Dynamics of semilinear wave equation
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Contents

CHAPTER I

Linear wave equation: classical theory

I.1. Presentation of the equation

The linear wave equation is the equation:

$$\partial_t^2 u - \Delta u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{LW}$$

where $N \geq 1$ is the spatial dimension (in this course, we will often assume $N = 3$), and

$$\Delta = \sum_{k=1}^{N} \frac{\partial^2}{\partial x_k^2}.$$

(We will use either the notations ∂_y or $\frac{\partial}{\partial y}$ for the derivative with respect to the variable $y \in \{t, x_1, \ldots, x_N\}$).

This is an evolution equation: we fix initial data at a certain time $t = t_0$, and are interested in the evolution of the equation over time t. Since the equation is of order 2, we actually fix an initial data for $\vec{u} = (u, \partial_t u)$:

$$\vec{u}_{\restriction t=t_0} = (u_0, u_1) \tag{I.1.1}$$

where (u_0, u_1) is to be taken in a certain functional space.

We will consider in this course initial data *with real values*. The passage to complex or vector values is immediate for most properties of the equation (LW) (by working coordinate by coordinate), but can induce drastic changes in the nonlinear case, if the nonlinearity mixes the components.

Equation (LW) is invariant under several obvious space-time transformations. If u is a solution, it is also the case of

$$\mu u(t - t_0, \lambda(Rx - x_0)),$$

where $\mu \in \mathbb{R}$, $t_0 \in \mathbb{R}$, $\lambda > 0$, $R \in \mathcal{O}_N(R)$, $x_0 \in \mathbb{R}^N$. It is in fact invariant under a larger group of linear transformations, the Lorentz group (cf Exercise I.10 p. 15).

As a consequence, we can limit ourselves, without loss of generality, to the case of an initial time $t_0 = 0$, i.e.

$$\vec{u}_{\restriction t=0} = (u_0, u_1) \tag{ID}$$

Furthermore, the equation is invariant under time inversion: if u is solution, it is also the case of $t \mapsto u(-t, x)$. It is thus a reversible equation.

We will also consider the equation with a force:

$$\partial_t^2 u - \Delta u = f, \tag{I.1.2}$$

(still with an initial condition of type (ID)), whose understanding will be crucial for the study of the nonlinear wave equation.

The Cauchy problem (LW), (ID) can be approached in at least 3 different ways:

- The classical approach which consists in finding an explicit formula to express the solution. It works when the initial data is sufficiently regular ($C^3 \times C^2$ in dimension 3 of space) and gives classical solutions (that is to say C^2 in (t, x) and satisfying (LW) in the sense of classical differentiation).
- The use of the Fourier transformation in space, which is very simple (once the Fourier transformation is known) and particularly effective in Sobolev spaces based on L^2 (which are natural spaces for the study of the equation due to the conservation of energy and other L^2-based quantities). This method allows to obtain weak solutions with degrees of regularity lower than the previous ones, and to use tools based on the Fourier transformation, which can be useful, for example, to prove certain dispersive properties of the equation.
- The "functional analysis" approach, by the theory of semi-groups, which gives the same type of solutions as the previous method.

In this chapter, we will detail the classical method, first by writing the explicit formula for solutions in dimension 1 of space, then in higher dimensions. We will study in the following chapter the equation in the energy space by the Fourier transformation. This chapter is partly based on Chapter 5 of the beautiful book by Folland on partial differential equations [**4**].

I.2. Explicit Formula in Dimension 1

In dimension 1, the equation (LW) can be written as:

$$(\partial_t^2 - \partial_x^2)u = 0, \tag{I.2.1}$$

which can be written $(\partial_t - \partial_x)(\partial_t + \partial_x)u = 0$. We thus make the change of variables $\eta = x + t$, $\xi = x - t$. Setting $v(\eta, \xi) = u\left(\frac{\eta-\xi}{2}, \frac{\eta+\xi}{2}\right)$, or $u(t,x) = v(t+x, t-x)$, we have:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 v}{\partial \eta^2} + \frac{\partial^2 v}{\partial \xi^2} + 2\frac{\partial^2 v}{\partial \xi \partial \eta},$$

and

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial \eta^2} + \frac{\partial^2 v}{\partial \xi^2} - 2\frac{\partial^2 v}{\partial \xi \partial \eta},$$

which gives:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -4\frac{\partial^2 v}{\partial \eta \partial \xi}.$$

Thus, we obtain:

$$\text{(LW)} \iff \frac{\partial^2 v}{\partial \eta \partial \xi} = 0.$$

Let u be a C^2 solution of (I.2.1), (ID). Thus, $u_1 \in C^1(\mathbb{R})$ and $u_0 \in C^2(\mathbb{R})$.

The equality $\frac{\partial^2 v}{\partial \eta \partial \xi} = 0$ shows that $\frac{\partial v}{\partial \xi}$ is a (class C^1) function $w(\xi)$ independent of η. Integrating with respect to ξ for η fixed, we deduce:

$$v(\eta, \xi) = \underbrace{\int_0^\xi w(\sigma) d\sigma}_{\varphi(\xi)} + \psi(\eta),$$

for a certain function ψ, necessarily C^2 since v is of class C^2 and w of class C^1. Thus, we necessarily have:

$$v(\eta, \xi) = \varphi(\xi) + \psi(\eta), \quad \varphi, \psi \in C^2(\mathbb{R}^2),$$

or equivalently:

$$u(t,x) = \varphi(x-t) + \psi(x+t). \tag{I.2.2}$$

Using the initial condition (ID), a direct calculation gives:

$$\psi(\eta) = \frac{1}{2}\int_0^\eta u_1(\sigma) d\sigma + \frac{1}{2}u_0(\eta) + c,$$

$$\varphi(\xi) = -\frac{1}{2}\int_0^\xi u_1(y) dy + \frac{1}{2}u_0(\xi) - c,$$

where $c \in \mathbb{R}$ (the choice of this constant is irrelevant). Hence, we deduce:

$$u(t,x) = \frac{1}{2}(u_0(x+t) + u_0(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} u_1(y) dy. \tag{I.2.3}$$

Conversely, it is easy to verify that formula (I.2.3) gives a C^2 solution of (I.2.1), (ID). Therefore, we have shown:

PROPOSITION I.2.1. *Let $(u_0, u_1) \in C^2(\mathbb{R}) \times C^1(\mathbb{R})$. Then, there exists a unique solution $u \in C^2(\mathbb{R} \times \mathbb{R})$ of* (LW) *satisfying the initial condition* (ID). *This solution satisfies formula* (I.2.3).

On formula (I.2.2), we observe that a solution of the wave equation in dimension 1 is the sum of two waves: one, $\varphi(x-t)$, moving at speed 1 to the right, and the other $\psi(x+t)$, moving at the same speed to the left.[1]

It is also possible to obtain a formula for the equation with the right-hand side (I.1.2). We leave this as an exercise to the reader. Further on, we will provide a general method giving the solution of the equation with the right-hand side in terms of the equation without the right-hand side.

[1]Note that the equations (LW), (I.2.1) have been normalized, so that the speed of propagation is exactly 1.

We can see from formula (I.2.3) that $u(t,x)$ depends only on the values of (u_0,u_1) over $\Big[x-|t|,x+|t|\Big]$. This is a first example of "finite speed of propagation" which holds in all spatial dimensions.

I.3. Integral on the Sphere and Divergence Theorem

We denote $S^{N-1}=\{x\in\mathbb{R}^N,\ |x|=1\}$, where $|\cdot|$ represents the Euclidean norm on $\mathbb{R}^N$:

$$|x|^2=\sum_{j=1}^N x_j^2.$$

More generally, S_R^{N-1} will denote the sphere of radius R: $\{x\in\mathbb{R}^N,\ |x|=R\}$.

We denote $d\sigma$ as the volume element on one of these spheres. Thus, the integral of a function $f\in\mathcal{L}^1(S_R^{N-1})$ (i.e., a function integrable on S_R^{N-1}) is written as

$$\int_{S_R^{N-1}} f(y)d\sigma(y).$$

This integral can be calculated using spherical coordinates. In dimension 3, this writes:

$$\int_{S_R^2} f(y)d\sigma(y)=R^2\int_0^{2\pi}\int_0^{\pi} f(R\sin\theta\cos\varphi,R\sin\theta\sin\varphi,R\sin\varphi)\sin(\theta)d\theta d\varphi.$$

We denote $B_R^N(x_0)$ as the ball centered at x_0 with radius R:

$$B_R^N(x_0)=\Big\{x\in\mathbb{R}^N,\ |x-x_0|<R\Big\}$$

and simply $B_R^N=B_R^N(0)$.

We will use the following formulas:

Scaling:

$$\int_{S_R^{N-1}} f(y)d\sigma(y)=R^{N-1}\int_{S^{N-1}} f(Ry)d\sigma(y)n\quad f\in\mathcal{L}^1(S_R^{N-1}).$$

Integral in radial coordinates: if $f\in\mathcal{L}^1(\{|x|\le R\})$,

$$\int_{B_R^N} f(x)dx=\int_0^R\int_{S_r^{N-1}} f(y)d\sigma(y)dr=\int_0^R\int_{S^{N-1}} f(r\omega)d\sigma(\omega)r^{N-1}dr$$

Divergence theorem: if $F\in C^1(\overline{B_R},\mathbb{R}^N)$,

$$\int_{|x|\le R}\nabla\cdot F(x)dx=\int_{S_R^{N-1}}\frac{y}{|y|}\cdot F(y)d\sigma(y),$$

where $\nabla\cdot F=\sum_{j=1}^N\partial_{x_j}F_j$ is the divergence of the vector field F.

I.4. Energy density. Uniqueness and finite speed of propagation

Before giving an explicit formula for the wave equation in dimension 3, we prove a uniqueness result valid in any dimension:

Theorem I.4.1. *Let* $(t_0,x_0)\in\mathbb{R}^{1+N}$, $t_1>t_0$, $R>0$. *We denote* $\Gamma=\Big\{(t,x)\in\mathbb{R}\times\mathbb{R}^N\ :\ t_0\le t\le t_1,\ |x-x_0|\le R-|t-t_0|\Big\}$. *Let* $u\in C^2(\Gamma)$ *be a solution of* (LW) *on* Γ. *We suppose* $(u,\partial_t u)(t_0,x)=0$ *for all* $x\in B_R(x_0)$. *Then* u *is identically zero on* Γ.

The proof of the theorem is based on a monotonicity law that has its own interest.

We define, for $(t,x)\in\Gamma$, the *density of energy* e_u as

$$e_u(t,x)=\frac{1}{2}|\nabla u(t,x)|^2+\frac{1}{2}(\partial_t u(t,x))^2,$$

where $|\nabla u|^2=\sum_{j=1}^N(\partial_{x_j}u)^2$, and we consider, for $t_0\le t\le t_1$, the local energy

$$E_{\text{loc}}(t)=\int_{B_{R-(t-t_0)}(x_0)} e_u(t,x)dx=\int_{|x-x_0|<R-(t-t_0)} e_u(t,x)dx.$$

Lemma I.4.2. *The function* E_{loc} *is decreasing on* $[t_0,t_1]$.

