Dynamics of semilinear wave equation
Master II course. 2024-2025

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Contents

CHAPTER I

Linear wave equation: classical theory

I.1. Presentation of the equation

The linear wave equation is the equation:

$$\partial_t^2 u - \Delta u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{LW}$$

where $N \geq 1$ is the spatial dimension (in this course, we will often assume $N = 3$), and

$$\Delta = \sum_{k=1}^{N} \frac{\partial^2}{\partial x_k^2}.$$

(We will use either the notations ∂_y or $\frac{\partial}{\partial y}$ for the derivative with respect to the variable $y \in \{t, x_1, \ldots, x_N\}$).

This is an evolution equation: we fix initial data at a certain time $t = t_0$, and are interested in the evolution of the equation over time t. Since the equation is of order 2, we actually fix an initial data for $\vec{u} = (u, \partial_t u)$:

$$\vec{u}_{\restriction t=t_0} = (u_0, u_1) \tag{I.1.1}$$

where (u_0, u_1) is to be taken in a certain functional space.

We will consider in this course initial data *with real values*. The passage to complex or vector values is immediate for most properties of the equation (LW) (by working coordinate by coordinate), but can induce drastic changes in the nonlinear case, if the nonlinearity mixes the components.

Equation (LW) is invariant under several obvious space-time transformations. If u is a solution, it is also the case of

$$\mu u(t - t_0, \lambda(Rx - x_0)),$$

where $\mu \in \mathbb{R}$, $t_0 \in \mathbb{R}$, $\lambda > 0$, $R \in \mathcal{O}_N(R)$, $x_0 \in \mathbb{R}^N$. It is in fact invariant under a larger group of linear transformations, the Lorentz group (cf Exercise I.10 p. 15).

As a consequence, we can limit ourselves, without loss of generality, to the case of an initial time $t_0 = 0$, i.e.

$$\vec{u}_{\restriction t=0} = (u_0, u_1) \tag{ID}$$

Furthermore, the equation is invariant under time inversion: if u is solution, it is also the case of $t \mapsto u(-t, x)$. It is thus a reversible equation.

We will also consider the equation with a force:

$$\partial_t^2 u - \Delta u = f, \tag{I.1.2}$$

(still with an initial condition of type (ID)), whose understanding will be crucial for the study of the nonlinear wave equation.

The Cauchy problem (LW), (ID) can be approached in at least 3 different ways:

- The classical approach which consists in finding an explicit formula to express the solution. It works when the initial data is sufficiently regular ($C^3 \times C^2$ in dimension 3 of space) and gives classical solutions (that is to say C^2 in (t,x) and satisfying (LW) in the sense of classical differentiation).
- The use of the Fourier transformation in space, which is very simple (once the Fourier transformation is known) and particularly effective in Sobolev spaces based on L^2 (which are natural spaces for the study of the equation due to the conservation of energy and other L^2-based quantities). This method allows to obtain weak solutions with degrees of regularity lower than the previous ones, and to use tools based on the Fourier transformation, which can be useful, for example, to prove certain dispersive properties of the equation.
- The "functional analysis" approach, by the theory of semi-groups, which gives the same type of solutions as the previous method.

In this chapter, we will detail the classical method, first by writing the explicit formula for solutions in dimension 1 of space, then in higher dimensions. We will study in the following chapter the equation in the energy space by the Fourier transformation. This chapter is partly based on Chapter 5 of the beautiful book by Folland on partial differential equations [**16**].

I.2. Explicit Formula in Dimension 1

In dimension 1, the equation (LW) can be written as:

$$(\partial_t^2 - \partial_x^2)u = 0, \tag{I.2.1}$$

which can be written $(\partial_t - \partial_x)(\partial_t + \partial_x)u = 0$. We thus make the change of variables $\eta = x + t$, $\xi = x - t$. Setting $v(\eta,\xi) = u\left(\frac{\eta-\xi}{2}, \frac{\eta+\xi}{2}\right)$, or $u(t,x) = v(t+x, t-x)$, we have:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 v}{\partial \eta^2} + \frac{\partial^2 v}{\partial \xi^2} + 2\frac{\partial^2 v}{\partial \xi \partial \eta},$$

and

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial \eta^2} + \frac{\partial^2 v}{\partial \xi^2} - 2\frac{\partial^2 v}{\partial \xi \partial \eta},$$

which gives:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -4\frac{\partial^2 v}{\partial \eta \partial \xi}.$$

Thus, we obtain:

$$\text{(LW)} \iff \frac{\partial^2 v}{\partial \eta \partial \xi} = 0.$$

Let u be a C^2 solution of (I.2.1), (ID). Thus, $u_1 \in C^1(\mathbb{R})$ and $u_0 \in C^2(\mathbb{R})$.

