

**Dynamics of semilinear wave equation**  
**Master II course. 2024-2025**

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## CHAPTER I

# Linear wave equation: classical theory

### I.1. Presentation of the equation

The linear wave equation is the equation:

$$(LW) \quad \partial_t^2 u - \Delta u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N,$$

where  $N \geq 1$  is the spatial dimension (in this course, we will often assume  $N = 3$ ), and

$$\Delta = \sum_{k=1}^N \frac{\partial^2}{\partial x_k^2}.$$

(We will use either the notations  $\partial_y$  or  $\frac{\partial}{\partial y}$  for the derivative with respect to the variable  $y \in \{t, x_1, \dots, x_N\}$ ).

This is an evolution equation: we fix initial data at a certain time  $t = t_0$ , and are interested in the evolution of the equation over time  $t$ . Since the equation is of order 2, we actually fix an initial data for  $\vec{u} = (u, \partial_t u)$ :

$$(I.1.1) \quad \vec{u}|_{t=t_0} = (u_0, u_1)$$

where  $(u_0, u_1)$  is to be taken in a certain functional space.

We will consider in this course initial data *with real values*. The passage to complex or vector values is immediate for most properties of the equation (LW) (by working coordinate by coordinate), but can induce drastic changes in the nonlinear case, if the nonlinearity mixes the components.

Equation (LW) is invariant under several obvious space-time transformations. If  $u$  is a solution, it is also the case of

$$\mu u(t - t_0, \lambda(Rx - x_0)),$$

where  $\mu \in \mathbb{R}$ ,  $t_0 \in \mathbb{R}$ ,  $\lambda > 0$ ,  $R \in \mathcal{O}_N(R)$ ,  $x_0 \in \mathbb{R}^N$ . It is in fact invariant under a larger group of linear transformations, the Lorentz group (cf Exercise I.9 p. 15).

As a consequence, we can limit ourselves, without loss of generality, to the case of an initial time  $t_0 = 0$ , i.e.

$$(ID) \quad \vec{u}|_{t=0} = (u_0, u_1)$$

Furthermore, the equation is invariant under time inversion: if  $u$  is solution, it is also the case of  $t \mapsto u(-t, x)$ . It is thus a reversible equation.

We will also consider the equation with a force:

$$(I.1.2) \quad \partial_t^2 u - \Delta u = f,$$

(still with an initial condition of type (ID)), whose understanding will be crucial for the study of the nonlinear wave equation.

The Cauchy problem (LW), (ID) can be approached in at least 3 different ways:

- The classical approach which consists in finding an explicit formula to express the solution. It works when the initial data is sufficiently regular ( $C^3 \times C^2$  in dimension 3 of space) and gives classical solutions (that is to say  $C^2$  in  $(t, x)$  and satisfying (LW) in the sense of classical differentiation).
- The use of the Fourier transformation in space, which is very simple (once the Fourier transformation is known) and particularly effective in Sobolev spaces based on  $L^2$  (which are natural spaces for the study of the equation due to the conservation of energy and other  $L^2$ -based quantities). This method allows to obtain weak solutions with degrees of regularity lower than the previous ones, and to use tools based on the Fourier transformation, which can be useful, for example, to prove certain dispersive properties of the equation.
- The "functional analysis" approach, by the theory of semi-groups, which gives the same type of solutions as the previous method.

In this chapter, we will detail the classical method, first by writing the explicit formula for solutions in dimension 1 of space, then in higher dimensions. We will study in the following chapter the equation in the energy space by the Fourier transformation. This chapter is partly based on Chapter 5 of the beautiful book by Folland on partial differential equations [1].

## I.2. Explicit Formula in Dimension 1

In dimension 1, the equation (LW) can be written as:

$$(I.2.1) \quad (\partial_t^2 - \partial_x^2)u = 0,$$

which can be written  $(\partial_t - \partial_x)(\partial_t + \partial_x)u = 0$ . We thus make the change of variables  $\eta = x + t$ ,  $\xi = x - t$ . Setting  $v(\eta, \xi) = u\left(\frac{\eta - \xi}{2}, \frac{\eta + \xi}{2}\right)$ , or  $u(t, x) = v(t + x, t - x)$ , we have:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 v}{\partial \eta^2} + \frac{\partial^2 v}{\partial \xi^2} + 2 \frac{\partial^2 v}{\partial \xi \partial \eta},$$

and

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial \eta^2} + \frac{\partial^2 v}{\partial \xi^2} - 2 \frac{\partial^2 v}{\partial \xi \partial \eta},$$

which gives:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -4 \frac{\partial^2 v}{\partial \eta \partial \xi}.$$

Thus, we obtain:

$$(LW) \iff \frac{\partial^2 v}{\partial \eta \partial \xi} = 0.$$

Let  $u$  be a  $C^2$  solution of (I.2.1), (ID). Thus,  $u_1 \in C^1(\mathbb{R})$  and  $u_0 \in C^2(\mathbb{R})$ .

The equality  $\frac{\partial^2 v}{\partial \eta \partial \xi} = 0$  shows that  $\frac{\partial v}{\partial \xi}$  is a (class  $C^1$ ) function  $w(\xi)$  independent of  $\eta$ . Integrating with respect to  $\xi$  for  $\eta$  fixed, we deduce:

$$v(\eta, \xi) = \underbrace{\int_0^\xi w(\sigma) d\sigma}_{\varphi(\xi)} + \psi(\eta),$$

for a certain function  $\psi$ , necessarily  $C^2$  since  $v$  is of class  $C^2$  and  $w$  of class  $C^1$ . Thus, we necessarily have:

$$v(\eta, \xi) = \varphi(\xi) + \psi(\eta), \quad \varphi, \psi \in C^2(\mathbb{R}^2),$$

or equivalently:

$$(I.2.2) \quad u(t, x) = \varphi(x - t) + \psi(x + t).$$

Using the initial condition (ID), a direct calculation gives:

$$\begin{aligned} \psi(\eta) &= \frac{1}{2} \int_0^\eta u_1(\sigma) d\sigma + \frac{1}{2} u_0(\eta) + c, \\ \varphi(\xi) &= -\frac{1}{2} \int_0^\xi u_1(y) dy + \frac{1}{2} u_0(\xi) - c, \end{aligned}$$

where  $c \in \mathbb{R}$  (the choice of this constant is irrelevant). Hence, we deduce:

$$(I.2.3) \quad u(t, x) = \frac{1}{2} (u_0(x + t) + u_0(x - t)) + \frac{1}{2} \int_{x-t}^{x+t} u_1(y) dy.$$