The lemma immediately implies Theorem I.4.1. Indeed, if $\vec{u}(t_0)$ vanishes on $B(x_0, R)$, then $E_{\text{loc}}(t_0) = 0$, and thus $E_{\text{loc}}(t) = 0$ for all $t \in [t_0, t_1]$, showing that u is zero on Γ.

Proof of Lemma I.4.2. We notice that

$$\frac{\partial e}{\partial t} = \sum_{j=1}^{N} \left(\partial_{x_j} u \partial_t \partial_{x_j} u + \partial^2_{x_j} u \partial_t u \right) = \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \left(\partial_{x_j} u \partial_t u \right) = \nabla \cdot \left(\partial_t u \nabla u \right), \tag{I.4.1}$$

where $\nabla u = (\partial_{x_i} u)_{1 \leq i \leq N}$ Without loss of generality, we can assume that $x_0 = 0$ and $t_0 = 0$. By the integration formula in radial coordinates,

$$E_{\text{loc}}(t) = \int_0^{R-t} s^{N-1} \int_{S^{N-1}} e_u(t, s\omega) d\sigma(\omega) ds.$$

By differentiation under the integral sign, we get that E_{loc} is differentiable and

$$E'_{\text{loc}}(t) = -(R-t)^{N-1} \int_{S^{N-1}} e_u(t, (R-t)\omega) d\sigma(\omega) + \int_{B^N_{R-t}} \frac{\partial e_u}{\partial t}(t, x) dx.$$

By formula (I.4.1), then the divergence formula

$$\int_{B^N_{R-t}} \frac{\partial e_u}{\partial t}(t, x) dx = \int_{B^N_{R-t}} \nabla \cdot \left(\partial_t u \nabla u \right)(t, x) dx = \int_{S^{N-1}_{R-t}} \frac{y}{|y|} \nabla u \partial_t u(t, y) d\sigma(y).$$

We thus have

$$E'_{\text{loc}}(t) = -\int_{S^{N-1}_{R-t}} \left(\frac{1}{2} |\nabla u|^2 + \frac{1}{2} (\partial_t u)^2 - \frac{y}{|y|} \nabla u \partial_t u(t, y) \right) d\sigma(y) \leq -\frac{1}{2} \int_{S^{N-1}_{R-t}} \left(\frac{y}{|y|} \nabla u + \partial_t u(t, y) \right)^2 d\sigma(y).$$

□

I.5. Explicit formulas.

This section is devoted to explicit formulas in space dimensions $N \geq 2$. In dimension $N = 3$, we will show that for any initial data $(u_0, u_1) \in C^2 \times C^3$, there exists a unique solution $u \in C^2(\mathbb{R}^{1+3})$ of (LW), (ID), and provide an explicit formula for this solution. We will also provide a formula in dimension $N = 2$. We refer the reader to [**4**, Chapter 5B] for expressions of solutions when $N \geq 4$.

5.a. The radial case in dimension 3. When the initial conditions depend only on the variable $r = |x|$, the explicit formula is very simple.

We start by observing that if f depends only on the variable r, then the function f is C^2 as a function on $\mathbb{R}^3$ if and only if it is C^2 as a function of the variable r on $[0, \infty[$, and satisfies $\frac{df}{dr}(0) = 0$. Moreover,

$$\Delta f = \frac{d^2 f}{dr^2} + \frac{2}{r} \frac{df}{dr}$$

(cf Exercise I.1). We notice that we can rewrite this formula as

$$r \Delta f = \frac{d^2}{dr^2}(rf).$$

Now let u be a C^2 solution of (LW), (ID) with initial data (u_0, u_1) assumed to be radial. We also assume that for all t, $u(t)$ is a radial function. We will show a posteriori that this second assumption is a consequence of the assumption on the initial data. The previous formula gives

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2} \right)(ru) = 0.$$

The function $(t, r) \mapsto ru(t, r)$ is thus a solution of the wave equation in dimension 1, on $\mathbb{R}_t \times]0, \infty[$. To obtain a function on $\mathbb{R}^2$, we extend $ru(t, r)$ to an odd function:

$$v(t, y) = yu(t, |y|).$$

One can verify (using Exercise I.1) that v is of class C^2 on $\mathbb{R}^2$, and that

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2} \right) v = 0.$$

Formula (I.2.3) then gives:

$$v(t,y)=\frac{1}{2}(v_0(y+t)+v_0(y-t))+\frac{1}{2}\int_{y-t}^{y+t}v_1(\sigma)d\sigma,$$

where $(v_0,v_1)=\vec{v}_{\restriction t=0}$, thus

$$u(t,r)=\frac{1}{2r}\Big((r+t)u_0(|r+t|)+(r-t)u_0(|r-t|)\Big)+\frac{1}{2r}\int_{r-t}^{r+t}\sigma u_1(|\sigma|)d\sigma. \tag{I.5.1}$$

Notice that when $t>0$ (to fix ideas),

$$\int_{r-t}^{r+t}\sigma u_1(|\sigma|)d\sigma=\int_{|r-t|}^{r+t}\sigma u_1(|\sigma|)d\sigma.$$

The finite speed of propagation is satisfied: the solution $u(t,r)$ depends only on the initial condition (u_0,u_1) on the ball centered at r with radius $|t|$.

The formula (I.5.1) defines a function $u(t,r)$ of class C^2 outside the origin $x=0$, as soon as the initial conditions (u_0,u_1) have the expected regularity $C^2\times C^1$. However, there is a subtle phenomenon of loss of regularity of the solution u compared to the initial data at the origin : there exist data $(u_0,u_1)\in C^2\times C^1$ such that u, defined by formula (I.5.1), cannot be extended by a C^2 function up to $r=0$. Indeed, it can be checked that (at fixed t),

$$\lim_{r\to 0}u(t,r)=u_0(t)+tu_0'(t)+tu_1(t), \tag{I.5.2}$$

which shows that if (u_0,u_1) are $C^k\times C^{k-1}$ functions, then $u(t,0)$ is only C^{k-1} in general (see also Exercise I.2). We can interpret this phenomenon physically as follows: a singularity on the circle $r=r_0$ at the initial time 0 that travels at speed 1 towards the origin will concentrate at the origin at time $t=r_0$, causing a stronger singularity.

The limit (I.5.2) suggests a maximal loss of regularity of a derivative with respect to the initial data, which is indeed the case:

Proposition I.5.1. *Let $(u_0,u_1)\in(C^3\times C^2)(\mathbb{R}^3)$ be radial functions. Then formula* (I.5.1) *extended by $u(t,0)=u_0(t)+tu_0'(t)+tu_1(t)$, defines a C^2 function on $\mathbb{R}\times\mathbb{R}^3$, radial with respect to the variable x, and satisfying* (LW), (ID).

The Proposition I.5.1 is left as an exercise to the reader. Combining with the uniqueness property (Theorem I.4.1), we obtain that (I.5.1) gives the unique solution of (LW) with initial data (u_0,u_1).

The formula (I.5.1) is remarkably simple. In higher space dimensions, we also have an explicit formula for radial solutions, which becomes more complicated as the dimension increases (see Exercise I.3). The loss of regularity observed in dimension 3 (and absent in dimension 1) increases with dimension, as the reader can verify on the formula obtained in Exercise I.3.

There is no simple formula in the radial case in even dimensions.

We also have explicit formulas (of course more complicated) without radiality assumptions, in all dimensions. We will explicitly state these formulas when $N=3$, then $N=2$.

5.b. General solutions in dimension 3: averaging over spheres. If $f\in C^0(\mathbb{R}^3)$, we define

$$(M_f)(t,x)=\frac{1}{4\pi}\int_{S^2}f(x+ty)d\sigma(y)=\frac{1}{4\pi t^2}\int_{S^2_{|t|}}f(x+z)d\sigma(z). \tag{I.5.3}$$

the average of f over the sphere of radius $|t|$ and center x. The function M_f inherits the regularity of f (cf exercise I.5).

Theorem I.5.2. *Let $(u_0,u_1)\in C^3(\mathbb{R}^3)\times C^2(\mathbb{R}^3)$. Then the unique C^2 solution of the wave equation* (LW) *with initial conditions* (ID) *is given by*

$$u(t,x)=tM_{u_1}(t,x)+\frac{\partial}{\partial t}(tM_{u_0}(t,x)).$$

Proof. We start by verifying that $tM_{u_1}(t,x)$ is the solution of the wave equation (LW), with initial condition $(0,u_1)$. By the theorem of differentiation under the integral sign, if $g\in C^2(\mathbb{R}^3)$,

$$\frac{\partial}{\partial t}\left(M_g(t,x)\right)=\frac{1}{4\pi}\int_{S^2}(y\cdot\nabla g)(x+ty)d\sigma(y).$$

Using the divergence formula,

$$\int_{S^2} (y\cdot\nabla g)(x+ty)d\sigma(y) = t\int_{|y|\leq 1} (\nabla\cdot(\nabla g))\,(x+ty)dy$$
$$= t\int_{|y|\leq 1} (\Delta g)(x+ty)dy \;=\frac{1}{t^2}\int_0^t\int_{S^2}(\Delta g)(x+sy)s^2d\sigma(y)ds.$$

Thus:

$$\frac{\partial}{\partial t}\left(tM_{u_1}(t,x)\right) = M_{u_1}(t,x) + \frac{1}{4\pi t}\int_0^t\int_{S^2}(\Delta u_1)(x+sy)d\sigma(y)s^2ds.$$

and therefore

$$\frac{\partial^2}{\partial t^2}\left(tM_{u_1}(t,x)\right) = \frac{1}{4\pi t^2}\int_0^t\int_{S^2}(\Delta u_1)(x+sy)d\sigma(y)s^2ds$$
$$-\frac{1}{4\pi t^2}\int_0^t\int_{S^2}(\Delta u_1)(x+sy)d\sigma(y)s^2ds + \frac{t}{4\pi}\int_{S^2}(\Delta u_1)(x+ty)d\sigma(y)\;=\Delta\left(tM_{u_1}(t,x)\right).$$

This shows that tM_{u_1} satisfies the wave equation (LW). Furthermore, since $M_{u_1}(0,x) = u_1(0,x)$, the initial condition at $t=0$ is indeed $(0,u_1)$.

Now let $v(t,x) = tM_{u_0}(t,x)$. Then, by the same reasoning, v is a solution of the wave equation (LW) with initial condition $(0,u_0)$. We deduce that $\partial_t v$ is a solution of the wave equation with initial condition $(u_0,0)$, which concludes the proof. □

Notice that we can rewrite the formula of the theorem as:

$$u(t,x) = tM_{u_1}(t,x) + M_{u_0}(t,x) + tM_{y\cdot\nabla u_0}(t,x). \tag{I.5.4}$$

We now give two important consequences of the previous formula.

COROLLARY I.5.3 (Strong Huygens' principle). *The solution $u(t,x)$ depends only on the values of u_0, ∇u_0, and u_1 on the sphere centered at x and of radius $|t|$.*

REMARK I.5.4. The strong Huygens' principle is a stronger version of the finite speed of propagation property, which states that $u(t,x)$ depends only on the values of (u_0,u_1) on the *ball* centered at x and of radius $|t|$. This principle remains valid in any odd dimension ≥ 3 (the number of derivatives of u_0 and u_1 in the statement increases with the dimension). In even dimension, solutions only satisfy the finite speed of propagation: see §5.c. In dimension 1, as shown by formula (I.2.3), only solutions that are even in time (i.e. with initial condition of the form $(u_0,0)$) satisfy the strong Huygens' principle.