The equality $\frac{\partial^2 v}{\partial \eta \partial \xi} = 0$ shows that $\frac{\partial v}{\partial \xi}$ is a (class C^1) function $w(\xi)$ independent of η. Integrating with respect to ξ for η fixed, we deduce:

$$v(\eta,\xi) = \underbrace{\int_0^\xi w(\sigma)d\sigma}_{\varphi(\xi)} + \psi(\eta),$$

for a certain function ψ, necessarily C^2 since v is of class C^2 and w of class C^1. Thus, we necessarily have:

$$v(\eta,\xi) = \varphi(\xi) + \psi(\eta), \quad \varphi, \psi \in C^2(\mathbb{R}^2),$$

or equivalently:

$$u(t,x) = \varphi(x-t) + \psi(x+t). \tag{I.2.2}$$

Using the initial condition (ID), a direct calculation gives:

$$\psi(\eta) = \frac{1}{2}\int_0^\eta u_1(\sigma)d\sigma + \frac{1}{2}u_0(\eta) + c,$$

$$\varphi(\xi) = -\frac{1}{2}\int_0^\xi u_1(y)dy + \frac{1}{2}u_0(\xi) - c,$$

where $c \in \mathbb{R}$ (the choice of this constant is irrelevant). Hence, we deduce:

$$u(t,x) = \frac{1}{2}(u_0(x+t) + u_0(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} u_1(y)dy. \tag{I.2.3}$$

Conversely, it is easy to verify that formula (I.2.3) gives a C^2 solution of (I.2.1), (ID). Therefore, we have shown:

PROPOSITION I.2.1. *Let $(u_0,u_1) \in C^2(\mathbb{R}) \times C^1(\mathbb{R})$. Then, there exists a unique solution $u \in C^2(\mathbb{R}\times\mathbb{R})$ of* (LW) *satisfying the initial condition* (ID). *This solution satisfies formula* (I.2.3).

On formula (I.2.2), we observe that a solution of the wave equation in dimension 1 is the sum of two waves: one, $\varphi(x-t)$, moving at speed 1 to the right, and the other $\psi(x+t)$, moving at the same speed to the left.[1]

It is also possible to obtain a formula for the equation with the right-hand side (I.1.2). We leave this as an exercise to the reader. Further on, we will provide a general method giving the solution of the equation with the right-hand side in terms of the equation without the right-hand side.

[1] Note that the equations (LW), (I.2.1) have been normalized, so that the speed of propagation is exactly 1.

We can see from formula (I.2.3) that $u(t,x)$ depends only on the values of (u_0,u_1) over $\Big[x-|t|,x+|t|\Big]$. This is a first example of "finite speed of propagation" which holds in all spatial dimensions.

I.3. Integral on the Sphere and Divergence Theorem

We denote $S^{N-1}=\{x\in\mathbb{R}^N,\ |x|=1\}$, where $|\cdot|$ represents the Euclidean norm on $\mathbb{R}^N$:

$$|x|^2=\sum_{j=1}^N x_j^2.$$

More generally, S_R^{N-1} will denote the sphere of radius R: $\{x\in\mathbb{R}^N,\ |x|=R\}$.

We denote $d\sigma$ as the volume element on one of these spheres. Thus, the integral of a function $f\in\mathcal{L}^1(S_R^{N-1})$ (i.e., a function integrable on S_R^{N-1}) is written as

$$\int_{S_R^{N-1}} f(y)d\sigma(y).$$

This integral can be calculated using spherical coordinates. In dimension 3, this writes:

$$\int_{S_R^2} f(y)d\sigma(y)=R^2\int_0^{2\pi}\int_0^{\pi} f(R\sin\theta\cos\varphi,R\sin\theta\sin\varphi,R\sin\varphi)\sin(\theta)d\theta d\varphi.$$

We denote $B_R^N(x_0)$ as the ball centered at x_0 with radius R:

$$B_R^N(x_0)=\big\{x\in\mathbb{R}^N,\ |x-x_0|<R\big\}$$

and simply $B_R^N=B_R^N(0)$.

We will use the following formulas:

Scaling:

$$\int_{S_R^{N-1}} f(y)d\sigma(y)=R^{N-1}\int_{S^{N-1}} f(Ry)d\sigma(y)n \quad f\in\mathcal{L}^1(S_R^{N-1}).$$

Integral in radial coordinates: if $f\in\mathcal{L}^1(\{|x|\leq R\})$,

$$\int_{B_R^N} f(x)dx=\int_0^R\int_{S_r^{N-1}} f(y)d\sigma(y)dr=\int_0^R\int_{S^{N-1}} f(r\omega)d\sigma(\omega)r^{N-1}dr$$

Divergence theorem: if $F\in C^1(\overline{B_R},\mathbb{R}^N)$,

$$\int_{|x|\leq R}\nabla\cdot F(x)dx=\int_{S_R^{N-1}}\frac{y}{|y|}\cdot F(y)d\sigma(y),$$

where $\nabla\cdot F=\sum_{j=1}^N\partial_{x_j}F_j$ is the divergence of the vector field F.