Conversely, it is easy to verify that formula (I.2.3) gives a  $C^2$  solution of (I.2.1), (ID). Therefore, we have shown:

**PROPOSITION I.2.1.** *Let  $(u_0, u_1) \in C^2(\mathbb{R}) \times C^1(\mathbb{R})$ . Then, there exists a unique solution  $u \in C^2(\mathbb{R} \times \mathbb{R})$  of (LW) satisfying the initial condition (ID). This solution satisfies formula (I.2.3).*

On formula (I.2.2), we observe that a solution of the wave equation in dimension 1 is the sum of two waves: one,  $\varphi(x - t)$ , moving at speed 1 to the right, and the other  $\psi(x + t)$ , moving at the same speed to the left.<sup>1</sup>

It is also possible to obtain a formula for the equation with the right-hand side (I.1.2). We leave this as an exercise to the reader. Further on, we will provide a general method giving the solution of the equation with the right-hand side in terms of the equation without the right-hand side.

<sup>1</sup>Note that the equations (LW), (I.2.1) have been normalized, so that the speed of propagation is exactly 1.

We can see from formula (I.2.3) that  $u(t, x)$  depends only on the values of  $(u_0, u_1)$  over  $[x - |t|, x + |t|]$ . This is a first example of "finite speed of propagation" which holds in all spatial dimensions.

### I.3. Integral on the Sphere and Divergence Theorem

We denote  $S^{N-1} = \{x \in \mathbb{R}^N, |x| = 1\}$ , where  $|\cdot|$  represents the Euclidean norm on  $\mathbb{R}^N$ :

$$|x|^2 = \sum_{j=1}^N x_j^2.$$

More generally,  $S_R^{N-1}$  will denote the sphere of radius  $R$ :  $\{x \in \mathbb{R}^N, |x| = R\}$ .

We denote  $d\sigma$  as the volume element on one of these spheres. Thus, the integral of a function  $f \in \mathcal{L}^1(S_R^{N-1})$  (i.e., a function integrable on  $S_R^{N-1}$ ) is written as

$$\int_{S_R^{N-1}} f(y) d\sigma(y).$$

This integral can be calculated using spherical coordinates. In dimension 3, this writes:

$$\int_{S_R^2} f(y) d\sigma(y) = R^2 \int_0^{2\pi} \int_0^\pi f(R \sin \theta \cos \varphi, R \sin \theta \sin \varphi, R \sin \theta) \sin(\theta) d\theta d\varphi.$$

We denote  $B_R^N(x_0)$  as the ball centered at  $x_0$  with radius  $R$ :

$$B_R^N(x_0) = \{x \in \mathbb{R}^N, |x - x_0| < R\}$$

and simply  $B_R^N = B_R^N(0)$ .

We will use the following formulas:

**Scaling:**

$$\int_{S_R^{N-1}} f(y) d\sigma(y) = R^{N-1} \int_{S^{N-1}} f(Ry) d\sigma(y) \quad f \in \mathcal{L}^1(S_R^{N-1}).$$

**Integral in radial coordinates:** if  $f \in \mathcal{L}^1(\{|x| \leq R\})$ ,

$$\int_{B_R^N} f(x) dx = \int_0^R \int_{S_r^{N-1}} f(y) d\sigma(y) dr = \int_0^R \int_{S^{N-1}} f(r\omega) d\sigma(\omega) r^{N-1} dr$$

**Divergence theorem:** if  $F \in C^1(\overline{B_R}, \mathbb{R}^N)$ ,

$$\int_{|x| \leq R} \nabla \cdot F(x) dx = \int_{S_R^{N-1}} \frac{y}{|y|} \cdot F(y) d\sigma(y),$$

where  $\nabla \cdot F = \sum_{j=1}^N \partial_{x_j} F_j$  is the divergence of the vector field  $F$ .

### I.4. Energy density. Uniqueness and finite speed of propagation

Before giving an explicit formula for the wave equation in dimension 3, we prove a uniqueness result valid in any dimension:

**THEOREM I.4.1.** *Let  $(t_0, x_0) \in \mathbb{R}^{1+N}$ ,  $t_1 > t_0$ ,  $R > 0$ . We denote  $\Gamma = \{(t, x) \in \mathbb{R} \times \mathbb{R}^N : t_0 \leq t \leq t_1, |x - x_0| \leq R - |t - t_0|\}$ . Let  $u \in C^2(\Gamma)$  be a solution of (LW) on  $\Gamma$ . We suppose  $(u, \partial_t u)(t_0, x) = 0$  for all  $x \in B_R(x_0)$ . Then  $u$  is identically zero on  $\Gamma$ .*

The proof of the theorem is based on a monotonicity law that has its own interest.

We define, for  $(t, x) \in \Gamma$ , the *density of energy*  $e_u$  as

$$e_u(t, x) = \frac{1}{2} |\nabla u(t, x)|^2 + \frac{1}{2} (\partial_t u(t, x))^2,$$

where  $|\nabla u|^2 = \sum_{j=1}^N (\partial_{x_j} u)^2$ , and we consider, for  $t_0 \leq t \leq t_1$ , the local energy

$$E_{\text{loc}}(t) = \int_{B_{R-(t-t_0)}(x_0)} e_u(t, x) dx = \int_{|x-x_0| < R-(t-t_0)} e_u(t, x) dx.$$

**LEMMA I.4.2.** *The function  $E_{\text{loc}}$  is decreasing on  $[t_0, t_1]$ .*

The lemma immediately implies Theorem I.4.1. Indeed, if  $\vec{u}(t_0)$  vanishes on  $B(x_0, R)$ , then  $E_{\text{loc}}(t_0) = 0$ , and thus  $E_{\text{loc}}(t) = 0$  for all  $t \in [t_0, t_1]$ , showing that  $u$  is zero on  $\Gamma$ .