The second consequence of the explicity formula proved above is an estimate related to the dispersive nature of the wave equation. We will denote

$$\|\varphi\|_{\dot{W}^{s,p}} = \sup_{|\alpha|=s}\|\partial_x^\alpha\varphi\|_{L^p(\mathbb{R}^N)}. \tag{I.5.5}$$

We prove:

THEOREM I.5.5 (Dispersion inequality). *Let $(u_0,u_1)\in(C^3\times C^2)(\mathbb{R}^3)$, with compact support and u the solution of* (LW), (ID). *Then for all $t>0$,*

$$\|u(t)\|_{L^\infty(\mathbb{R}^3)} \lesssim \frac{1}{t}\left(\|u_0\|_{\dot{W}^{2,1}} + \|u_1\|_{\dot{W}^{1,1}}\right).$$

PROOF. By space translation invariance it is sufficient to bound $|u(t,0)|$. We have

$$4\pi u(t,0) = t\int_{S^2}u_1(ty)d\sigma(y) + \int_{S^2}u_0(ty)d\sigma(y) + t\int_{S^2}y\cdot\nabla u_0(ty)d\sigma(y).$$

By the divergence theorem (denoting by B^3 the unit ball of $\mathbb{R}^3$),

$$t\int_{S^2}u_1(ty)d\sigma(y) = t\int_{B^3}\nabla\cdot\left(yu_1(ty)\right)dy = 3t\int_{B^3}u_1(ty)dy + t^2\int_{B^3}y\cdot\nabla u_1(ty)dy. \tag{I.5.6}$$

We have

$$\left|\int_{B^3}y\cdot\nabla u_1(ty)dy\right| \leq \frac{1}{t^3}\int_{tB^3}|\nabla u_1(y)|dy \leq \frac{3}{t^3}\|u_1\|_{\dot{W}^{1,1}}, \tag{I.5.7}$$

and

$$\int_{B^3} |u_1(ty)|dy \leq t \int_{\mathbb{R}^3} |\partial_{x_1} u_1(ty)|dy \leq \frac{1}{t^2}\|u_1\|_{\dot{W}^{1,1}}, \tag{I.5.8}$$

where we have used the inequality $\int_{B^3} |\varphi|dx \lesssim \int_{\mathbb{R}^3} |\partial_{x_1}\varphi|$, that follows immediately from the formula $\varphi(x_1, x_2, x_3) = \int_{-\infty}^{x_1} \partial_{x_1}\varphi(s, x_2, x_3)ds$. Combining (I.5.6), (I.5.7) and (I.5.8), we obtain

$$\left| t \int_{S^2} u_1(ty)d\sigma(y) \right| \lesssim \frac{1}{t}\|u_1\|_{\dot{W}^{1,1}}. \tag{I.5.9}$$

By the same proof, using also the inequality $\int_{B^3} |\varphi| \lesssim \int_{\mathbb{R}^3} |\partial_{x_1}\partial_{x_2}\varphi|$, we have

$$\left| \int_{S^2} u_0(ty)d\sigma(y) \right| + \left| \int_{S^2} y \cdot \nabla u_0(ty)d\sigma(y) \right| \lesssim \frac{1}{t}\|u_0\|_{\dot{W}^{2,1}}. \tag{I.5.10}$$

This concludes the proof of the dispersion inequality. □

5.c. Dimension $1 + 2$. A solution u of equation (LW) with $N = 2$ is also a solution of the same equation with $N = 3$, constant with respect to the 3rd spatial coordinate. From Theorem I.5.2, one can derive an expression of u from the initial data. This strategy is called "descent method".

THEOREM I.5.6. *Let $(u_0, u_1) \in (C^3 \times C^2)(\mathbb{R}^2)$. Then equation* (LW) *has a unique C^2 solution on $\mathbb{R} \times \mathbb{R}^2$, given by the formula*

$$u(t, x) = \frac{1}{2\pi}\left[\frac{\partial}{\partial_t}\left(t\int_{|y|\leq 1} \frac{u_0(x + ty)}{\sqrt{1 - |y|^2}}dy\right) + t\int_{|y|\leq 1} \frac{u_1(x + ty)}{\sqrt{1 - |y|^2}}dy\right]. \tag{I.5.11}$$

PROOF. Uniqueness follows from Theorem I.4.1. Moreover, as in the proof of Theorem I.5.2, the formula for even solutions in time (with initial condition $(u_0, 0)$) can be easily deduced from the formula for odd solutions in time (with initial condition $(0, u_1)$). So we only consider this second case.

Let u be a C^2 solution of (LW) on $\mathbb{R} \times \mathbb{R}^2$, with initial data $(u, \partial_t u)(0) = (0, u_1)$, where $u_1 \in C^2(\mathbb{R}^2)$. By Theorem I.5.2, considering u as a solution on $\mathbb{R} \times \mathbb{R}^3$, we obtain:

$$u(t, x_1, x_2) = \frac{t}{4\pi}\int_{S^2} \tilde{u}_1((x_1, x_2, 0) + ty)d\sigma(y)dy,$$

where by definition $\tilde{u}_1(x_1, x_2, x_3) = u_1(x_1, x_2)$. Passing to spherical coordinates, we get

$$\begin{aligned}
&\int_{S^2} \tilde{u}_1((x_1, x_2, 0) + ty)d\sigma(y) \\
&\qquad = \int_0^{2\pi}\int_0^{\pi} u_1(x_1 + t\sin\theta\cos\varphi, x_2 + t\sin\theta\sin\varphi)\sin\theta d\theta d\varphi \\
&\qquad\qquad = 2\int_0^{2\pi}\int_0^{\pi/2} u_1(x_1 + t\sin\theta\cos\varphi, x_2 + t\sin\theta\sin\varphi)\sin\theta d\theta d\varphi.
\end{aligned}$$

The announced formula then follows from the change of variable $y_1 = t\sin\theta\cos\varphi$, $y_2 = t\sin\theta\sin\varphi$. □

It can be seen from the formula in Theorem I.5.6 that the strong Huygens principle is not verified in dimension $1 + 2$: the solution $u(t, x)$ depends on the values of the initial condition over the entire ball $B^2_{|t|}(x)$, not just on the sphere $\{y \in \mathbb{R}^2 \ : \ |x - y| = |t|\}$.

I.6. Conservation Laws

The energy of a solution u on $\mathbb{R} \times \mathbb{R}^N$ is defined as:

$$E(\vec{u}(t)) = \int_{\mathbb{R}^N} e_u(t, x)dx = \frac{1}{2}\int_{\mathbb{R}^N} \Big((\partial_t u(t, x))^2 + |\nabla u(t)|^2\Big)dx.$$

This is the global version of the local energy considered in §I.4. The energy of a solution is conserved over time.

THEOREM I.6.1. *Let $u \in C^2(\mathbb{R}^{1+N})$ be a solution of* (LW), (ID). *Assume (u_0, u_1) has finite energy. Then for any t, $E(\vec{u}(t))$ is finite and $E(\vec{u}(t)) = E(u_0, u_1)$.*

PROOF. One might be tempted to write

$$\frac{d}{dt}(E(\vec{u}(t))) = \int \partial_t e_u(t,x)dx = \int \nabla \cdot (\partial_t u \nabla u)dx = 0,$$

but the last equality, obtained by integration by parts ignoring the "boundary" term (i.e., when $|x| \to \infty$) is purely formal. To justify the preceding calculation, we can use the decay of the local energy (Lemma I.4.2). For $R > 0$, we define:

$$E_{<R}(\vec{u}(t)) = \int_{|x|<R} e_u(t,x)dx.$$

Notice that this quantity is finite as soon as $u \in C^1(\mathbb{R}^{1+N})$. Fix $t > 0$. By Lemma I.4.2, for any $R > t$,

$$E_{<R-t}(\vec{u}(t)) \leq E_{<R}(\vec{u}(0)) \leq E(u_0, u_1).$$

As we let R tend to $+\infty$, we obtain that $E(\vec{u}(t))$ is finite, and

$$E(\vec{u}(t)) \leq E(u_0, u_1).$$

Reversing the direction of time, we also obtain the inequality

$$E(u_0, u_1) \leq E(\vec{u}(t)).$$

We have shown that the energy is conserved for $t \geq 0$. By applying this result to the solution $(t,x) \mapsto u(-t,x)$, we obtain energy conservation for $t \leq 0$, which concludes the proof. □

There exists another (vectorial) conserved quantity, the momentum, defined as

$$P(\vec{u}(t)) = \int \partial_t u(t,x) \nabla u(t,x) dx \in \mathbb{R}^N.$$

PROPOSITION I.6.2. *Let $u \in C^2(\mathbb{R}^{1+N})$ be a solution of* (LW) *with finite energy. Then*

$$\forall t \in \mathbb{R}, \quad P(\vec{u}(t)) = P(u_0, u_1).$$

The proof of this proposition is left as an exercise (see Exercise I.7).

I.7. Equation with a source term

We now consider the equation with a source term (I.1.2). We will express the solution of this equation in terms of the propagator of the free equation (LW). For $(u_0, u_1) \in C^3 \times C^2(\mathbb{R}^3)$, let $S_L(t)(u_0, u_1)$ denote the solution of (LW) with initial data (u_0, u_1) at $t = 0$. We denote $S(t)u_1 = S_L(t)(0, u_1)$, so that

$$S_L(t)(u_0, u_1) = \frac{\partial}{\partial t}(S(t)u_0) + S(t)u_1.$$

For $u_1 \in C^2$, we recall that

$$(S(t)u_1)(x) = tM_{u_1}(t,x) = t\int_{S^2} u_1(x+ty)d\sigma(y).$$

THEOREM I.7.1 (Duhamel's Formula). *Let $(u_0, u_1) \in (C^2 \times C^3)(\mathbb{R}^3)$ and $f \in C^2(\mathbb{R} \times \mathbb{R}^3)$. Then the equation* (I.1.2), (ID) *has a unique C^2 solution, given by the formula:*

$$u(t) = S_L(t)(u_0, u_1) + \int_0^t S(t-s)f(s)ds.$$

REMARK I.7.2. The Duhamel term $\int_0^t S(t-s)f(s)ds$ can be explicited, see (I.7.1).

PROOF OF THEOREM I.7.1. Uniqueness follows immediately from Theorem I.4.1, since the difference of 2 solutions of (I.1.2) with the same source term f is a solution of (LW). For existence, taking into account Theorem I.5.2, it is sufficient to check that the function

$$U : (t,x) \mapsto \int_0^t S(t-s)f(s)ds$$

is C^2 and satisfies equation (I.1.2) with zero initial conditions.

We have:

$$U(t,x) = \frac{1}{4\pi}\int_0^t (t-s)\int_{S^2} f(s, x+(t-s)y)d\sigma(y)ds, \tag{I.7.1}$$

and the fact that U is C^2 follows from the theorem on differentiation under the integral sign.