I.4. Energy density. Uniqueness and finite speed of propagation

Before giving an explicit formula for the wave equation in dimension 3, we prove a uniqueness result valid in any dimension:

Theorem I.4.1. *Let $(t_0,x_0)\in\mathbb{R}^{1+N}$, $t_1>t_0$, $R>0$. We denote $\Gamma=\Big\{(t,x)\in\mathbb{R}\times\mathbb{R}^N\ :\ t_0\leq t\leq t_1,\ |x-x_0|\leq R-|t-t_0|\Big\}$. Let $u\in C^2(\Gamma)$ be a solution of* (LW) *on Γ. We suppose $(u,\partial_t u)(t_0,x)=0$ for all $x\in B_R(x_0)$. Then u is identically zero on Γ.*

The proof of the theorem is based on a monotonicity law that has its own interest.

We define, for $(t,x)\in\Gamma$, the *density of energy* e_u as

$$e_u(t,x)=\frac{1}{2}|\nabla u(t,x)|^2+\frac{1}{2}(\partial_t u(t,x))^2,$$

where $|\nabla u|^2=\sum_{j=1}^N(\partial_{x_j}u)^2$, and we consider, for $t_0\leq t\leq t_1$, the local energy

$$E_{\text{loc}}(t)=\int_{B_{R-(t-t_0)}(x_0)} e_u(t,x)dx=\int_{|x-x_0|<R-(t-t_0)} e_u(t,x)dx.$$

Lemma I.4.2. *The function E_{loc} is decreasing on $[t_0,t_1]$.*

The lemma immediately implies Theorem I.4.1. Indeed, if $\vec{u}(t_0)$ vanishes on $B(x_0,R)$, then $E_{\text{loc}}(t_0)=0$, and thus $E_{\text{loc}}(t)=0$ for all $t\in[t_0,t_1]$, showing that u is zero on Γ.

Proof of Lemma I.4.2. We notice that

$$\frac{\partial e}{\partial t}=\sum_{j=1}^{N}\left(\partial_{x_j}u\partial_t\partial_{x_j}u+\partial_{x_j}^2u\partial_t u\right)=\sum_{j=1}^{N}\frac{\partial}{\partial x_j}\left(\partial_{x_j}u\partial_t u\right)=\nabla\cdot\left(\partial_t u\nabla u\right), \tag{I.4.1}$$

where $\nabla u=(\partial_{x_i}u)_{1\leq i\leq N}$ Without loss of generality, we can assume that $x_0=0$ and $t_0=0$. By the integration formula in radial coordinates,

$$E_{\text{loc}}(t)=\int_0^{R-t}s^{N-1}\int_{S^{N-1}}e_u(t,s\omega)d\sigma(\omega)ds.$$

By differentiation under the integral sign, we get that E_{loc} is differentiable and

$$E_{\text{loc}}'(t)=-(R-t)^{N-1}\int_{S^{N-1}}e_u(t,(R-t)\omega)d\sigma(\omega)+\int_{B^N_{R-t}}\frac{\partial e_u}{\partial t}(t,x)dx.$$

By formula (I.4.1), then the divergence formula

$$\int_{B^N_{R-t}}\frac{\partial e_u}{\partial t}(t,x)dx=\int_{B^N_{R-t}}\nabla\cdot\left(\partial_t u\nabla u\right)(t,x)dx=\int_{S^{N-1}_{R-t}}\frac{y}{|y|}\nabla u\partial_t u(t,y)d\sigma(y).$$

We thus have

$$E_{\text{loc}}'(t)=-\int_{S^{N-1}_{R-t}}\left(\frac{1}{2}|\nabla u|^2+\frac{1}{2}(\partial_t u)^2-\frac{y}{|y|}\nabla u\partial_t u(t,y)\right)d\sigma(y)\leq-\frac{1}{2}\int_{S^{N-1}_{R-t}}\left(\frac{y}{|y|}\nabla u+\partial_t u(t,y)\right)^2d\sigma(y).$$

□

I.5. Explicit formulas.