PROOF OF LEMMA I.4.2. We notice that

$$(I.4.1) \quad \frac{\partial e}{\partial t} = \sum_{j=1}^N \left( \partial_{x_j} u \partial_t \partial_{x_j} u + \partial_{x_j}^2 u \partial_t u \right) = \sum_{j=1}^N \frac{\partial}{\partial x_j} (\partial_{x_j} u \partial_t u) = \nabla \cdot (\partial_t u \nabla u),$$

where  $\nabla u = (\partial_{x_i} u)_{1 \leq i \leq N}$ . Without loss of generality, we can assume that  $x_0 = 0$  and  $t_0 = 0$ . By the integration formula in radial coordinates,

$$E_{\text{loc}}(t) = \int_0^{R-t} s^{N-1} \int_{S^{N-1}} e_u(t, s\omega) d\sigma(\omega) ds.$$

By differentiation under the integral sign, we get that  $E_{\text{loc}}$  is differentiable and

$$E'_{\text{loc}}(t) = -(R-t)^{N-1} \int_{S^{N-1}} e_u(t, (R-t)\omega) d\sigma(\omega) + \int_{B_{R-t}^N} \frac{\partial e_u}{\partial t}(t, x) dx.$$

By formula (I.4.1), then the divergence formula

$$\int_{B_{R-t}^N} \frac{\partial e_u}{\partial t}(t, x) dx = \int_{B_{R-t}^N} \nabla \cdot (\partial_t u \nabla u)(t, x) dx = \int_{S_{R-t}^{N-1}} \frac{y}{|y|} \nabla u \partial_t u(t, y) d\sigma(y).$$

We thus have

$$E'_{\text{loc}}(t) = - \int_{S_{R-t}^{N-1}} \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{2} (\partial_t u)^2 - \frac{y}{|y|} \nabla u \partial_t u(t, y) \right) d\sigma(y) \leq - \frac{1}{2} \int_{S_{R-t}^{N-1}} \left( \frac{y}{|y|} \nabla u + \partial_t u(t, y) \right)^2 d\sigma(y).$$

□

### I.5. Explicit formulas.

This section is devoted to explicit formulas in space dimensions  $N \geq 2$ . In dimension  $N = 3$ , we will show that for any initial data  $(u_0, u_1) \in C^2 \times C^3$ , there exists a unique solution  $u \in C^2(\mathbb{R}^{1+3})$  of (LW), (ID), and provide an explicit formula for this solution. We will also provide a formula in dimension  $N = 2$ . We refer the reader to [1, Chapter 5B] for expressions of solutions when  $N \geq 4$ .

**5.a. The radial case in dimension 3.** When the initial conditions depend only on the variable  $r = |x|$ , the explicit formula is very simple.

We start by observing that if  $f$  depends only on the variable  $r$ , then the function  $f$  is  $C^2$  as a function on  $\mathbb{R}^3$  if and only if it is  $C^2$  as a function of the variable  $r$  on  $[0, \infty[$ , and satisfies  $\frac{df}{dr}(0) = 0$ . Moreover,

$$\Delta f = \frac{d^2 f}{dr^2} + \frac{2}{r} \frac{df}{dr}$$

(cf Exercise I.1). We notice that we can rewrite this formula as

$$r \Delta f = \frac{d^2}{dr^2} (rf).$$

Now let  $u$  be a  $C^2$  solution of (LW), (ID) with initial data  $(u_0, u_1)$  assumed to be radial. We also assume that for all  $t$ ,  $u(t)$  is a radial function. We will show a posteriori that this second assumption is a consequence of the assumption on the initial data. The previous formula gives

$$\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2} \right) (ru) = 0.$$

The function  $(t, r) \mapsto ru(t, r)$  is thus a solution of the wave equation in dimension 1, on  $\mathbb{R}_t \times ]0, \infty[$ . To obtain a function on  $\mathbb{R}^2$ , we extend  $ru(t, r)$  to an odd function:

$$v(t, y) = yu(t, |y|).$$

One can verify (using Exercise I.1) that  $v$  is of class  $C^2$  on  $\mathbb{R}^2$ , and that

$$\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2} \right) v = 0.$$



Formula (I.2.3) then gives:

$$v(t, y) = \frac{1}{2}(v_0(y+t) + v_0(y-t)) + \frac{1}{2} \int_{y-t}^{y+t} v_1(\sigma) d\sigma,$$

where  $(v_0, v_1) = \vec{v}|_{t=0}$ , thus

$$(I.5.1) \quad u(t, r) = \frac{1}{2r} \left( (r+t)u_0(|r+t|) + (r-t)u_0(|r-t|) \right) + \frac{1}{2r} \int_{r-t}^{r+t} \sigma u_1(|\sigma|) d\sigma.$$

Notice that when  $t > 0$  (to fix ideas),

$$\int_{r-t}^{r+t} \sigma u_1(|\sigma|) d\sigma = \int_{|r-t|}^{r+t} \sigma u_1(|\sigma|) d\sigma.$$

The finite speed of propagation is satisfied: the solution  $u(t, r)$  depends only on the initial condition  $(u_0, u_1)$  on the ball centered at  $r$  with radius  $|t|$ .

The formula (I.5.1) defines a function  $u(t, r)$  of class  $C^2$  outside the origin  $x = 0$ , as soon as the initial conditions  $(u_0, u_1)$  have the expected regularity  $C^2 \times C^1$ . However, there is a subtle phenomenon of loss of regularity of the solution  $u$  compared to the initial data at the origin : there exist data  $(u_0, u_1) \in C^2 \times C^1$  such that  $u$ , defined by formula (I.5.1), cannot be extended by a  $C^2$  function up to  $r = 0$ . Indeed, it can be checked that (at fixed  $t$ ),

$$(I.5.2) \quad \lim_{r \rightarrow 0} u(t, r) = u_0(t) + tu'_0(t) + tu_1(t),$$

which shows that if  $(u_0, u_1)$  are  $C^k \times C^{k-1}$  functions, then  $u(t, 0)$  is only  $C^{k-1}$  in general. We can interpret this phenomenon physically as follows: a singularity on the circle  $r = r_0$  at the initial time 0 that travels at speed 1 towards the origin will concentrate at the origin at time  $t = r_0$ , causing a stronger singularity.

The limit (I.5.2) suggests a maximal loss of regularity of a derivative with respect to the initial data, which is indeed the case:

**PROPOSITION I.5.1.** *Let  $(u_0, u_1) \in (C^3 \times C^2)(\mathbb{R}^3)$  be radial functions. Then formula (I.5.1) extended by  $u(t, 0) = u_0(t) + tu'_0(t) + tu_1(t)$ , defines a  $C^2$  function on  $\mathbb{R} \times \mathbb{R}^3$ , radial with respect to the variable  $x$ , and satisfying (LW), (ID).*

The Proposition I.5.1 is left as an exercise to the reader. Combining with the uniqueness property (Theorem I.4.1), we obtain that (I.5.1) gives the unique solution of (LW) with initial data  $(u_0, u_1)$ .