Furthermore, using that $S(0)g=0$ for any function g,

$$\frac{\partial U}{\partial t}=\int_0^t\frac{\partial}{\partial t}\Big(S(t-s)f(s)\Big)ds.$$

Upon further differentiation, we obtain

$$\frac{\partial^2 U}{\partial t^2}=\frac{\partial}{\partial t}\Big(S(t-s)f(s)\Big)_{\restriction s=t}+\int_0^t\frac{\partial^2}{\partial t^2}\Big(S(t-s)f(s)\Big)ds=f(t)+\int_0^t\Delta\Big(S(t-s)f(s)\Big)ds=f(t)+\Delta U.$$

where we used that $\frac{\partial}{\partial t}(S(t)g)_{\restriction t=0}=g$ for any function g of class C^2. □

Remark I.7.3. Duhamel's formula is certainly not specific to dimension 3, as shown by the calculation leading to this formula, which is completely independent of dimension. The reader is invited to explicitly rewrite the solution of equation (I.1.2) when $N=1$ and $N=2$.

From Duhamel's formula, we deduce the energy inequality:

Proposition I.7.4. *Let u be a C^2 solution of (I.1.2) with $N=3$ with initial data (u_0,u_1), such that $f\in C^2(\mathbb{R}^{1+3})$. Suppose furthermore that (u_0,u_1) has finite energy, and for all $T>0$,*

$$\int_{[-T,+T]}\sqrt{\int_{\mathbb{R}^3}|f(t,x)|^2dx}dt<\infty.$$

Then for all $t>0$,

$$\sqrt{2E(\vec{u}(t))}\leq\sqrt{2E(u_0,u_1)}+\int_0^t\sqrt{\int_{\mathbb{R}^3}|f(s,x)|^2dx}ds.$$

Proof. To lighten notations, we will denote:

$$\|\vec{u}(t)\|^2_{\dot{H}^1\times L^2}=\int_{\mathbb{R}^3}|\nabla u(t,x)|^2dx+\int_{\mathbb{R}^3}|\partial_t u(t,x)|^2dx,\quad \|f\|_{L^1(I,L^2)}=\int_I\|f(t)\|_{L^2(\mathbb{R}^3)}dt$$

($\|\cdot\|_{\dot{H}^1}$ is the norm defining the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^3)$, see Section II.2 below). From Duhamel's formula and the conservation of energy for the free equation (LW), it suffices to verify that for all $t>0$,

$$\left\|\left(\int_0^t S(t-s)f(s)ds,\partial_t\int_0^t S(t-s)f(s)ds\right)\right\|_{\dot{H}^1\times L^2}\leq\|f\|_{L^1([0,t],L^2)}\tag{I.7.2}$$

By conservation of energy (Theorem I.6.1), we have

$$\Big\|(S(t-s)f(s),\partial_t(S(t-s)f(s)))\Big\|_{\dot{H}^1\times L^2}=\|f(s)\|_{L^2},$$

which implies directly (I.7.2) □

I.8. Exercises

Exercice I.1. Let $f:\mathbb{R}^N\to\mathbb{R}$ ($N\geq 1$). Suppose f is radial (i.e. That it depends only on the variable $r=|x|=\sqrt{x_1^2+x_2^2+\ldots+x_N^2}$). Denote $f(x)=g(|x|)$, where $g:[0,\infty[\to\mathbb{R}$.

(1) Show that f is continuous on $\mathbb{R}^N$ if and only if g is continuous on $[0,\infty[$.
(2) Show that f is C^1 on $\mathbb{R}^N$ if and only if g is C^1 on $[0,\infty[$ and $g'(0)=0$.
(3) Show that for any $k\geq 2$, f is C^k on $\mathbb{R}^N$ if and only if g is C^k on $\mathbb{R}^N$ and $g^{(j)}(0)=0$ for all odd integers $j\leq k$.
(4) Assuming f is C^1, compute $\frac{\partial f}{\partial x_j}$ in terms of g', $j=1,\ldots,N$. Compute $g'(r)$ in terms of ∇f.
(5) Assuming f is C^2 on $\mathbb{R}^N$, prove the formula

$$\Delta f(x)=g''(|x|)+\frac{N-1}{|x|}g'(|x|).$$

To lighten notation, we use the same notation (f) for functions f and g, and denote $g'=\frac{df}{dr}$, etc...

Exercice I.2 (Loss of regularity for the radial wave equation in dimension 1+3). Let $k\geq 0$ and $f\in C^k(\mathbb{R}^3)$ be a *radial* function. Define a function u on $\mathbb{R}\times(\mathbb{R}^3\setminus\{0\})$, radial with respect to the space variable, by

$$u(t,x)=\frac{1}{2r}\Big((r+t)f(|r+t|)+(r-t)f(|r-t|)\Big),$$

where $r=|x|$. Note that this defines a function of class C^k on $\mathbb{R}\times\big(\mathbb{R}^3\setminus\{0\}\big)$.

(1) Suppose that f is supported in the annulus $\{\frac{1}{2} \leq |x| \leq 2\}$ and is such that for $|\eta - 1| \leq 1/10$,

$$f(\eta) = \begin{cases} 2-\eta & \text{if } \eta > 1 \\ \eta & \text{if } \eta < 1 \end{cases}.$$

Calculate $\lim_{r\to 0} u(t,r)$ when $t=1$, $t>1$, and $t<1$ (close to 1). Conclude that u cannot be extended to a continuous function on $\mathbb{R}\times\mathbb{R}^3$.

(2) Similarly, give an example of a C^2 function f such that u cannot be extended to a C^2 function on $\mathbb{R}\times\mathbb{R}^3$.

(3) Assume f is C^3. Show that u defines a C^2 function on $\mathbb{R}\times\mathbb{R}^3$.

(4) Let g be a C^2 radial function on $\mathbb{R}^3$. Show that

$$u(t,r) = \frac{1}{2r}\int_{r-t}^{r+t} \sigma g(|\sigma|)d\sigma,$$

extends to a C^2 function on $\mathbb{R}^3$.

Exercice I.3 (Explicit solutions of the radial wave equation in odd space dimension). Let $N \geq 3$ be an odd integer, written as $N = 2k+1$. Let T_k be the operator defined by

$$T_k\phi = \left(r^{-1}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\phi(r)\right).$$

(1) Show that

$$T_k\varphi = \sum_{j=0}^{k-1} c_j r^{j+1}\phi^{(j)}r,$$

for some $c_j \in \mathbb{R}$. Determine c_0 and c_{k-1}.

(2) Show that for any function $\varphi \in C^{k+1}([0,+\infty[)$,

$$\frac{d^2}{dr^2}(T_k\varphi) = \left(r^{-1}\frac{d}{dr}\right)^k (r^{2k}\varphi'(r)).$$

Hint: You can start by verifying that the formula is true when $\varphi(r) = r^m$ for any integer m.

(3) Consider a solution $u(t,x)$ of the linear wave equation in space dimension N, radial with respect to the space variable. Suppose u is C^{k+1} on $\mathbb{R}^{1+N}$. Show prove

$$(\partial_t^2 - \partial_r^2)(T_k u) = 0.$$

Deduce an expression of $T_k u$ in terms of u_0 and u_1.

(4) Express $u(t,r)$ in terms of u_0 and u_1 when $N=5$. What regularity of u_0 and u_1 is required for u to be C^2 on $\mathbb{R}^{1+5}$?

****

Exercice I.4. Let u be a solution of the wave equation (LW) in spce dimension $N \geq 3$, radial with respect to the space variable. Recall that $\Delta u = \frac{d^2}{dr^2} + \frac{N-1}{r}\frac{d}{dr}$. Suppose $u \in C^2(\mathbb{R}^{1+N})$, with compactly supported initial data. Let

$$v(t,r) = \int_r^\infty \rho\partial_t u(t,\rho)d\rho.$$

Show that v defines a radial solution, of class C^2, to the wave equation in space dimension $N-2$.

Exercice I.5. Let $f \in C^k(\mathbb{R}^3)$. Show that the function M_f, defined by (I.5.3), is also of class C^k.

Exercice I.6. Let $u \in C^2(\mathbb{R}\times\mathbb{R}^N)$ be a solution of (LW) with finite energy. Show

$$\forall \varepsilon > 0,\ \exists R > 0,\ \forall t \in \mathbb{R},\quad \int_{|x|>R+|t|} e_u(t,x)dx \leq \varepsilon.$$

Exercice I.7 (Conservation of momentum). (1) Let u be a C^2 solution of (LW) on $\mathbb{R}\times\mathbb{R}^N$, and $j \in 1,\dots N$. Let $p_{j,u}(t,x) = \partial_{x_j}u(t,x)\partial_t u(t,x)$. Show

$$\frac{\partial p_{j,u}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x_j}\left((\partial_t u)^2 - |\nabla u|^2\right) + \nabla\cdot V,$$

where V is a certain C^1 vector field to be specified.

(2) Assume that (u_0, u_1) has finite energy. Justify that
$$P_j(\vec{u}(t)) = \int_{\mathbb{R}^N} p_{j,u}(t,x)dx$$
is defined for all times. Show that this quantity is independent of time. You can start by considering a local version of the momentum
$$\int_{[-R,R]^N} p_{j,u}(t,x)dx \text{ or } \int_{\mathbb{R}^N} p_{j,u}(t,x)\varphi\left(\frac{x}{R}\right)dx$$
then let R tend to $+\infty$. Here φ denotes a C^2 function with compact support equal to 1 in a neighborhood of the origin.

EXERCICE I.8. (1) Let $u_1 \in C^2(\mathbb{R}^3)$ such that
$$\forall t \geq 0,\ \forall x \in \mathbb{R}^3, \quad u_1(x) \geq 0.$$
Assume $u_0 = 0$. Let u be the corresponding solution of (LW). Prove
$$\forall t \geq 0,\ \forall x \in \mathbb{R}^3, \quad u(t,x) \geq 0.$$

(2) Suppose now $N = 1$ or $N = 2$. Let u be the solution of (LW), (ID), with $(u_0, u_1) \in C^3 \times C^2$ (if $N = 2$) or $C^2 \times C^1$ (if $N = 1$).
Show that if $u_1 \geq 0$ and $u_0 = 0$ then $u(t,x)$ has the sign of t for all x and $t \neq 0$.
When $N = 1$, give a weaker sufficient condition on (u_0, u_1) such that:
$$\forall t \geq 0,\ \forall x \in \mathbb{R}, \quad u(t,x) \geq 0.$$

EXERCICE I.9. Assume $N = 1$ or $N = 2$. Let u be a solution of (I.1.2), with $u_0 = u_1 = 0$, and f of class C^1 (if $N = 1$) or C^2 (if $N = 2$). Express u in terms of f.

EXERCICE I.10. The *Minkowski spacetime* of dimension N is the space $\mathbb{R}^{1+N}$, equipped with the quadratic form of signature $(1, N)$:
$$g(X) = x_0^2 - \sum_{j=1}^{N} x_j^2 = t^2 - |x|^2 = {}^tXJX,$$
where ${}^tX$ is the transpose of X,
$$X = (x_0, x_1, \ldots, x_N),\ t = x_0,\ x = (x_1, \ldots, x_N),$$
and $J = [J_{\mu,\nu}]_{0\leq\mu\ \nu\leq N}$ is the matrix such that $J_{0,0} = 1$, $J_{\ell,\ell} = -1$ if $\ell \in 1, \ldots, N$, and $J_{\mu,\nu} = 0$ if $\mu \neq \nu$.

