Rx + x_0)$$

Let $\underline{g}$, $\tilde g$ be given by the extension result, so that:

$$\underline{g}(x) = g(x), \quad x \in B_1(0) \qquad \tilde g(x) = g(x) \quad x \in {}^c B_1(0),$$

$$\|\underline{g}\|_{\dot H^1(\mathbb{R}^3)} \le M \left[\| \mathbb{1}_{B_1(0)} |Dg| \|_{L^2} + \| \mathbb{1}_{B_1(0)} g \|_{L^6} \right]$$

$$\|\tilde g\|_{\dot H^1(\mathbb{R}^3)} \le M \| \mathbb{1}_{{}^c B_1(0)} |Dg| \|_{L^2}$$

Let $\quad \underline{f}(x) = \underline{g}\left(\frac{x - x_0}{R}\right) \qquad \tilde f(x) = \tilde g\left(\frac{x-x_0}{R}\right)$

Then $\quad \underline{g}(x) = \underline{f}(Rx + x_0) \qquad \tilde g(x) = \tilde f(Rx + x_0)$

Thus $\quad f(Rx + x_0) = \underline{f}(Rx + x_0) \quad$ for $x \in B_1(0)$

i.e. $\quad f(y) = \underline{f}(y) \quad$ for $y \in B_R(x_0)$

and similarly $\quad f(y) = \tilde f(y) \quad$ for $y \in {}^c B_R(x_0)$

Furthermore: $\|\underline{f}\|_{\dot{H}^1(\mathbb{R}^3)} = R^{\frac{1}{2}} \|\underline{g}\|_{\dot{H}^1(\mathbb{R}^3)}$

$$\|\tilde{f}\|_{\dot{H}^1(\mathbb{R}^3)} = R^{\frac{1}{2}} \|\tilde{g}\|_{\dot{H}^1(\mathbb{R}^3)}$$

$$\|\mathbb{1}_{B_R(x_0)} |\nabla f|\|^2_{L^2(\mathbb{R}^3)} = \int_{|x-x_0|<R} \frac{1}{R^2} \left|\nabla g\left(\frac{x-x_0}{R}\right)\right|^2 dx$$

$$= R \int_{|y|<1} |\nabla g(y)|^2 dy$$

Hence $\|\mathbb{1}_{B_R(x_0)} |\nabla f|\|_{L^2(\mathbb{R}^3)} = R^{\frac{1}{2}} \|\mathbb{1}_{B_1(0)} |\nabla g|\|_{L^2(\mathbb{R}^3)}$

and similarly $\|\mathbb{1}_{B_R(x_0)} f\|_{L^6(\mathbb{R}^3)} = R^{\frac{1}{2}} \|\mathbb{1}_{B_1(0)} g\|_{L^6(\mathbb{R}^3)}$

etc...

Using the inequalities between the norms of $g, \underline{g}, \tilde{g}$, we obtain the same inequalities on $f, \underline{f}, \tilde{f}$.

3.2) Assume

$$\|\mathbb{1}_{B_R(x_0)} |\nabla u_0|\|_{L^2} + \|\mathbb{1}_{B_R(x_0)} u_0\|_{L^6} + \|\mathbb{1}_{B_R(x_0)} u_1\|_{L^2} = \varepsilon \leq \delta_0$$

Let $\underline{u}_0$ be given by the question 3.1, such that $\underline{u}_0(x) = u_0(x)$ on $B_R(x_0)$ and $\|\underline{u}_0\|_{\dot{H}^1(\mathbb{R}^3)} \leq M\delta_0$

Let $\underline{u}_1(x) = u_1(x)$ if $x \in B_R(x_0)$ $\underline{u}_1(x) = 0$ if $x \in {}^c B_R(x_0)$

Then $\|(\underline{u}_0, \underline{u}_1)\|_{\dot{H}^1} \leq (M+1)\delta_0$

Let $\underline{u}$ be the solution with initial data $(\underline{u}_0, \underline{u}_1)$ of (6)

By the small data theory, choosing δ_0 small enough,

$\|\underline{u}\|_{L^5 L^{10}} \leq C_0 \varepsilon$ for some constant $\varepsilon > 0$.

By finite speed of propagation $u(t,x) = \underline{u}(t,x)$ for $(t,x) \in \Gamma_R(x_0)$. Thus

$$\| \mathbb{1}_{\Gamma_R(x_0)} u \|_{L^5 L^{10}} = \| \mathbb{1}_{\Gamma_R(x_0)} \underline{u} \|_{L^5 L^{10}} \leq C_0 \varepsilon$$

3.3) We use the same argument, replacing $\underline{u}_0$ by $\tilde{u}_0$, $\underline{u}_1$ with $\tilde{u}_1$ defined by

$\tilde{u}_1(x) = u_1(x)$ if $|x - x_0| > R$

$\tilde{u}_1(x) = 0$ if $|x - x_0| \leq R$.

and u by $\tilde{u}$, solution with initial data $(\tilde{u}_0, \tilde{u}_1)$

3.4) Let $(u_0, u_1) \in \dot{H}^1$ such that

$$\sup_{x_0 \in \mathbb{R}^3} \| \mathbb{1}_{B_R(x_0)} |\nabla u_0| \|_{L^2} + \| \mathbb{1}_{B_R(x_0)} u_0 \|_{L^6} + \| \mathbb{1}_{B_R(x_0)} u_1 \|_{L^2} \leq \delta_0$$

Let $R_1 > 0$ such that

$$\| \mathbb{1}_{{}^c B_{R_1}(0)} |\nabla u_0| \|_{L^2} + \| \mathbb{1}_{{}^c B_{R_1}(0)} u_0 \|_{L^2} \leq \delta_1$$

Then $\mathbb{1}_{\Gamma_{R_1}(0)} u \in L^5 L^{10}$ by 3.3,

Assume (by contradiction) that $T_+(u) < R$

Let $K = \{ (t,x) \in [0, T_+(u)],\ |x| \leq R_1 + |t| \}$

This is a compact subset of $\mathbb{R}^4$.

We have $K \subset \bigcup_{\substack{|x_0|\le R_1+2R \\ x_0\in\mathbb{R}^3}} \overline{\Gamma_R(x_0)}$

Since $\overline{\Gamma_R(x_0)} \cap K$ is open in K, this is a covering of K by open subsets of K. By compactness, we can extract a finite covering

$$K \subset \bigcup_{i=1,\dots,J} \overline{\Gamma_R(x_i)}$$

Hence

$$[0,T_+(u)] \subset \bigcup_{i=1,\dots,J} \overline{\Gamma_R(x_i)} \cup \overline{\tilde{\Gamma}_{R_1}(0)}$$

Since $\mathbb{1}_{\tilde{\Gamma}_{R_1}(0)}\, u \in L^5;L^{10}$ and, by 3.2), $\mathbb{1}_{\Gamma_R(x_i)}\, u \in L^5;L^{10}$ for all i, we obtain $u \in L^5([0,T_+(u)[\,;L^{10}(\mathbb{R}^3))$ which contradicts the blow-up criterion

Thus $R \le T_+(u)$. Using the same argument for negative times, we obtain $(-R,+R) \subset I_{max}(u)$.