The formula (I.5.1) is remarkably simple. In higher space dimensions, we also have an explicit formula for radial solutions, which becomes more complicated as the dimension increases (see Exercise I.2). The loss of regularity observed in dimension 3 (and absent in dimension 1) increases with dimension, as the reader can verify on the formula obtained in Exercise I.2.

There is no simple formula in the radial case in even dimensions.

We also have explicit formulas (of course more complicated) without radially assumptions, in all dimensions. We will explicitly state these formulas when  $N = 3$ , then  $N = 2$ .

**5.b. General solutions in dimension 3: averaging over spheres.** If  $f \in C^0(\mathbb{R}^3)$ , we define

$$(I.5.3) \quad (M_f)(t, x) = \frac{1}{4\pi} \int_{S^2} f(x + ty) d\sigma(y) = \frac{1}{4\pi t^2} \int_{S^2_{|t|}} f(x + z) d\sigma(z).$$

the average of  $f$  over the sphere of radius  $|t|$  and center  $x$ . The function  $M_f$  inherits the regularity of  $f$  (cf exercise I.4).

**THEOREM I.5.2.** *Let  $(u_0, u_1) \in C^3(\mathbb{R}^3) \times C^2(\mathbb{R}^3)$ . Then the unique  $C^2$  solution of the wave equation (LW) with initial conditions (ID) is given by*

$$u(t, x) = tM_{u_1}(t, x) + \frac{\partial}{\partial t}(tM_{u_0}(t, x)).$$

**PROOF.** We start by verifying that  $tM_{u_1}(t, x)$  is the solution of the wave equation (LW), with initial condition  $(0, u_1)$ . By the theorem of differentiation under the integral sign, if  $g \in C^2(\mathbb{R}^3)$ ,

$$\frac{\partial}{\partial t}(M_g(t, x)) = \frac{1}{4\pi} \int_{S^2} (y \cdot \nabla g)(x + ty) d\sigma(y).$$

Using the divergence formula,

$$\begin{aligned} \int_{S^2} (y \cdot \nabla g)(x + ty) d\sigma(y) &= t \int_{|y| \leq 1} (\nabla \cdot (\nabla g))(x + ty) dy \\ &= t \int_{|y| \leq 1} (\Delta g)(x + ty) dy = \frac{1}{t^2} \int_0^t \int_{S^2} (\Delta g)(x + sy) s^2 d\sigma(y) ds. \end{aligned}$$

Thus:

$$\frac{\partial}{\partial t} (tM_{u_1}(t, x)) = M_{u_1}(t, x) + \frac{1}{4\pi t} \int_0^t \int_{S^2} (\Delta u_1)(x + sy) d\sigma(y) s^2 ds.$$

and therefore

$$\begin{aligned} \frac{\partial^2}{\partial t^2} (tM_{u_1}(t, x)) &= \frac{1}{4\pi t^2} \int_0^t \int_{S^2} (\Delta u_1)(x + sy) d\sigma(y) s^2 ds \\ &\quad - \frac{1}{4\pi t^2} \int_0^t \int_{S^2} (\Delta u_1)(x + sy) d\sigma(y) s^2 ds + \frac{t}{4\pi} \int_{S^2} (\Delta u_1)(x + ty) d\sigma(y) = \Delta (tM_{u_1}(t, x)). \end{aligned}$$

This shows that  $tM_{u_1}$  satisfies the wave equation (LW). Furthermore, since  $M_{u_1}(0, x) = u_1(0, x)$ , the initial condition at  $t = 0$  is indeed  $(0, u_1)$ .

Now let  $v(t, x) = tM_{u_0}(t, x)$ . Then, by the same reasoning,  $v$  is a solution of the wave equation (LW) with initial condition  $(0, u_0)$ . We deduce that  $\partial_t v$  is a solution of the wave equation with initial condition  $(u_0, 0)$ , which concludes the proof.  $\square$

Notice that we can rewrite the formula of the theorem as:

$$(I.5.4) \quad u(t, x) = tM_{u_1}(t, x) + M_{u_0}(t, x) + tM_{y \cdot \nabla u_0}(t, x).$$

We now give two important consequences of the previous formula.

**COROLLARY I.5.3** (Strong Huygens' principle). *The solution  $u(t, x)$  depends only on the values of  $u_0, \nabla u_0$ , and  $u_1$  on the sphere centered at  $x$  and of radius  $|t|$ .*

**REMARK I.5.4.** The strong Huygens' principle is a stronger version of the finite speed of propagation property, which states that  $u(t, x)$  depends only on the values of  $(u_0, u_1)$  on the ball centered at  $x$  and of radius  $|t|$ . This principle remains valid in any odd dimension  $\geq 3$  (the number of derivatives of  $u_0$  and  $u_1$  in the statement increases with the dimension). In even dimension, solutions only satisfy the finite speed of propagation: see §5.c. In dimension 1, as shown by formula (I.2.3), only solutions that are even in time (i.e. with initial condition of the form  $(u_0, 0)$ ) satisfy the strong Huygens' principle.

The second consequence of the explicit formula proved above is an estimate related to the dispersive nature of the wave equation. We will denote

$$(I.5.5) \quad \|\varphi\|_{\dot{W}^{s,p}} = \sup_{|\alpha|=s} \|\partial_x^\alpha \varphi\|_{L^p(\mathbb{R}^N)}.$$

We prove:

**THEOREM I.5.5** (Dispersion inequality). *Let  $(u_0, u_1) \in (C^3 \times C^2)(\mathbb{R}^3)$ , with compact support and  $u$  the solution of (LW), (ID). Then for all  $t > 0$ ,*

$$\|u(t)\|_{L^\infty(\mathbb{R}^3)} \lesssim \frac{1}{t} (\|u_0\|_{\dot{W}^{2,1}} + \|u_1\|_{\dot{W}^{1,1}}).$$

**PROOF.** By space translation invariance it is sufficient to bound  $|u(t, 0)|$ . We have

$$4\pi u(t, 0) = t \int_{S^2} u_1(ty) d\sigma(y) + \int_{S^2} u_0(ty) d\sigma(y) + t \int_{S^2} y \cdot \nabla u_0(ty) d\sigma(y).$$