The Lorentz group $\mathrm{O}(1, N)$ is the group of real square matrices P of size $1 + N$ which leave the quadratic form g invariant, i.e., such that $g(PX) = g(X)$ for all X in $\mathbb{R}^{1+N}$. In other words, if P is a $(1 + N) \times (1 + N)$ matrix,
$$P \in \mathrm{O}(1, N) \iff {}^tPJP = J.$$

(1) Prove that a function v of class C^2 on $\mathbb{R}^{1+N}$ satisfies the wave equation (LW) if and only if $\mathrm{Tr}(Jv'') = 0$, where v'' is the Hessian matrix $\left[\partial_{x_\mu}\partial_{x_\nu} v\right]_{0\leq {\mu \atop \nu} \leq N}$.

(2) Let $P \in \mathrm{O}(1, N)$, $v \in C^2(\mathbb{R}^{1+N})$, and $w(X) = v(PX)$. Then
$$(\partial_t^2 - \Delta)v = 0 \iff (\partial_t^2 - \Delta)w = 0.$$

(3) Prove that the space rotations:
$$\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & R \end{bmatrix}, \quad R \in \mathrm{O}(N)$$
and the Lorentz boosts
$$\mathcal{R}\sigma = \begin{bmatrix} R_\sigma & \mathbf{0} \\ \mathbf{0} & I_{N-1}. \end{bmatrix}, \quad R_\sigma = \begin{bmatrix} \cosh(\sigma) & \sinh(\sigma) \\ \sinh(\sigma) & \cosh(\sigma) \end{bmatrix},$$
where I_{N-1} denotes the identity matrix $(N - 1) \times (N - 1)$ and $\sigma \in \mathbb{R}$ are Lorentz transformations. In these formulas, $\mathbf{0}$ always denotes the zero matrix of appropriate size.

EXERCICE I.11. In all Chapter I, we considered the Cauchy problem with initial conditions on a hyperplane in $\mathbb{R}^{1+N}$ of the form $\{t = t_0\}$. We now seek to solve the same problem by prescribing an initial condition on other hyperplanes. Therefore, we consider a hyperplane of the form
$$\Pi = \{X \in \mathbb{R}^{1+N}\ :\ {}^tAX = 0\}$$
where $A \in \mathbb{R}^{1+N} \setminus \{0\}$, $A = (a_0, a_1, \ldots, a_N) = (a_0, a)$.

(1) Prove that if $|a_0| > |a|$, there exists a transformation $P \in \mathrm{O}(1,N)$ such that

$$\Pi = P\left(\{(0,x),\ x \in \mathbb{R}^N\}\right).$$

Hint: use compositions of transformations defined in Question (3) *of Exercise I.10.*

(2) If the condition $|a_0| > |a|$ is satisfied, we can therefore reduce the Cauchy problem with an initial condition

$$u_{\restriction \Pi} = u_0, \quad A \cdot \nabla u_{\restriction \Pi} = u_1,$$

to a Cauchy problem with initial conditions at $t = 0$ as treated above. The hyperplane Π is called *timelike* when $A = (a_0, a)$ with $a_0 \in \mathbb{R}$, $A \in \mathbb{R}^N$, and $|a_0| > |a|$.

Prove that Π is timelike if and only if the restriction of the quadratic form g to Π is negatively defined.

(3) Under what condition on A does there exist $B = (b_0, b_1, \dots, b_N) \in \mathbb{R}^{N+1}$ such that the function

$$e^{A \cdot X + iB \cdot X}$$

is a solution of (LW)?

(4) Now assume that the hyperplane Π is not timelike. Let $Y \notin \Pi$. Construct a sequence of solutions $(u_n)_n$ of (LW) such that $u_n(X) = 0$ on Π, such that for any differential operator $D = \prod_{j=1}^N \partial_{x_1}^{\alpha_1} \dots \partial_{x_N}^{\alpha_N}$ (of arbitrarily large order), there exists $C > 0$ such that $|Du_n(X)| \leq Ce^{-n}$ on Π, but $|u_n(Y)| \to +\infty$ as $n \to \infty$.

CHAPTER II

The linear equation in Sobolev spaces

II.1. Reminders on the Fourier transform

Here, we recall the definition and basic properties of the Fourier transform on $\mathbb{R}^N$, in the most general framework possible, that of tempered distributions. We omit the proofs. For more details, one can consult, for example, the foundational writings of Laurent Schwartz [**6**], the course of Jean-Michel Bony [**2**], as well as [**1**, Section 1.2] for a quick introduction, and [**5**] for a more in-depth exposition (the first two references are in French).

We begin by introducing a notation: a *multi-index* is an element $\alpha = (\alpha_1, \dots, \alpha_N)$ of $\mathbb{N}^N$. The order of α is $|\alpha| = \sum_{j=1}^N \alpha_j$. The derivative with respect to α of a function f of class $C^{|\alpha|}$ on $\mathbb{R}^N$ is then defined by:

$$\partial_x^\alpha f = \prod_{j=1}^N \partial_{x_j}^{\alpha_j} f.$$

1.a. Fourier Transform on $\mathcal{S}$.

Definition II.1.1. The Schwartz space $\mathcal{S}(\mathbb{R}^N)$ is the space of functions f of class C^∞ on $\mathbb{R}^N$ such that for every $p \in \mathbb{N}$,

$$N_p(f) := \sup_{x\in\mathbb{R}^N \ |\alpha|\leq p} (1+|x|)^p |\partial_x^\alpha f(x)| < \infty.$$

It can be observed that each N_p is a norm on $\mathcal{S}(\mathbb{R}^N)$, but N_p is not complete for any of these norms.

We equip $\mathcal{S}(\mathbb{R}^N)$ with the distance function

$$d(\varphi, \psi) = \sum_{p\geq 0} \frac{1}{2^p} \min\Big(N_p(\varphi - \psi), 1\Big). \tag{II.1.1}$$

Notice that $d(\varphi_n, \varphi)$ tends to 0 as n tends to infinity if and only if $N_p(\varphi_n - \varphi)$ tends to 0 for every p.

The metric space $(\mathcal{S}, d)$ is complete.[1]

The Fourier transform of an element φ of $\mathcal{S}$ is defined by the formula

$$\widehat{\varphi}(\xi) = \mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} \varphi(x)dx. \tag{II.1.2}$$

One easily checks that $\mathcal{F}$ is a continuous application from $\mathcal{S}$ into $\mathcal{S}$.

Fubini's theorem immediately implies the duality formula:

$$\int_{\mathbb{R}^N} \widehat{\varphi}(\xi)\psi(\xi)d\xi = \int_{\mathbb{R}^N} \varphi(x)\widehat{\psi}(x)dx, \tag{II.1.3}$$

for $\varphi, \psi \in \mathcal{S}(\mathbb{R}^N)$.

The Fourier transformation is a bijection of $\mathcal{S}$: by defining

$$\overline{\mathcal{F}}(\psi)(x) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{ix\cdot\xi}\psi(\xi)d\xi = \frac{1}{(2\pi)^N}\widehat{\psi}(-x), \tag{II.1.4}$$

we have the *Fourier inversion formula*: for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$,

$$\mathcal{F}\overline{\mathcal{F}}\varphi = \overline{\mathcal{F}}\mathcal{F}\varphi = \varphi. \tag{II.1.5}$$

[1]Such a vector space, equipped with a countable family of semi-norms, and which is complete as a metric space (where the distance function is defined as in (II.1.1)), is called a *Fréchet space*. It is a natural generalization of a Banach space when a unique norm is not sufficient to ensure completeness.

By combining the Fourier inversion formula (II.1.5) and the duality formula (II.1.3), we obtain the Plancherel theorem: for all φ, ψ in $\mathcal{S}$,

$$\int_{\mathbb{R}^N} \varphi(x)\overline{\psi}(x)dx = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} \widehat{\varphi}(\xi)\overline{\widehat{\psi}(\xi)}d\xi. \tag{II.1.6}$$

The Fourier transform exchanges multiplication by powers of x with differentiation. For all $\varphi \in \mathcal{S}(\mathbb{R}^N)$

$$\forall \alpha \in \mathbb{N}^N, \quad \mathcal{F}\partial_x^\alpha \varphi = i^{|\alpha|}\xi^\alpha \widehat{\varphi}(\xi), \quad \mathcal{F}(x^\alpha \varphi) = i^{|\alpha|}\partial_\xi^\alpha \widehat{\varphi}(\xi). \tag{II.1.7}$$

1.b. Fourier Transform of Tempered Distributions.

DEFINITION II.1.2. The space $\mathcal{S}'(\mathbb{R}^N)$ of *tempered distributions* is the topological dual of $\mathcal{S}(\mathbb{R}^N)$, i.e., the vector space of continuous linear forms on $\mathcal{S}$.

In the definition, continuity must be interpreted in the sense of the topology induced by the distance d defined by (II.1.1). Using the definition of this topology, one sees that a linear form f on $\mathcal{S}$ is an element of $\mathcal{S}'$ if and only if:

$$\exists p \in \mathbb{N}, \quad \forall \varphi \in \mathcal{S}, \quad |\langle f, \varphi\rangle| \leq CN_p(\varphi).$$

We equip $\mathcal{S}'$ with the topology of pointwise convergence: a sequence $(f_n)_n$ of elements of $\mathcal{S}'$ converges to f in $\mathcal{S}'$ if and only if

$$\forall \varphi \in \mathcal{S}, \quad \lim_{n\to\infty} \langle f_n, \varphi\rangle = \langle f, \varphi\rangle.$$

Several function spaces continuously embed into $\mathcal{S}'(\mathbb{R}^N)$ in the following manner. If f is a measurable, locally integrable function on f such that

$$\forall R > 0, \quad \int_{|x|\leq R} |f(x)|dx \leq C(1+R)^C$$

for some constant $C > 0$, we define an element L_f of $\mathcal{S}'(\mathbb{R}^N)$ by

$$\langle L_f, \varphi\rangle = \int_{\mathbb{R}^N} f(x)\varphi(x)dx.$$

The preceding application is injective, i.e., L_f is null if and only if f is null almost everywhere on $\mathbb{R}^N$. We then identify f with the linear form L_f, also denoted f. The preceding identification allows us to consider $\mathcal{S}$, Lebesgue spaces $L^p(\mathbb{R}^N)$ $(1 \leq p \leq \infty)$, C_b^k (the space of C^k functions on $\mathbb{R}^N$ that are bounded along with all their derivatives up to order k) as subspaces of $\mathcal{S}'$.

Examples of tempered distributions that are not functions are given by the (improperly named) Dirac delta function at a, denoted δ_a and defined by $\langle \delta_a, \varphi\rangle = \varphi(a)$, as well as the surface measure σ on the sphere S^{N-1}, defined by:

$$\langle \sigma, \varphi\rangle = \int_{S^{N-1}} \varphi(y)d\sigma(y).$$

By duality, several actions can be defined on the elements of $\mathcal{S}'$.