This section is devoted to explicit formulas in space dimensions $N\geq 2$. In dimension $N=3$, we will show that for any initial data $(u_0,u_1)\in C^2\times C^3$, there exists a unique solution $u\in C^2(\mathbb{R}^{1+3})$ of (LW), (ID), and provide an explicit formula for this solution. We will also provide a formula in dimension $N=2$. We refer the reader to [**16**, Chapter 5B] for expressions of solutions when $N\geq 4$.

5.a. The radial case in dimension 3. When the initial conditions depend only on the variable $r=|x|$, the explicit formula is very simple.

We start by observing that if f depends only on the variable r, then the function f is C^2 as a function on $\mathbb{R}^3$ if and only if it is C^2 as a function of the variable r on $[0,\infty[$, and satisfies $\frac{df}{dr}(0)=0$. Moreover,

$$\Delta f=\frac{d^2f}{dr^2}+\frac{2}{r}\frac{df}{dr}$$

(cf Exercise I.1). We notice that we can rewrite this formula as

$$r\Delta f=\frac{d^2}{dr^2}(rf).$$

Now let u be a C^2 solution of (LW), (ID) with initial data (u_0,u_1) assumed to be radial. We also assume that for all t, $u(t)$ is a radial function. We will show a posteriori that this second assumption is a consequence of the assumption on the initial data. The previous formula gives

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial r^2}\right)(ru)=0.$$

The function $(t,r)\mapsto ru(t,r)$ is thus a solution of the wave equation in dimension 1, on $\mathbb{R}_t\times]0,\infty[$. To obtain a function on $\mathbb{R}^2$, we extend $ru(t,r)$ to an odd function:

$$v(t,y)=yu(t,|y|).$$

One can verify (using Exercise I.1) that v is of class C^2 on $\mathbb{R}^2$, and that

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial y^2}\right)v=0.$$

Formula (I.2.3) then gives:

$$v(t,y)=\frac{1}{2}(v_0(y+t)+v_0(y-t))+\frac{1}{2}\int_{y-t}^{y+t}v_1(\sigma)d\sigma,$$

where $(v_0,v_1)=\vec{v}_{\restriction t=0}$, thus

$$u(t,r)=\frac{1}{2r}\Big((r+t)u_0(|r+t|)+(r-t)u_0(|r-t|)\Big)+\frac{1}{2r}\int_{r-t}^{r+t}\sigma u_1(|\sigma|)d\sigma. \tag{I.5.1}$$

Notice that when $t>0$ (to fix ideas),

$$\int_{r-t}^{r+t}\sigma u_1(|\sigma|)d\sigma=\int_{|r-t|}^{r+t}\sigma u_1(|\sigma|)d\sigma.$$

The finite speed of propagation is satisfied: the solution $u(t,r)$ depends only on the initial condition (u_0,u_1) on the ball centered at r with radius $|t|$.

The formula (I.5.1) defines a function $u(t,r)$ of class C^2 outside the origin $x=0$, as soon as the initial conditions (u_0,u_1) have the expected regularity $C^2\times C^1$. However, there is a subtle phenomenon of loss of regularity of the solution u compared to the initial data at the origin : there exist data $(u_0,u_1)\in C^2\times C^1$ such that u, defined by formula (I.5.1), cannot be extended by a C^2 function up to $r=0$. Indeed, it can be checked that (at fixed t),

$$\lim_{r\to 0}u(t,r)=u_0(t)+tu_0'(t)+tu_1(t), \tag{I.5.2}$$

which shows that if (u_0,u_1) are $C^k\times C^{k-1}$ functions, then $u(t,0)$ is only C^{k-1} in general (see also Exercise I.2). We can interpret this phenomenon physically as follows: a singularity on the circle $r=r_0$ at the initial time 0 that travels at speed 1 towards the origin will concentrate at the origin at time $t=r_0$, causing a stronger singularity.

The limit (I.5.2) suggests a maximal loss of regularity of a derivative with respect to the initial data, which is indeed the case:

Proposition I.5.1. *Let $(u_0,u_1)\in(C^3\times C^2)(\mathbb{R}^3)$ be radial functions. Then formula* (I.5.1) *extended by $u(t,0)=u_0(t)+tu_0'(t)+tu_1(t)$, defines a C^2 function on $\mathbb{R}\times\mathbb{R}^3$, radial with respect to the variable x, and satisfying* (LW)*,* (ID).

The Proposition I.5.1 is left as an exercise to the reader. Combining with the uniqueness property (Theorem I.4.1), we obtain that (I.5.1) gives the unique solution of (LW) with initial data (u_0,u_1).

The formula (I.5.1) is remarkably simple. In higher space dimensions, we also have an explicit formula for radial solutions, which becomes more complicated as the dimension increases (see Exercise I.3). The loss of regularity observed in dimension 3 (and absent in dimension 1) increases with dimension, as the reader can verify on the formula obtained in Exercise I.3.