By the divergence theorem (denoting by  $B^3$  the unit ball of  $\mathbb{R}^3$ ),

$$(I.5.6) \quad t \int_{S^2} u_1(ty) d\sigma(y) = t \int_{B^3} \nabla \cdot (yu_1(ty)) dy = 3t \int_{B^3} u_1(ty) dy + t^2 \int_{B^3} y \cdot \nabla u_1(ty) dy.$$

We have

$$(I.5.7) \quad \left| \int_{B^3} y \cdot \nabla u_1(ty) dy \right| \leq \frac{1}{t^3} \int_{tB^3} |\nabla u_1(y)| dy \leq \frac{3}{t^3} \|u_1\|_{\dot{W}^{1,1}},$$

and

$$(I.5.8) \quad \int_{B^3} |u_1(ty)| dy \leq t \int_{\mathbb{R}^3} |\partial_{x_1} u_1(ty)| dy \leq \frac{1}{t^2} \|u_1\|_{\dot{W}^{1,1}},$$

where we have used the inequality  $\int_{B^3} |\varphi| dx \lesssim \int_{\mathbb{R}^3} |\partial_{x_1} \varphi|$ , that follows immediately from the formula  $\varphi(x_1, x_2, x_3) = \int_{-\infty}^{x_1} \partial_{x_1} \varphi(s, x_2, x_3) ds$ . Combining (I.5.6), (I.5.7) and (I.5.8), we obtain

$$(I.5.9) \quad \left| t \int_{S^2} u_1(ty) d\sigma(y) \right| \lesssim \frac{1}{t} \|u_1\|_{\dot{W}^{1,1}}.$$

By the same proof, using also the inequality  $\int_{B^3} |\varphi| \lesssim \int_{\mathbb{R}^3} |\partial_{x_1} \partial_{x_2} \varphi|$ , we have

$$(I.5.10) \quad \left| \int_{S^2} u_0(ty) d\sigma(y) \right| + \left| \int_{S^2} y \cdot \nabla u_0(ty) d\sigma(y) \right| \lesssim \frac{1}{t} \|u_0\|_{\dot{W}^{2,1}}.$$

This concludes the proof of the dispersion inequality.  $\square$

**5.c. Dimension 1 + 2.** A solution  $u$  of equation (LW) with  $N = 2$  is also a solution of the same equation with  $N = 3$ , constant with respect to the 3rd spatial coordinate. From Theorem I.5.2, one can derive an expression of  $u$  from the initial data. This strategy is called "descent method".

**THEOREM I.5.6.** *Let  $(u_0, u_1) \in (C^3 \times C^2)(\mathbb{R}^2)$ . Then equation (LW) has a unique  $C^2$  solution on  $\mathbb{R} \times \mathbb{R}^2$ , given by the formula*

$$(I.5.11) \quad u(t, x) = \frac{1}{2\pi} \left[ \frac{\partial}{\partial t} \left( t \int_{|y| \leq 1} \frac{u_0(x + ty)}{\sqrt{1 - |y|^2}} dy \right) + t \int_{|y| \leq 1} \frac{u_1(x + ty)}{\sqrt{1 - |y|^2}} dy \right].$$

**PROOF.** Uniqueness follows from Theorem I.4.1. Moreover, as in the proof of Theorem I.5.2, the formula for even solutions in time (with initial condition  $(u_0, 0)$ ) can be easily deduced from the formula for odd solutions in time (with initial condition  $(0, u_1)$ ). So we only consider this second case.

Let  $u$  be a  $C^2$  solution of (LW) on  $\mathbb{R} \times \mathbb{R}^2$ , with initial data  $(u, \partial_t u)(0) = (0, u_1)$ , where  $u_1 \in C^2(\mathbb{R}^2)$ . By Theorem I.5.2, considering  $u$  as a solution on  $\mathbb{R} \times \mathbb{R}^3$ , we obtain:

$$u(t, x_1, x_2) = \frac{t}{4\pi} \int_{S^2} \tilde{u}_1((x_1, x_2, 0) + ty) d\sigma(y) dy,$$

where by definition  $\tilde{u}_1(x_1, x_2, x_3) = u_1(x_1, x_2)$ . Passing to spherical coordinates, we get

$$\begin{aligned} \int_{S^2} \tilde{u}_1((x_1, x_2, 0) + ty) d\sigma(y) &= \int_0^{2\pi} \int_0^\pi u_1(x_1 + t \sin \theta \cos \varphi, x_2 + t \sin \theta \sin \varphi) \sin \theta d\theta d\varphi \\ &= 2 \int_0^{2\pi} \int_0^{\pi/2} u_1(x_1 + t \sin \theta \cos \varphi, x_2 + t \sin \theta \sin \varphi) \sin \theta d\theta d\varphi. \end{aligned}$$

The announced formula then follows from the change of variable  $y_1 = \sin \theta \cos \varphi$ ,  $y_2 = \sin \theta \sin \varphi$ .  $\square$

It can be seen from the formula in Theorem I.5.6 that the strong Huygens principle is not verified in dimension 1 + 2: the solution  $u(t, x)$  depends on the values of the initial condition over the entire ball  $B_{|t|}^2(x)$ , not just on the sphere  $\{y \in \mathbb{R}^2 : |x - y| = |t|\}$ .

## I.6. Conservation Laws

The energy of a solution  $u$  on  $\mathbb{R} \times \mathbb{R}^N$  is defined as:

$$E(\vec{u}(t)) = \int_{\mathbb{R}^N} e_u(t, x) dx = \frac{1}{2} \int_{\mathbb{R}^N} \left( (\partial_t u(t, x))^2 + |\nabla u(t, x)|^2 \right) dx.$$

This is the global version of the local energy considered in §I.4. The energy of a solution is conserved over time.