Differentiation. Let $\alpha \in \mathbb{N}^N$ and $f \in \mathcal{S}'$. The *derivative* of f of order α is the element ∂_x^α of $\mathcal{S}'$ defined by:

$$\forall \varphi \in \mathcal{S}, \quad \langle \partial_x^\alpha f, \varphi\rangle = (-1)^{|\alpha|}\langle f, \partial_x^\alpha \varphi\rangle.$$

The integration by parts formula shows that if $f \in C_b^{|\alpha|}$, its derivative of order α in the sense of distributions coincides with its derivative in the classical sense.

Multiplication by a Function. We denote by $\mathcal{P} = \mathcal{P}(\mathbb{R}^N)$ the space of C^∞ functions with *slow growth*, i.e., such that

$$\forall \alpha, \quad \exists M, C > 0 \quad \forall x \in \mathbb{R}^N, \quad |\partial_x^\alpha g(x)| \leq C(1+|x|)^M. \tag{II.1.8}$$

It is easy to check that the multiplication by an element of $\mathcal{P}$ defines a continuous mapping from $\mathcal{S}$ into $\mathcal{S}$. We then define, for $f \in \mathcal{S}'$ and $g \in \mathcal{P}$, the product fg by:

$$\langle fg, \varphi\rangle = \langle f, g\varphi\rangle.$$

The product fg is an element of $\mathcal{S}'$. Fixing $g \in \mathcal{P}$, $f \mapsto fg$ is a continuous mapping from $\mathcal{S}'$ into $\mathcal{S}'$.

Fourier Transform. We define the Fourier transform of an element f of $\mathcal{S}'$ by

$$\forall \varphi \in \mathcal{S}, \quad \left\langle \widehat{f}, \varphi \right\rangle = \langle f, \widehat{\varphi} \rangle$$

The duality formula (II.1.3) shows that if $f \in \mathcal{S}$, its Fourier transform according to formula (II.1.2) and its Fourier transform in the sense of $\mathcal{S}'$ coincide.

We recall that $L^1(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N)$ are identified with subspaces of $\mathcal{S}'(\mathbb{R}^N)$. The Fourier transform on $\mathcal{S}'$ thus applies to elements of these two spaces. On $L^1(\mathbb{R}^N)$, we recover the Fourier transform in the classical sense.

Proposition II.1.3 (Fourier Transform in L^1). *Let $f \in L^1(\mathbb{R}^N)$, and $\widehat{f}$ be its Fourier transform in $\mathcal{S}'$. Then $\widehat{f}$ can be identified with the continuous function given by the formula:*

$$\widehat{f}(\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x) dx.$$

The second proposition immediately follows from the Plancherel theorem:

Proposition II.1.4 (Fourier Transform in L^2). *Let $f \in L^2(\mathbb{R}^N)$ then $\widehat{f} \in L^2(\mathbb{R}^N)$ and*

$$\|f\|_{L^2} = \frac{1}{(2\pi)^{N/2}} \|\widehat{f}\|_{L^2}.$$

Indeed, the Fourier inversion formula in $\mathcal{S}'$ (see below) implies that $f \mapsto \frac{1}{(2\pi)^{N/2}} \hat{f}$ is an isometry of $L^2(\mathbb{R}^N)$. The properties of the Fourier transform on $\mathcal{S}$ are transmitted by duality to the Fourier transform:

- We define the inverse Fourier transform $\overline{F}$ of an element f of $\mathcal{S}'$ by

  $$\left\langle \overline{F} f, \varphi \right\rangle = \left\langle f, \overline{F} \varphi \right\rangle.$$

  Then we have the Fourier inversion formula:

  $$\forall f \in \mathcal{S}', \quad \overline{\mathcal{F}}\mathcal{F} f = \mathcal{F}\overline{\mathcal{F}} f = f.$$

- Property (II.1.7) remains valid for $\varphi \in \mathcal{S}'$.

II.2. Sobolev Spaces

2.a. Definition. (cf [**1**, Section 1.3]) We mainly focus on Sobolev spaces on $\mathbb{R}^N$, of Hilbert type (i.e. based on L^2 spaces). In this section, we consider homogeneous Sobolev spaces $\dot{H}^\sigma$. We refer to the exercise sheet for classical Sobolev spaces H^σ.

The Hilbertian Sobolev spaces on $\mathbb{R}^N$ are easily defined using the Fourier transform:

Definition II.2.1. Let $\sigma \in \mathbb{R}$. The Sobolev space $\dot{H}^\sigma(\mathbb{R}^N)$ is the set of $f \in \mathcal{S}'(\mathbb{R}^N)$ such that $\widehat{f}$ can be identified with a function in $L^1(K)$ for every compact set K, such that the following quantity is finite:

$$\|f\|^2_{\dot{H}^\sigma} = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} |\xi|^{2\sigma} |\widehat{f}(\xi)|^2 d\xi.$$

The space $\dot{H}^\sigma$, equipped with the inner product:

$$(f, g)_{\dot{H}^\sigma} = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} |\xi|^{2\sigma} \widehat{f}(\xi) \overline{\widehat{g}(\xi)} d\xi$$

is a pre-Hilbert space.

Theorem II.2.2. *The space $\dot{H}^\sigma(\mathbb{R}^N)$ is complete if and only if $\sigma < N/2$. In this case, the vector space $\mathcal{S}_0$ of functions in $\mathcal{S}$ whose Fourier transform vanishes in a neighborhood of 0 is dense in $\dot{H}^\sigma(\mathbb{R}^N)$.*

Note that $\dot{H}^0$ is exactly the space L^2.

2.b. Sobolev Inequalities. We have the following Sobolev embedding on $\mathbb{R}^N$.

THEOREM II.2.3. *Let $\sigma \in]0, N/2[$, and $p \in (2,\infty)$ such that $\frac{1}{p} = \frac{1}{2} - \frac{\sigma}{N}$. Then $\dot{H}^\sigma(\mathbb{R}^N)$ is contained in L^p with continuous embedding.*

The result is well-known. We give a proof based on the Fourier transform, which yields a slightly stronger result that we will use later in this course.

By the density result in Theorem II.2.2, it suffices to show that there exists a constant $C > 0$ such that

$$\forall f \in \mathcal{S}(\mathbb{R}^N), \quad \|f\|_{L^p(\mathbb{R}^N)} \leq C\|f\|_{\dot{H}^\sigma(\mathbb{R}^N)}. \tag{II.2.1}$$

Let $f \in \mathcal{S}$. We denote[2]

$$\|f\|^2_{\dot{B}^\sigma} = \sup_{k\in\mathbb{Z}} \frac{1}{(2\pi)^N} \int_{2^k \leq |x| \leq 2^{k+1}} |\xi|^{2\sigma} |\widehat{f}(\xi)|^2 d\xi,$$

and observe that $\|f\|_{\dot{B}^\sigma} \leq \|f\|_{\dot{H}^\sigma}$. We will prove the following result, which implies (II.2.1):

THEOREM II.2.4 (Improved Sobolev Inequality). *Let σ and p be as in the previous theorem. Then there exists a constant $C > 0$ such that*

$$\forall f \in \mathcal{S}(\mathbb{R}^N), \quad \|f\|^p_{L^p} \leq \|f\|^{p-2}_{\dot{B}^\sigma} \|f\|^2_{\dot{H}^\sigma}.$$

NOTATION II.2.5. Let φ be a function on $\mathbb{R}^N$. For $u \in \mathcal{S}'(\mathbb{R}^N)$, we denote

$$\varphi(D)u = \overline{\mathcal{F}}\left(\varphi(\xi)\widehat{u}(\xi)\right).$$

The operator $\varphi(D)$ is called Fourier multiplier (with symbol φ).

The tempered distribution $\varphi(D)u$ is not well-defined for all functions φ and $u \in \mathcal{S}'$: we need $\varphi\,\widehat{u}$ to define a tempered distribution. This is for example the case if $\varphi \in L^\infty$ and $u \in \dot{H}^\sigma$ (in this case $\varphi(D)u \in \dot{H}^\sigma$), or if $\varphi \in \mathcal{P}(\mathbb{R}^N)$ (the space of C^∞ functions with slow growth i.e. that satisfy (II.1.8)).

PROOF. We use a method introduced by Chemin and Xu in [**3**]. We fix a parameter $A > 0$ and decompose f into a *high-frequency* part $f_{>A}$ and a *low-frequency* part $f_{<A}$:

$$f_{>A} = \overline{\mathcal{F}}\left(\mathbb{1}_{|\xi|>A}\widehat{f}(\xi)\right) = \mathbb{1}_{|D|>A} f, \quad f_{<A} = \mathbb{1}_{|D|<A} f = 1 - f.$$

Let $k(A)$ be the largest integer such that $2^{k(A)} \leq A$. By using the Cauchy-Schwarz inequality, then the fact that $\sigma < N/2$, we obtain:

$$\begin{aligned}
|f_{<A}(x)| = \frac{1}{(2\pi)^N}\left|\int_{|\xi|<A} e^{ix\cdot\xi}\widehat{f}(\xi)d\xi\right| &\leq \frac{1}{(2\pi)^N}\sum_{k\leq k(A)}\int_{2^k\leq|\xi|\leq 2^{k+1}} |\widehat{f}(\xi)|d\xi \\
&\leq \frac{1}{(2\pi)^N}\sum_{k\leq k(A)} 2^{k(N/2-\sigma)}\left(\int_{2^k\leq|\xi|\leq 2^{k+1}} |\xi|^{2\sigma}|\widehat{f}(\xi)|^2 d\xi\right)^{1/2} \leq C_N A^{N/2-\sigma}\|f\|_{\dot{B}^\sigma},
\end{aligned}$$

where C_N depends only on the dimension N. Then we write (using Fubini's equality):

$$\|f\|^p_{L^p} = \int |f(x)|^p dx = \int_{\mathbb{R}^N} p \int_0^{|f(x)|} \lambda^{p-1} d\lambda dx = p\int_0^{+\infty} \lambda^{p-1}\left|\left\{x \in \mathbb{R}^N \;:\; |f(x)| \geq \lambda\right\}\right| d\lambda,$$

where $|S|$ denotes the Lebesgue measure of the measurable subset S of $\mathbb{R}^N$. Let $A(\lambda)$ be such that

$$C_N A(\lambda)^{\frac{N}{2}-\sigma}\|f\|_{\dot{B}^\sigma} = \lambda/2.$$

For any x in $\mathbb{R}^N$,

$$|f_{<A(\lambda)}(x)| \leq \frac{\lambda}{2}.$$

Thus $|f(x)| > \lambda \Longrightarrow |f_{>A(\lambda)}(x)| > \lambda/2$. Hence:

$$\|f\|^p_{L^p} \leq p\int_0^\infty \lambda^{p-1}\left|\left\{x \in \mathbb{R}^N \;:\; |f_{>A(\lambda)}(x)| > \lambda/2\right\}\right| d\lambda$$

By integrating $|f_{>A(\lambda)}|^2$ over the set $\left\{x \in \mathbb{R}^N \;:\; |f_{>A(\lambda)}(x)| > \lambda/2\right\}$, we get

$$\left|\left\{x \in \mathbb{R}^N \;:\; |f_{>A(\lambda)}(x)| > \lambda/2\right\}\right| \leq \frac{4}{\lambda^2}\|f_{>A(\lambda)}\|^2_{L^2}.$$

[2]This norm defines the Besov space $\dot{B}^\sigma_{2,\infty}$. See [**1**, Section 2.3] for the definition of general Besov spaces.