There is no simple formula in the radial case in even dimensions.

We also have explicit formulas (of course more complicated) without radiality assumptions, in all dimensions. We will explicitly state these formulas when $N=3$, then $N=2$.

5.b. General solutions in dimension 3: averaging over spheres. If $f\in C^0(\mathbb{R}^3)$, we define

$$(M_f)(t,x)=\frac{1}{4\pi}\int_{S^2}f(x+ty)d\sigma(y)=\frac{1}{4\pi t^2}\int_{S^2_{|t|}}f(x+z)d\sigma(z). \tag{I.5.3}$$

the average of f over the sphere of radius $|t|$ and center x. The function M_f inherits the regularity of f (cf exercise I.5).

Theorem I.5.2. *Let $(u_0,u_1)\in C^3(\mathbb{R}^3)\times C^2(\mathbb{R}^3)$. Then the unique C^2 solution of the wave equation* (LW) *with initial conditions* (ID) *is given by*

$$u(t,x)=tM_{u_1}(t,x)+\frac{\partial}{\partial t}(tM_{u_0}(t,x)).$$

Proof. We start by verifying that $tM_{u_1}(t,x)$ is the solution of the wave equation (LW), with initial condition $(0,u_1)$. By the theorem of differentiation under the integral sign, if $g\in C^2(\mathbb{R}^3)$,

$$\frac{\partial}{\partial t}\left(M_g(t,x)\right)=\frac{1}{4\pi}\int_{S^2}(y\cdot\nabla g)(x+ty)d\sigma(y).$$

Using the divergence formula,

$$\int_{S^2} (y\cdot\nabla g)(x+ty)d\sigma(y) = t\int_{|y|\leq 1} \left(\nabla\cdot(\nabla g)\right)(x+ty)dy$$
$$= t\int_{|y|\leq 1}(\Delta g)(x+ty)dy \; = \frac{1}{t^2}\int_0^t\int_{S^2}(\Delta g)(x+sy)s^2d\sigma(y)ds.$$

Thus:

$$\frac{\partial}{\partial t}\left(tM_{u_1}(t,x)\right) = M_{u_1}(t,x) + \frac{1}{4\pi t}\int_0^t\int_{S^2}(\Delta u_1)(x+sy)d\sigma(y)s^2ds.$$

and therefore

$$\frac{\partial^2}{\partial t^2}\left(tM_{u_1}(t,x)\right) = \frac{1}{4\pi t^2}\int_0^t\int_{S^2}(\Delta u_1)(x+sy)d\sigma(y)s^2ds$$
$$-\frac{1}{4\pi t^2}\int_0^t\int_{S^2}(\Delta u_1)(x+sy)d\sigma(y)s^2ds + \frac{t}{4\pi}\int_{S^2}(\Delta u_1)(x+ty)d\sigma(y) \; = \Delta\left(tM_{u_1}(t,x)\right).$$

This shows that tM_{u_1} satisfies the wave equation (LW). Furthermore, since $M_{u_1}(0,x) = u_1(0,x)$, the initial condition at $t=0$ is indeed $(0,u_1)$.

Now let $v(t,x) = tM_{u_0}(t,x)$. Then, by the same reasoning, v is a solution of the wave equation (LW) with initial condition $(0,u_0)$. We deduce that $\partial_t v$ is a solution of the wave equation with initial condition $(u_0,0)$, which concludes the proof. □

Notice that we can rewrite the formula of the theorem as:

$$u(t,x) = tM_{u_1}(t,x) + M_{u_0}(t,x) + tM_{y\cdot\nabla u_0}(t,x). \tag{I.5.4}$$

We now give two important consequences of the previous formula.

COROLLARY I.5.3 (Strong Huygens' principle). *The solution $u(t,x)$ depends only on the values of u_0, ∇u_0, and u_1 on the sphere centered at x and of radius $|t|$.*

REMARK I.5.4. The strong Huygens' principle is a stronger version of the finite speed of propagation property, which states that $u(t,x)$ depends only on the values of (u_0,u_1) on the *ball* centered at x and of radius $|t|$. This principle remains valid in any odd dimension ≥ 3 (the number of derivatives of u_0 and u_1 in the statement increases with the dimension). In even dimension, solutions only satisfy the finite speed of propagation: see §5.c. In dimension 1, as shown by formula (I.2.3), only solutions that are even in time (i.e. with initial condition of the form $(u_0,0)$) satisfy the strong Huygens' principle.