**THEOREM I.6.1.** *Let  $u \in C^2(\mathbb{R}^{1+N})$  be a solution of (LW), (ID). Assume  $(u_0, u_1)$  has finite energy. Then for any  $t$ ,  $E(\vec{u}(t))$  is finite and  $E(\vec{u}(t)) = E(u_0, u_1)$ .*

PROOF. One might be tempted to write

$$\frac{d}{dt}(E(\vec{u}(t))) = \int \partial_t e_u(t, x) dx = \int \nabla \cdot (\partial_t u \nabla u) dx = 0,$$

but the last equality, obtained by integration by parts ignoring the "boundary" term (i.e., when  $|x| \rightarrow \infty$ ) is purely formal. To justify the preceding calculation, we can use the decay of the local energy (Lemma I.4.2). For  $R > 0$ , we define:

$$E_{<R}(\vec{u}(t)) = \int_{|x| < R} e_u(t, x) dx.$$

Notice that this quantity is finite as soon as  $u \in C^1(\mathbb{R}^{1+N})$ . Fix  $t > 0$ . By Lemma I.4.2, for any  $R > t$ ,

$$E_{<R-t}(\vec{u}(t)) \leq E_{<R}(\vec{u}(0)) \leq E(u_0, u_1).$$

As we let  $R$  tend to  $+\infty$ , we obtain that  $E(\vec{u}(t))$  is finite, and

$$E(\vec{u}(t)) \leq E(u_0, u_1).$$

Reversing the direction of time, we also obtain the inequality

$$E(u_0, u_1) \leq E(\vec{u}(t)).$$

We have shown that the energy is conserved for  $t \geq 0$ . By applying this result to the solution  $(t, x) \mapsto u(-t, x)$ , we obtain energy conservation for  $t \leq 0$ , which concludes the proof.  $\square$

There exists another (vectorial) conserved quantity, the momentum, defined as

$$P(\vec{u}(t)) = \int \partial_t u(t, x) \nabla u(t, x) dx \in \mathbb{R}^N.$$

PROPOSITION I.6.2. *Let  $u \in C^2(\mathbb{R}^{1+N})$  be a solution of (LW) with finite energy. Then*

$$\forall t \in \mathbb{R}, \quad P(\vec{u}(t)) = P(u_0, u_1).$$

The proof of this proposition is left as an exercise (see Exercise I.6).

### I.7. Equation with a source term

We now consider the equation with a source term (I.1.2). We will express the solution of this equation in terms of the propagator of the free equation (LW). For  $(u_0, u_1) \in C^3 \times C^2(\mathbb{R}^3)$ , let  $S_L(t)(u_0, u_1)$  denote the solution of (LW) with initial data  $(u_0, u_1)$  at  $t = 0$ . We denote  $S(t)u_1 = S_L(t)(0, u_1)$ , so that

$$S_L(t)(u_0, u_1) = \frac{\partial}{\partial t} (S(t)u_0) + S(t)u_1.$$

For  $u_1 \in C^2$ , we recall that

$$(S(t)u_1)(x) = tM_{u_1}(t, x) = t \int_{S^2} u_1(x + ty) d\sigma(y).$$

THEOREM I.7.1 (Duhamel's Formula). *Let  $(u_0, u_1) \in (C^2 \times C^3)(\mathbb{R}^3)$  and  $f \in C^2(\mathbb{R} \times \mathbb{R}^3)$ . Then the equation (I.1.2), (ID) has a unique  $C^2$  solution, given by the formula:*

$$u(t) = S_L(t)(u_0, u_1) + \int_0^t S(t-s)f(s)ds.$$

REMARK I.7.2. The Duhamel term  $\int_0^t S(t-s)f(s)ds$  can be explicitated, see (I.7.1).

PROOF OF THEOREM I.7.1. Uniqueness follows immediately from Theorem I.4.1, since the difference of 2 solutions of (I.1.2) with the same source term  $f$  is a solution of (LW). For existence, taking into account Theorem I.5.2, it is sufficient to check that the function

$$U : (t, x) \mapsto \int_0^t S(t-s)f(s)ds$$

is  $C^2$  and satisfies equation (I.1.2) with zero initial conditions.

We have:

$$(I.7.1) \quad U(t, x) = \frac{1}{4\pi} \int_0^t (t-s) \int_{S^2} f(s, x + (t-s)y) d\sigma(y) ds,$$

and the fact that  $U$  is  $C^2$  follows from the theorem on differentiation under the integral sign.

Furthermore, using that  $S(0)g = 0$  for any function  $g$ ,

$$\frac{\partial U}{\partial t} = \int_0^t \frac{\partial}{\partial t} (S(t-s)f(s)) ds.$$

Upon further differentiation, we obtain

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial}{\partial t} (S(t-s)f(s)) \Big|_{s=t} + \int_0^t \frac{\partial^2}{\partial t^2} (S(t-s)f(s)) ds = f(t) + \int_0^t \Delta (S(t-s)f(s)) ds = f(t) + \Delta U.$$

where we used that  $\frac{\partial}{\partial t} (S(t)g) \Big|_{t=0} = g$  for any function  $g$  of class  $C^2$ .  $\square$

REMARK I.7.3. Duhamel's formula is certainly not specific to dimension 3, as shown by the calculation leading to this formula, which is completely independent of dimension. The reader is invited to explicitly rewrite the solution of equation (I.1.2) when  $N = 1$  and  $N = 2$ .

From Duhamel's formula, we deduce the energy inequality:

PROPOSITION I.7.4. *Let  $u$  be a  $C^2$  solution of (I.1.2) with  $N = 3$  with initial data  $(u_0, u_1)$ , such that  $f \in C^2(\mathbb{R}^{1+3})$ . Suppose furthermore that  $(u_0, u_1)$  has finite energy, and for all  $T > 0$ ,*

$$\int_{[-T, +T]} \sqrt{\int_{\mathbb{R}^3} |f(t, x)|^2 dx} dt < \infty.$$

Then for all  $t > 0$ ,

$$\sqrt{2E(\vec{u}(t))} \leq \sqrt{2E(u_0, u_1)} + \int_0^t \sqrt{\int_{\mathbb{R}^3} |f(s, x)|^2 dx} ds.$$

PROOF. To lighten notations, we will denote:

$$\|\vec{u}(t)\|_{\dot{H}^1 \times L^2}^2 = \int_{\mathbb{R}^3} |\nabla u(t, x)|^2 dx + \int_{\mathbb{R}^3} |\partial_t u(t, x)|^2 dx, \quad \|f\|_{L^1(I, L^2)} = \int_I \|f(t)\|_{L^2(\mathbb{R}^3)} dt$$

( $\|\cdot\|_{\dot{H}^1}$  is the norm defining the homogeneous Sobolev space  $\dot{H}^1(\mathbb{R}^3)$ , that we will define in the next chapter). From Duhamel's formula and the conservation of energy for the free equation (LW), it suffices to verify that for all  $t > 0$ ,

$$(I.7.2) \quad \left\| \left( \int_0^t S(t-s)f(s) ds, \partial_t \int_0^t S(t-s)f(s) ds \right) \right\|_{\dot{H}^1 \times L^2} \leq \|f\|_{L^1([0, t], L^2)}$$