Combining with the Plancherel theorem, then Fubini's theorem, we obtain

$$\|f\|_{L^p}^p \leq \frac{4p}{(2\pi)^N} \int_0^\infty \lambda^{p-3} \int_{|\xi|>A(\lambda)} |\widehat{f}(\xi)|^2 d\xi d\lambda$$
$$= \frac{4p}{(2\pi)^N} \int_{\mathbb{R}^N} \left|\widehat{f}(\xi)\right|^2 \int_0^{c(f,\xi)} \lambda^{p-3} d\lambda d\xi = C_{p,N} \int_{\mathbb{R}^N} \left|\widehat{f}(\xi)\right|^2 c(f,\xi)^{p-2} d\xi,$$

where $c(f,\xi) = 2C_N \|f\|_{\dot{B}^\sigma} |\xi|^{\frac{N}{2}-s}$, and $C_{p,N}$ depends only on N and p. It can be easily verified that $(\frac{N}{2} - \sigma)(p-2) = 2\sigma$, which proves the announced inequality. □

We will focus more particularly on the case $s = 1$. According to the above, the Sobolev space $\dot{H}^1(\mathbb{R}^N)$, $N \geq 3$, is a Hilbert space, contained in $L^{\frac{2N}{N-2}}$, which can be defined as the closure of the space $\mathcal{S}(\mathbb{R}^N)$ (or $C_0^\infty(\mathbb{R}^N)$) for the $\dot{H}^1(\mathbb{R}^N)$-norm. We can characterize this norm with the first-order partial derivatives of f. Indeed,

$$\|f\|_{\dot{H}^1}^2 = \frac{1}{(2\pi)^N} \int |\xi|^2 \left|\widehat{f}(\xi)\right|^2 d\xi = \sum_{j=1}^N \int \left|\xi_j \widehat{f}(\xi)\right|^2 d\xi,$$

which shows by Plancherel's theorem and formula (II.1.7)

$$\|f\|_{\dot{H}^1}^2 = \int |\nabla f(x)|^2 dx.$$

The attentive reader will have noticed that the space $\dot{H}^1(\mathbb{R}^N)$ is not the set of $\varphi \in \mathcal{S}'(\mathbb{R}^N)$ such that for all j, $\partial_{x_j}\varphi \in L^2(\mathbb{R}^N)$: indeed, nonzero constant functions are in this space, but not in $\dot{H}^1(\mathbb{R}^N)$ (the Fourier transform $\hat{c}$ of a nonzero constant function is the multiple of a Dirac function, which does not satisfies the assumption of local integrability in the definition of $\dot{H}^1$).

The density result of Theorem II.2.2 implies that $\dot{H}^1(\mathbb{R}^N)$ is the closure of $C_0^\infty(\mathbb{R}^N)$ for the norm $\|\cdot\|_{\dot{H}^1}^2$. An other characterization, using the Sobolev inequality, is given by

$$\dot{H}^1(\mathbb{R}^N) = \left\{ f \in L^{\frac{2N}{N-2}}(\mathbb{R}^N),\ |\nabla f| \in L^2(\mathbb{R}^N) \right\}. \tag{II.2.2}$$

The proof of (II.2.2) is left to the reader.

II.3. The wave equation in the Schwartz space

Let $(u_0, u_1) \in \mathcal{S}(\mathbb{R}^N)$. We will write the solution u of (LW), (ID) using the Fourier transformation. We start with a formal calculation, assuming that $u(t) \in \mathcal{S}$ for all t (which we will prove later). We denote $\widehat{u}(t)$ as the Fourier transform of u with respect to the spatial variable, i.e.,

$$\widehat{u}(t,\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} u(t,x) dx.$$

Thus, we have

$$\widehat{\Delta u}(t,\xi) = -|\xi|^2 \widehat{u}(t,\xi),$$

and the wave equation (LW) is formally equivalent to the linear differential equation

$$\partial_t^2 \widehat{u}(t,\xi) + |\xi|^2 \widehat{u}(t,\xi) = 0,$$

where the variable ξ is considered as a parameter. The solution to this equation, with initial conditions $(\widehat{u}(0), \partial_t \widehat{u}(0)) = (u_0, u_1)$, yields

$$\widehat{u}(t,\xi) = \cos(t|\xi|)\widehat{u}_0(\xi) + \frac{\sin(t|\xi|)}{|\xi|} \widehat{u}_1(\xi),$$

or, with the previously introduced notation,

$$u(t) = \cos(t|D|)u_0 + \frac{\sin(t|D|)}{|D|} u_1. \tag{II.3.1}$$

THEOREM II.3.1. *Let* $(u_0, u_1) \in \mathcal{S}(\mathbb{R}^N)^2$. *Then* u *defined by* (II.3.1) *is an element of* $C^\infty(\mathbb{R} \times \mathbb{R}^N)$. *It is the unique* C^2 *solution of* (LW), (ID).

Proof. Uniqueness follows from Theorem I.4.1. Hence, it suffices to prove that u, defined by (II.3.1), is C^∞ and satisfies (LW), (ID). We have

$$u(t,x)=\frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{ix\cdot\xi}\left(\cos(t|\xi|)\widehat{u}_0(\xi)+\frac{\sin(t|\xi|)}{|\xi|}\widehat{u_1}(\xi)\right)d\xi.$$

By writing

$$\frac{\sin(t|\xi|)}{|\xi|}=t\sum_{k\geq 0}\frac{(-1)^k(t|\xi|)^{2k}}{(2k+1)!},$$

we see that it is a C^∞ function of (t,ξ). Moreover, $\frac{\left|\partial_t^j\sin(t|\xi|)\right|}{|\xi|}\leq |t||\xi|^j$. Similarly, $(t,\xi)\mapsto\cos(t|\xi|)$ is C^∞ and $\left|\partial_t^j\cos(t|\xi|)\right|\leq|\xi|^j$. Using the fact that $\widehat{u}_0$ and $\widehat{u}_1$ are elements of $\mathcal{S}(\mathbb{R}^N)$, by the theorem of differentiation under the integral sign, we obtain that u is C^∞ and satisfies (LW). The Fourier inversion formula shows that u also satisfies the initial conditions (ID). □

II.4. The wave equation in Sobolev spaces

4.a. The equation in general homogeneous Sobolev spaces. Let $(u_0,u_1)\in\dot{H}^\sigma\times\dot{H}^{\sigma-1}$, $\sigma<N/2$. We define as before u by (II.3.1). We also define the formal derivative of u with respect to time:

$$u'(t,x)=\cos(t|D|)u_1-|D|\sin(t|D|)u_0.$$

Then u and u' satisfy the following properties:

Claim II.4.1. *$u\in C^0(\mathbb{R},\dot{H}^\sigma)$, $u'\in C^0(\mathbb{R},\dot{H}^{\sigma-1})$, $u(0)=u_0$, $u'(0)=u_1$.*

Proof. Using that $\widehat{u}_0\in L^2(|\xi|^{2\sigma}d\xi)$ and $\widehat{u}_1\in L^2(|\xi|^{2\sigma-2}d\xi)$, it is easy to see that

$$\widehat{u}\in C^0(\mathbb{R},L^2(|\xi|^{2\sigma}d\xi)),\quad \widehat{u'}\in C^0(\mathbb{R},L^2(|\xi|^{2\sigma-2}d\xi)),\tag{II.4.1}$$

which yields the announced continuity property. The facts that $u(0)=u_0$ and $u'(0)=u_1$ follow immediately from the definition. □

Claim II.4.2. *$\forall t,\quad \|(u(t),u'(t))\|_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}}=\|(u_0,u_1)\|_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}}$.*

Proof.

$$\begin{aligned}\int_{\mathbb{R}^N}|\widehat{u}(t,\xi)|^2|\xi|^{2\sigma}d\xi+\int_{\mathbb{R}^N}\widehat{u'}(t,\xi)|\xi|^{2\sigma-2}d\xi&\\ =\int_{\mathbb{R}^N}\left|\cos(t|\xi|)\widehat{u}_0(\xi)+\frac{\sin(t|\xi|)}{|\xi|}\widehat{u}_1(\xi)\right|^2|\xi|^{2\sigma}d\xi&\\ +\int_{\mathbb{R}^N}\left|-|\xi|\sin(t|\xi|)\widehat{u}_0(\xi)+\cos(t|\xi|)\widehat{u}_1(\xi)\right|^2|\xi|^{2\sigma-2}d\xi&\\ =\int_{\mathbb{R}^N}\left(|\widehat{u}_0(\xi)|^2+|\widehat{u}_1(\xi)|^2|\xi|^{-2}\right)|\xi|^{2\sigma}d\xi,&\end{aligned}$$

which gives the desired property. □

Claim II.4.3. *Let $(u_{0,n},u_{1,n})\in(\mathcal{S}_0(\mathbb{R}^N))^2$ such that $(u_{0,n},u_{1,n})$ converges to (u_0,u_1) in $\dot{H}^\sigma\times\dot{H}^{\sigma-1}$. Let u_n be the solution of* (LW) *with data $(u_{0,n},u_{1,n})$. Then*

$$\lim_{n\to\infty}\sup_{t\in\mathbb{R}}\|u_n(t)-u(t)\|_{\dot{H}^\sigma}+\|\partial_t u_n(t)-u'(t)\|_{\dot{H}^{\sigma-1}}=0.$$

Proof. It follows immediately from the preceding claim, applied to $(u-u_n,u'-\partial_t u_n)$. □

Claim II.4.4. *One can identify u with a distribution on $\mathbb{R}\times\mathbb{R}^N$, and it satisfies the wave equation* (LW) *in the distributional sense. Furthermore $u'=\partial_t u$ in the sense of distribution.*

Proof. We first give a "concrete" proof of these facts for the reader which is not familiar with the theory of distributions, assuming that σ is large enough so that the object considered are all functions on $\mathbb{R}\times\mathbb{R}^N$.

Let $\sigma\geq 0$. We let u_n be as in Claim II.4.3. Using that u_n is a C^∞ solution of (LW) and integrating by parts, we obtain

$$\iint u_n(t,x)(\partial_t^2-\Delta)\varphi\,dx\,dt=0.$$

Using the Sobolev embedding $\dot{H}^\sigma \subset L^p$, $\frac{1}{p} = \frac{1}{2} - \frac{\sigma}{N}$, and the point (II.4.3), we see that

$$\lim_{n\to\infty} \|u - u_n\|_{L^p(K)} = 0,$$

for all compact K of $\mathbb{R}^N$. This implies

$$0 = \lim_{n\to\infty} \iint u_n(t,x)(\partial_t^2 - \Delta)\varphi dxdt = \lim_{n\to\infty} \iint u(t,x)(\partial_t^2 - \Delta)\varphi dxdt,$$

and thus

$$\forall \varphi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^N), \quad \iint u(\partial_t^2 - \Delta)\varphi dtdx = 0,$$

which is precisely the meaning of $\partial_t^2 u - \Delta u = 0$ in the distributional sense.