The second consequence of the explicity formula proved above is an estimate related to the dispersive nature of the wave equation. We will denote

$$\|\varphi\|_{\dot{W}^{s,p}} = \sup_{|\alpha|=s}\|\partial_x^\alpha\varphi\|_{L^p(\mathbb{R}^N)}. \tag{I.5.5}$$

We prove:

THEOREM I.5.5 (Dispersion inequality). *Let $(u_0,u_1)\in(C^3\times C^2)(\mathbb{R}^3)$, with compact support and u the solution of* (LW), (ID). *Then for all $t>0$,*

$$\|u(t)\|_{L^\infty(\mathbb{R}^3)} \lesssim \frac{1}{t}\left(\|u_0\|_{\dot{W}^{2,1}} + \|u_1\|_{\dot{W}^{1,1}}\right).$$

PROOF. By space translation invariance it is sufficient to bound $|u(t,0)|$. We have

$$4\pi u(t,0) = t\int_{S^2}u_1(ty)d\sigma(y) + \int_{S^2}u_0(ty)d\sigma(y) + t\int_{S^2}y\cdot\nabla u_0(ty)d\sigma(y).$$

By the divergence theorem (denoting by B^3 the unit ball of $\mathbb{R}^3$),

$$t\int_{S^2}u_1(ty)d\sigma(y) = t\int_{B^3}\nabla\cdot\left(yu_1(ty)\right)dy = 3t\int_{B^3}u_1(ty)dy + t^2\int_{B^3}y\cdot\nabla u_1(ty)dy. \tag{I.5.6}$$

We have

$$\left|\int_{B^3}y\cdot\nabla u_1(ty)dy\right| \leq \frac{1}{t^3}\int_{tB^3}|\nabla u_1(y)|dy \leq \frac{3}{t^3}\|u_1\|_{\dot{W}^{1,1}}, \tag{I.5.7}$$

and

$$\int_{B^3} |u_1(ty)|dy \leq t \int_{\mathbb{R}^3} |\partial_{x_1} u_1(ty)|dy \leq \frac{1}{t^2} \|u_1\|_{\dot{W}^{1,1}}, \tag{I.5.8}$$

where we have used the inequality $\int_{B^3} |\varphi|dx \lesssim \int_{\mathbb{R}^3} |\partial_{x_1}\varphi|$, that follows immediately from the formula $\varphi(x_1, x_2, x_3) = \int_{-\infty}^{x_1} \partial_{x_1}\varphi(s, x_2, x_3)ds$. Combining (I.5.6), (I.5.7) and (I.5.8), we obtain

$$\left| t \int_{S^2} u_1(ty)d\sigma(y) \right| \lesssim \frac{1}{t} \|u_1\|_{\dot{W}^{1,1}}. \tag{I.5.9}$$

By the same proof, using also the inequality $\int_{B^3} |\varphi| \lesssim \int_{\mathbb{R}^3} |\partial_{x_1}\partial_{x_2}\varphi|$, we have

$$\left| \int_{S^2} u_0(ty)d\sigma(y) \right| + \left| \int_{S^2} y \cdot \nabla u_0(ty)d\sigma(y) \right| \lesssim \frac{1}{t} \|u_0\|_{\dot{W}^{2,1}}. \tag{I.5.10}$$

This concludes the proof of the dispersion inequality. □

5.c. Dimension $1 + 2$. A solution u of equation (LW) with $N = 2$ is also a solution of the same equation with $N = 3$, constant with respect to the 3rd spatial coordinate. From Theorem I.5.2, one can derive an expression of u from the initial data. This strategy is called "descent method".

THEOREM I.5.6. *Let $(u_0, u_1) \in (C^3 \times C^2)(\mathbb{R}^2)$. Then equation* (LW) *has a unique C^2 solution on $\mathbb{R} \times \mathbb{R}^2$, given by the formula*

$$u(t, x) = \frac{1}{2\pi} \left[\frac{\partial}{\partial_t} \left(t \int_{|y|\leq 1} \frac{u_0(x + ty)}{\sqrt{1 - |y|^2}} dy \right) + t \int_{|y|\leq 1} \frac{u_1(x + ty)}{\sqrt{1 - |y|^2}} dy \right]. \tag{I.5.11}$$

PROOF. Uniqueness follows from Theorem I.4.1. Moreover, as in the proof of Theorem I.5.2, the formula for even solutions in time (with initial condition $(u_0, 0)$) can be easily deduced from the formula for odd solutions in time (with initial condition $(0, u_1)$). So we only consider this second case.