By conservation of energy (Theorem I.6.1), we have

$$\left\| (S(t-s)f(s), \partial_t(S(t-s)f(s))) \right\|_{\dot{H}^1 \times L^2} = \|f(s)\|_{L^2},$$

which implies directly (I.7.2)  $\square$

## I.8. Exercises

EXERCICE I.1. Let  $f : \mathbb{R}^N \rightarrow \mathbb{R}$  ( $N \geq 1$ ). Suppose  $f$  is radial (i.e. That it depends only on the variable  $r = |x| = \sqrt{x_1^2 + x_2^2 + \dots + x_N^2}$ ). Denote  $f(x) = g(|x|)$ , where  $g : [0, \infty[ \rightarrow \mathbb{R}$ .

- (1) Show that  $f$  is continuous on  $\mathbb{R}^N$  if and only if  $g$  is continuous on  $[0, \infty[$ .
- (2) Show that  $f$  is  $C^1$  on  $\mathbb{R}^N$  if and only if  $g$  is  $C^1$  on  $[0, \infty[$  and  $g'(0) = 0$ .
- (3) Show that for any  $k \geq 2$ ,  $f$  is  $C^k$  on  $\mathbb{R}^N$  if and only if  $g$  is  $C^k$  on  $\mathbb{R}^N$  and  $g^{(j)}(0) = 0$  for all odd integers  $j \leq k$ .
- (4) Assuming  $f$  is  $C^1$ , compute  $\frac{\partial f}{\partial x_j}$  in terms of  $g'$ ,  $j = 1, \dots, N$ . Compute  $g'(r)$  in terms of  $\nabla f$ .
- (5) Assuming  $f$  is  $C^2$  on  $\mathbb{R}^N$ , prove the formula

$$\Delta f(x) = g''(|x|) + \frac{N-1}{|x|} g'(|x|).$$

To lighten notation, we use the same notation ( $f$ ) for functions  $f$  and  $g$ , and denote  $g' = \frac{df}{dr}$ , etc...

EXERCICE I.2 (Explicit solutions of the radial wave equation in odd space dimension). Let  $N \geq 3$  be an odd integer, written as  $N = 2k + 1$ . Let  $T_k$  be the operator defined by

$$T_k \phi = \left( r^{-1} \frac{d}{dr} \right)^{k-1} (r^{2k-1} \phi(r)).$$

- (1) Show that

$$T_k \varphi = \sum_{j=0}^{k-1} c_j r^{j+1} \phi^{(j)} r,$$

for some  $c_j \in \mathbb{R}$ . Determine  $c_0$  and  $c_{k-1}$ .

- (2) Show that for any function  $\varphi \in C^{k+1}([0, +\infty[)$ ,

$$\frac{d^2}{dr^2}(T_k \varphi) = \left(r^{-1} \frac{d}{dr}\right)^k (r^{2k} \varphi'(r)).$$

Hint: You can start by verifying that the formula is true when  $\varphi(r) = r^m$  for any integer  $m$ .

- (3) Consider a solution  $u(t, x)$  of the linear wave equation in space dimension  $N$ , radial with respect to the space variable. Suppose  $u$  is  $C^{k+1}$  on  $\mathbb{R}^{1+N}$ . Show prove

$$(\partial_t^2 - \partial_r^2)(T_k u) = 0.$$

Deduce an expression of  $T_k u$  in terms of  $u_0$  and  $u_1$ .

- (4) Express  $u(t, r)$  in terms of  $u_0$  and  $u_1$  when  $N = 5$ . What regularity of  $u_0$  and  $u_1$  is required for  $u$  to be  $C^2$  on  $\mathbb{R}^{1+5}$ ?

EXERCICE I.3. Let  $u$  be a solution of the wave equation (LW) in space dimension  $N \geq 3$ , radial with respect to the space variable. Recall that  $\Delta u = \frac{d^2}{dr^2} + \frac{N-1}{r} \frac{d}{dr}$ . Suppose  $u \in C^2(\mathbb{R}^{1+N})$ , with compactly supported initial data. Let

$$v(t, r) = \int_r^\infty \rho \partial_t u(t, \rho) d\rho.$$

Show that  $v$  defines a radial solution, of class  $C^2$ , to the wave equation in space dimension  $N - 2$ .

EXERCICE I.4. Let  $f \in C^k(\mathbb{R}^3)$ . Show that the function  $M_f$ , defined by (I.5.3), is also of class  $C^k$ .

EXERCICE I.5. Let  $u \in C^2(\mathbb{R} \times \mathbb{R}^N)$  be a solution of (LW) with finite energy. Show

$$\forall \varepsilon > 0, \exists R > 0, \forall t \in \mathbb{R}, \int_{|x| > R+|t|} e_u(t, x) dx \leq \varepsilon.$$

EXERCICE I.6 (Conservation of momentum). (1) Let  $u$  be a  $C^2$  solution of (LW) on  $\mathbb{R} \times \mathbb{R}^N$ , and  $j \in 1, \dots, N$ . Let  $p_{j,u}(t, x) = \partial_{x_j} u(t, x) \partial_t u(t, x)$ . Show

$$\frac{\partial p_{j,u}}{\partial t} = \frac{1}{2} \frac{\partial}{\partial x_j} ((\partial_t u)^2 - |\nabla u|^2) + \nabla \cdot V,$$

where  $V$  is a certain  $C^1$  vector field to be specified.

- (2) Assume that  $(u_0, u_1)$  has finite energy. Justify that

$$P_j(\vec{u}(t)) = \int_{\mathbb{R}^N} p_{j,u}(t, x) dx$$

is defined for all times. Show that this quantity is independent of time. You can start by considering a local version of the momentum

$$\int_{[-R, R]^N} p_{j,u}(t, x) dx \text{ or } \int_{\mathbb{R}^N} p_{j,u}(t, x) \varphi\left(\frac{x}{R}\right) dx$$

then let  $R$  tend to  $+\infty$ . Here  $\varphi$  denotes a  $C^2$  function with compact support equal to 1 in a neighborhood of the origin.