Let $\sigma \geq 1$. The equality

$$\partial_t u_n = -|D|\sin(t|D|)u_{0,n} + \cos(t|D|)u_{1,n}.$$

holds by differentiation below the integral sign. By integration by parts,

$$\forall \varphi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^N), \quad \iint \partial_t u_n \varphi dtdx = -\iint u_n \partial_t \varphi dtdx,$$

Letting $n \to \infty$, we obtain

$$\forall \varphi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^N), \quad \iint u'\varphi dtdx = -\iint u\partial_t\varphi dtdx,$$

which means that $u' = \partial_t u$ in the distributional sense.

The proof for general σ is essentially the same, and can be skipped by the reader who is not familiar with distributions.

If $\varphi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^N)$ (the space of smooth functions with compact support on $\mathbb{R}\times\mathbb{R}^N$), one defines the action of u on φ by

$$\langle u, \varphi \rangle = \int_{-\infty}^{+\infty} \langle u(t), \varphi(t)\rangle_{\mathcal{S}',\mathcal{S}}\, dt,$$

where $\varphi(t)$ is the function $t \mapsto \varphi(t,\cdot)$. It is a straightforward exercise to prove that u is well-defined and that is is a distribution on $\mathbb{R}\times\mathbb{R}^N$. The facts that u satisfies the wave equation in the distributional sense and that $u'(t) = \partial_t u(t)$ follow immediately from Claim II.4.3, that implies that $\lim u_n = u$ in the distributional sense, where u_n is a in Claim II.4.3. This last fact is an immediate consequence of Claim II.4.3. □

From now on, we will use the formula (II.1.2) as the definition of the solution u of (LW), (ID) with $(u_0, u_1) \in \dot{H}^\sigma \times \dot{H}^{\sigma-1}$. The preceding claims show that such a u is a limit of smooth, classical solutions of (LW), (ID), and that it satisfies (LW) in a weak sense. Also, we have

$$\partial_t u = -|D|\sin(t|D|)u_0 + \cos(t|D|)u_1$$

in the sense of distribution. In the sequel, we will always use the notation $\partial_t u$ to denote this quantity.

4.b. The wave equation in the energy space. Of particular interest for us is the case $s = 1$. We will call "finite energy solutions" the weak solutions with initial data $\dot{H}^1 \times L^2$ given by the preceding subsection in the case $s = 1$, $N \geq 3$. We will focus on the case $N = 3$. We note that if $(u_0, u_1) \in (C^3 \times C^2)(\mathbb{R}^3) \cap \left(\dot{H}^1 \times L^2\right)(\mathbb{R}^3)$, we have two ways of defining the solution u: by integrals on spheres, as in Theorem I.5.2, and using the Fourier transform, i.e. by formula (II.3.1). Let us prove that these two definitions coincide:

Proposition II.4.5. *Let $u \in C^2(\mathbb{R}\times\mathbb{R}^3)$ be a solution of* (LW), (ID). *Assume furthermore $u_0 = u(0) \in \dot{H}^1$, $u_1 = \partial_t u(0) \in L^2$. Then*

$$u(t) = \cos(t|D|)u_0 + \frac{\sin(t|D|)}{|D|}u_1, \quad \partial_t u(t) = -|D|\sin(t|D|)u_0 + \cos(t|D|)u_1.$$

Proof. Let $(u_{0,n}, u_{1,n}) \in \left(\mathcal{S}(\mathbb{R}^N)\right)^2$ with

$$\lim_{n\to\infty} \|u_{0,n} - u_0\|_{\dot{H}^1} + \|u_{1,n} - u_1\|_{L^2} = 0.$$

Let u_n be the corresponding solution of (LW) given by (II.3.1) (note that by uniqueness it is also the solution given by Theorem I.5.2). Since $u - u_n$ is a C^2, finite energy solution of (LW), Theorem I.6.1 yields

$$\forall t, \quad \|u(t) - u_n(t)\|_{\dot{H}^1}^2 + \|\partial_t u(t) - \partial_t u_n(t)\|_{L^2}^2 = \|u_0 - u_{0,n}\|_{\dot{H}^1}^2 + \|u_1 - u_{1,n}\|_{L^2}^2,$$

which tends to 0 as n goes to infinity. This proves the result, since $u_n(t)$ converges to $\cos(t|D|)u_0 + \frac{\sin(t|D|)}{|D|}u_1$ in $\dot{H}^1(\mathbb{R}^3)$ and $\partial_t u_n(t)$ converges to $-|D|\sin(t|D|)u_0 + \cos(t|D|)u_1$ in L^2 by Claim II.4.3. □

Using the approximation of finite energy solutions by solutions with initial data in $\mathcal{S}$, we can transfer several results of Chapter I to general finite energy solutions. This is the case of the decay of energy on past wave cones, which imply finite speed of propagation. If u is a finite energy solution (in any dimension $N \geq 3$) and $R > 0$, $x_0 \in \mathbb{R}^N$, $t_0 \in \mathbb{R}$, we denote by

$$E_{\text{loc}}(t) = \int_{|x-x_0|<R-|t-t_0|} e_u(t,x)dx.$$

Then

Theorem II.4.6. *$E_{\text{loc}}(t)$ is nonincreasing for $t \geq t_0$.*

Proof. From Theorem I.4.1, this quantity is nonincreasing when $(u_0, u_1) \in \mathcal{S}$. Considering the approximation given by Claim II.4.3, we obviously have, as a consequence of this claim,

$$\forall t, \quad \lim_{n\to\infty} \int_{|x-x_0|<R-|t-t_0|} e_{u_n}(t,x)dx = \int_{|x-x_0|<R-|t-t_0|} e_u(t,x)dx.$$

This gives the desired monotonicity property. □

We note that for general finite energy solution the integration by parts used in the proof of Theorem I.4.1 is no longer valid (since the boundary terms are not always well-defined).

4.c. Equation with a source term. We next consider the wave equation with a source term (I.1.2). By linearity, it is sufficient to study the equation with zero initial data:

$$\partial_t^2 u - \Delta u = f, \quad \vec{u}_{\restriction t=0} = (0,0). \tag{II.4.2}$$

Proposition II.4.7. *Assume $f \in C^0\left(\mathbb{R}, \mathcal{S}(\mathbb{R}^N)\right)$. Then u defined by*

$$u(t) = \int_0^s \frac{\sin\left((t-s)|D|\right)}{|D|} f(s)ds \tag{II.4.3}$$

is the unique solution of (II.4.2).

Proof. The uniqueness follows as usual by Theorem I.4.1. It is thus sufficient to check that u defined by (II.4.3) is of class C^2, and is a solution of (II.4.2). We consider F the function defined on $\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N$ by

$$F(t,s,x) = \left(\frac{\sin\left((t-s)|D|\right)}{|D|} f(s)\right)(x).$$

Thus

$$F(t,s,x) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{ix\cdot\xi} \frac{\sin\left((t-s)|\xi|\right)}{|\xi|} \widehat{f}(s,\xi)d\xi$$

Using that $\widehat{f} \in C^0\Big(\mathbb{R}, \mathcal{S}(\mathbb{R}^N)\Big)$, it is easy to check that F is continuous and C^∞ with respect to the variable (t,x), and that one can differentiate below the integral sign. The result follows since by integration by parts in the ξ variable,

$$\Delta F(t,s,x) = -\frac{1}{(2\pi)^N} \int |\xi|^2 e^{ix\cdot\xi} \frac{\sin\left((t-s)|\xi|\right)}{|\xi|} \widehat{f}(s,\xi)d\xi$$

□

We note that Duhamel formula (II.4.3) is still valid when $f \in L^1([-T,+T], \dot{H}^{\sigma-1})$ for all T, where σ is a fixed real number (assumed to be $< N/2$ for simplicity), and that it yields a function $u \in C^0(\mathbb{R}, \dot{H}^\sigma)$ with $\partial_t u \in C^0(\mathbb{R}, \dot{H}^{\sigma-1})$,

$$\partial_t u = \int_0^t \cos\left((t-s)|D|\right) f(s)ds, \tag{II.4.4}$$

in the sense of distribution, and such thatœ

$$\|\vec{u}(t)\|_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}} \leq \int_0^t \|f(s)\|_{\dot{H}^{\sigma-1}} ds. \tag{II.4.5}$$

Note that (II.4.5) is exactly the energy inequality proved in Chapter I when $\sigma = 1$.

We can approximate such an f by a sequence of functions (f_n) with

$$f_n \in C^0(\mathbb{R}, \mathcal{S}), \quad \forall t, \quad \lim_{n\to\infty} \int_{-T}^{+T} \|f(s) - f_n(s)\|_{\dot{H}^{\sigma-1}} ds = 0.$$

The corresponding solutions u_n defined by

$$u_n(t) = \int_0^t \frac{\sin\big((t-s)|D|\big)}{|D|} f_n(s) ds$$

are C^2 solutions of (II.4.2) and satisfy

$$\sup_{-T\leq t\leq T} \|\vec{u}_n(t) - \vec{u}(t)\|_{\dot{H}^\sigma \times \dot{H}^{\sigma-1}} \underset{n\to\infty}{\longrightarrow} 0. \tag{II.4.6}$$

As in the case of the free wave equation, this proves that u satisfies (LW) in the sense of distribution. In this situation, we will take the formula (II.4.3) as a definition of the solution u of (LW).

Exercice II.1. Assume that $\sigma = 1$. Let f defined on $\mathbb{R} \times \mathbb{R}^N$, such that $f \in L^1([-T, +T, L^2(\mathbb{R}^N))$. Prove that there exists a sequence of functions $f_n \in C_0^\infty(\mathbb{R} \times \mathbb{R}^N)$ such that

$$\forall T > 0, \quad \lim_{n\to\infty} \|f_n - f\|_{L^1([-T,+T], L^2(\mathbb{R}^N))} = 0.$$

Exercice II.2. Let u be a C^2 solution of (LW) for some $f \in C^0(\mathbb{R}\times\mathbb{R}^N)$. Assume that $f \in L^1([-T, +T], L^2(\mathbb{R}^N))$ for all $T > 0$. Show that u satisfies (II.4.3).

Bibliography

[1] Bahouri, H., Chemin, J.-Y., and Danchin, R. *Fourier analysis and nonlinear partial differential equations*, vol. 343 of *Grundlehren Math. Wiss.* Berlin: Springer, 2011.

[2] Bony, J.-M. *Cours d'analyse. Théorie des distributions et analyse de Fourier.* Palaiseau: Les Éditions de l'École Polytechnique, 2001.

[3] Chemin, J.-Y., and Xu, C. Sobolev imbeddings, Weyl-Hörmander calculus and subelliptic vector fields. *Ann. Sci. Éc. Norm. Supér. (4) 30*, 6 (1997), 719–751.

[4] Folland, G. B. *Introduction to partial differential equations.*, 2nd ed. ed. Princeton, NJ: Princeton University Press, 1995.

[5] Hörmander, L. *The analysis of linear partial differential operators. I: Distribution theory and Fourier analysis.*, reprint of the 2nd edition 1990 ed. Class. Math. Berlin: Springer, 2003.

[6] Schwartz, L. Théorie des distributions. Nouv. éd., entièrement corr., ref. + augm. (Nouv. tirage). Paris: Hermann. 436 p. (1984)., 1984.