Let u be a C^2 solution of (LW) on $\mathbb{R} \times \mathbb{R}^2$, with initial data $(u, \partial_t u)(0) = (0, u_1)$, where $u_1 \in C^2(\mathbb{R}^2)$. By Theorem I.5.2, considering u as a solution on $\mathbb{R} \times \mathbb{R}^3$, we obtain:

$$u(t, x_1, x_2) = \frac{t}{4\pi} \int_{S^2} \tilde{u}_1((x_1, x_2, 0) + ty)d\sigma(y)dy,$$

where by definition $\tilde{u}_1(x_1, x_2, x_3) = u_1(x_1, x_2)$. Passing to spherical coordinates, we get

$$\begin{aligned} &\int_{S^2} \tilde{u}_1((x_1, x_2, 0) + ty)d\sigma(y) \\ &\qquad = \int_0^{2\pi} \int_0^{\pi} u_1(x_1 + t \sin\theta \cos\varphi, x_2 + t \sin\theta \sin\varphi) \sin\theta d\theta d\varphi \\ &\qquad\qquad = 2 \int_0^{2\pi} \int_0^{\pi/2} u_1(x_1 + t \sin\theta \cos\varphi, x_2 + t \sin\theta \sin\varphi) \sin\theta d\theta d\varphi. \end{aligned}$$

The announced formula then follows from the change of variable $y_1 = t \sin\theta \cos\varphi$, $y_2 = t \sin\theta \sin\varphi$. □

It can be seen from the formula in Theorem I.5.6 that the strong Huygens principle is not verified in dimension $1 + 2$: the solution $u(t, x)$ depends on the values of the initial condition over the entire ball $B^2_{|t|}(x)$, not just on the sphere $\{y \in \mathbb{R}^2 \ : \ |x - y| = |t|\}$.

I.6. Conservation Laws

The energy of a solution u on $\mathbb{R} \times \mathbb{R}^N$ is defined as:

$$E(\vec{u}(t)) = \int_{\mathbb{R}^N} e_u(t, x)dx = \frac{1}{2} \int_{\mathbb{R}^N} \left((\partial_t u(t, x))^2 + |\nabla u(t)|^2 \right) dx.$$

This is the global version of the local energy considered in §I.4. The energy of a solution is conserved over time.

THEOREM I.6.1. *Let $u \in C^2(\mathbb{R}^{1+N})$ be a solution of* (LW), (ID). *Assume (u_0, u_1) has finite energy. Then for any t, $E(\vec{u}(t))$ is finite and $E(\vec{u}(t)) = E(u_0, u_1)$.*

Proof. One might be tempted to write

$$\frac{d}{dt}(E(\vec{u}(t))) = \int \partial_t e_u(t,x)dx = \int \nabla \cdot (\partial_t u \nabla u)dx = 0,$$

but the last equality, obtained by integration by parts ignoring the "boundary" term (i.e., when $|x| \to \infty$) is purely formal. To justify the preceding calculation, we can use the decay of the local energy (Lemma I.4.2). For $R > 0$, we define:

$$E_{<R}(\vec{u}(t)) = \int_{|x|<R} e_u(t,x)dx.$$

Notice that this quantity is finite as soon as $u \in C^1(\mathbb{R}^{1+N})$. Fix $t > 0$. By Lemma I.4.2, for any $R > t$,

$$E_{<R-t}(\vec{u}(t)) \leq E_{<R}(\vec{u}(0)) \leq E(u_0, u_1).$$

As we let R tend to $+\infty$, we obtain that $E(\vec{u}(t))$ is finite, and

$$E(\vec{u}(t)) \leq E(u_0, u_1).$$

Reversing the direction of time, we also obtain the inequality

$$E(u_0, u_1) \leq E(\vec{u}(t)).$$

We have shown that the energy is conserved for $t \geq 0$. By applying this result to the solution $(t,x) \mapsto u(-t,x)$, we obtain energy conservation for $t \leq 0$, which concludes the proof. □

There exists another (vectorial) conserved quantity, the momentum, defined as

$$P(\vec{u}(t)) = \int \partial_t u(t,x) \nabla u(t,x) dx \in \mathbb{R}^N.$$

Proposition I.6.2. *Let $u \in C^2(\mathbb{R}^{1+N})$ be a solution of* (LW) *with finite energy. Then*

$$\forall t \in \mathbb{R}, \quad P(\vec{u}(t)) = P(u_0, u_1).$$

The proof of this proposition is left as an exercise (see Exercise I.7).