EXERCICE I.7. (1) Let  $u_1 \in C^2(\mathbb{R}^3)$  such that

$$\forall t \geq 0, \forall x \in \mathbb{R}^3, \quad u_1(x) \geq 0.$$

Assume  $u_0 = 0$ . Let  $u$  be the corresponding solution of (LW). Prove

$$\forall t \geq 0, \forall x \in \mathbb{R}^3, \quad u(t, x) \geq 0.$$

- (2) Suppose now  $N = 1$  or  $N = 2$ . Let  $u$  be the solution of (LW), (ID), with  $(u_0, u_1) \in C^3 \times C^2$  (if  $N = 2$ ) or  $C^2 \times C^1$  (if  $N = 1$ ).

Show that if  $u_1 \geq 0$  and  $u_0 = 0$  then  $u(t, x)$  has the sign of  $t$  for all  $x$  and  $t \neq 0$ .

When  $N = 1$ , give a weaker sufficient condition on  $(u_0, u_1)$  such that:

$$\forall t \geq 0, \forall x \in \mathbb{R}, \quad u(t, x) \geq 0.$$

EXERCICE I.8. Assume  $N = 1$  or  $N = 2$ . Let  $u$  be a solution of (I.1.2), with  $u_0 = u_1 = 0$ , and  $f$  of class  $C^1$  (if  $N = 1$ ) or  $C^2$  (if  $N = 2$ ). Express  $u$  in terms of  $f$ .

EXERCICE I.9. The *Minkowski spacetime* of dimension  $N$  is the space  $\mathbb{R}^{1+N}$ , equipped with the quadratic form of signature  $(1, N)$ :

$$g(X) = x_0^2 - \sum_{j=1}^N x_j^2 = t^2 - |x|^2 = {}^tXJX,$$

where  ${}^tX$  is the transpose of  $X$ ,

$$X = (x_0, x_1, \dots, x_N), \quad t = x_0, \quad x = (x_1, \dots, x_N),$$

and  $J = [J_{\mu, \nu}]_{0 \leq \mu, \nu \leq N}$  is the matrix such that  $J_{0,0} = 1$ ,  $J_{\ell, \ell} = -1$  if  $\ell \in 1, \dots, N$ , and  $J_{\mu, \nu} = 0$  if  $\mu \neq \nu$ .

The Lorentz group  $O(1, N)$  is the group of real square matrices  $P$  of size  $1 + N$  which leave the quadratic form  $g$  invariant, i.e., such that  $g(PX) = g(X)$  for all  $X$  in  $\mathbb{R}^{1+N}$ . In other words, if  $P$  is a  $(1 + N) \times (1 + N)$  matrix,

$$P \in O(1, N) \iff {}^tPJP = J.$$

- (1) Prove that a function  $v$  of class  $C^2$  on  $\mathbb{R}^{1+N}$  satisfies the wave equation (LW) if and only if  $\text{Tr}(Jv'') = 0$ , where  $v''$  is the Hessian matrix  $[\partial_{x_\mu} \partial_{x_\nu} v]_{0 \leq \mu, \nu \leq N}$ .

- (2) Let  $P \in O(1, N)$ ,  $v \in C^2(\mathbb{R}^{1+N})$ , and  $w(X) = v(PX)$ . Then

$$(\partial_t^2 - \Delta)v = 0 \iff (\partial_t^2 - \Delta)w = 0.$$

- (3) Prove that the space rotations:

$$\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & R \end{bmatrix}, \quad R \in O(N)$$

and the Lorentz boosts

$$\mathcal{R}_\sigma = \begin{bmatrix} R_\sigma & \mathbf{0} \\ \mathbf{0} & I_{N-1} \end{bmatrix}, \quad R_\sigma = \begin{bmatrix} \cosh(\sigma) & \sinh(\sigma) \\ \sinh(\sigma) & \cosh(\sigma) \end{bmatrix},$$

where  $I_{N-1}$  denotes the identity matrix  $(N-1) \times (N-1)$  and  $\sigma \in \mathbb{R}$  are Lorentz transformations. In these formulas,  $\mathbf{0}$  always denotes the zero matrix of appropriate size.

EXERCICE I.10. In all Chapter I, we considered the Cauchy problem with initial conditions on a hyperplane in  $\mathbb{R}^{1+N}$  of the form  $\{t = t_0\}$ . We now seek to solve the same problem by prescribing an initial condition on other hyperplanes. Therefore, we consider a hyperplane of the form

$$\Pi = \{X \in \mathbb{R}^{1+N} : {}^tAX = 0\}$$

where  $A \in \mathbb{R}^{1+N} \setminus \{0\}$ ,  $A = (a_0, a_1, \dots, a_N) = (a_0, a)$ .

- (1) Prove that if  $|a_0| > |a|$ , there exists a transformation  $P \in O(1, N)$  such that

$$\Pi = P(\{(0, x), x \in \mathbb{R}^N\}).$$

*Hint: use compositions of transformations defined in Question (3) of Exercise I.9.*

- (2) If the condition  $|a_0| > |a|$  is satisfied, we can therefore reduce the Cauchy problem with an initial condition

$$u|_\Pi = u_0, \quad A \cdot \nabla u|_\Pi = u_1,$$

to a Cauchy problem with initial conditions at  $t = 0$  as treated above. The hyperplane  $\Pi$  is called *timelike* when  $A = (a_0, a)$  with  $a_0 \in \mathbb{R}$ ,  $A \in \mathbb{R}^N$ , and  $|a_0| > |a|$ .

Prove that  $\Pi$  is timelike if and only if the restriction of the quadratic form  $g$  to  $\Pi$  is negatively defined.

- (3) Under what condition on  $A$  does there exist  $B = (b_0, b_1, \dots, b_N) \in \mathbb{R}^{N+1}$  such that the function

$$e^{A \cdot X + iB \cdot X}$$

is a solution of (LW)?

- (4) Now assume that the hyperplane  $\Pi$  is not timelike. Let  $Y \notin \Pi$ . Construct a sequence of solutions  $(u_n)_n$  of (LW) such that  $u_n(X) = 0$  on  $\Pi$ , such that for any differential operator  $D = \prod_{j=1}^N \partial_{x_1}^{\alpha_1} \dots \partial_{x_N}^{\alpha_N}$  (of arbitrarily large order), there exists  $C > 0$  such that  $|Du_n(X)| \leq Ce^{-n}$  on  $\Pi$ , but  $|u_n(Y)| \rightarrow +\infty$  as  $n \rightarrow \infty$ .





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