I.7. Equation with a source term

We now consider the equation with a source term (I.1.2). We will express the solution of this equation in terms of the propagator of the free equation (LW). For $(u_0, u_1) \in C^3 \times C^2(\mathbb{R}^3)$, let $S_L(t)(u_0, u_1)$ denote the solution of (LW) with initial data (u_0, u_1) at $t = 0$. We denote $S(t)u_1 = S_L(t)(0, u_1)$, so that

$$S_L(t)(u_0, u_1) = \frac{\partial}{\partial t}(S(t)u_0) + S(t)u_1.$$

For $u_1 \in C^2$, we recall that

$$(S(t)u_1)(x) = tM_{u_1}(t,x) = t\int_{S^2} u_1(x+ty)d\sigma(y).$$

Theorem I.7.1 (Duhamel's Formula). *Let $(u_0, u_1) \in (C^2 \times C^3)(\mathbb{R}^3)$ and $f \in C^2(\mathbb{R} \times \mathbb{R}^3)$. Then the equation* (I.1.2)*,* (ID) *has a unique C^2 solution, given by the formula:*

$$u(t) = S_L(t)(u_0, u_1) + \int_0^t S(t-s)f(s)ds.$$

Remark I.7.2. The Duhamel term $\int_0^t S(t-s)f(s)ds$ can be explicited, see (I.7.1).

Proof of Theorem I.7.1. Uniqueness follows immediately from Theorem I.4.1, since the difference of 2 solutions of (I.1.2) with the same source term f is a solution of (LW). For existence, taking into account Theorem I.5.2, it is sufficient to check that the function

$$U : (t,x) \mapsto \int_0^t S(t-s)f(s)ds$$

is C^2 and satisfies equation (I.1.2) with zero initial conditions.

We have:

$$U(t,x) = \frac{1}{4\pi}\int_0^t (t-s)\int_{S^2} f(s, x+(t-s)y)d\sigma(y)ds, \tag{I.7.1}$$

and the fact that U is C^2 follows from the theorem on differentiation under the integral sign.

Furthermore, using that $S(0)g = 0$ for any function g,

$$\frac{\partial U}{\partial t} = \int_0^t \frac{\partial}{\partial t}\Big(S(t-s)f(s)\Big)ds.$$

Upon further differentiation, we obtain

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial}{\partial t}\Big(S(t-s)f(s)\Big)_{\restriction s=t} + \int_0^t \frac{\partial^2}{\partial t^2}\Big(S(t-s)f(s)\Big)ds = f(t) + \int_0^t \Delta\Big(S(t-s)f(s)\Big)ds = f(t) + \Delta U.$$

where we used that $\frac{\partial}{\partial t}(S(t)g)_{\restriction t=0} = g$ for any function g of class C^2. □

Remark I.7.3. Duhamel's formula is certainly not specific to dimension 3, as shown by the calculation leading to this formula, which is completely independent of dimension. The reader is invited to explicitly rewrite the solution of equation (I.1.2) when $N = 1$ and $N = 2$.

From Duhamel's formula, we deduce the energy inequality:

Proposition I.7.4. *Let u be a C^2 solution of* (I.1.2) *with $N = 3$ with initial data (u_0, u_1), such that $f \in C^2(\mathbb{R}^{1+3})$. Suppose furthermore that (u_0, u_1) has finite energy, and for all $T > 0$,*

$$\int_{[-T,+T]} \sqrt{\int_{\mathbb{R}^3} |f(t,x)|^2 dx dt} < \infty.$$

Then for all $t > 0$,

$$\sqrt{2E(\vec{u}(t))} \leq \sqrt{2E(u_0,u_1)} + \int_0^t \sqrt{\int_{\mathbb{R}^3} |f(s,x)|^2 dx ds}.$$

Proof. To lighten notations, we will denote:

$$\|\vec{u}(t)\|^2_{\dot{H}^1\times L^2} = \int_{\mathbb{R}^3} |\nabla u(t,x)|^2 dx + \int_{\mathbb{R}^3} |\partial_t u(t,x)|^2 dx, \quad \|f\|_{L^1(I,L^2)} = \int_I \|f(t)\|_{L^2(\mathbb{R}^3)} dt$$

($\|\cdot\|_{\dot{H}^1}$ is the norm defining the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^3)$, see Section II.2 below). From Duhamel's formula and the conservation of energy for the free equation (LW), it suffices to verify that for all $t > 0$,

$$\left\|\left(\int_0^t S(t-s)f(s)ds, \partial_t \int_0^t S(t-s)f(s)ds\right)\right\|_{\dot{H}^1\times L^2} \leq \|f\|_{L^1([0,t],L^2)} \tag{I.7.2}$$

By conservation of energy (Theorem I.6.1), we have

$$\Big\|\,(S(t-s)f(s), \partial_t(S(t-s)f(s)))\,\Big\|_{\dot{H}^1\times L^2} = \|f(s)\|_{L^2},$$

which implies directly (I.7.2) □

I.8. Exercises

Exercice I.1. Let $f : \mathbb{R}^N \to \mathbb{R}$ ($N \geq 1$). Suppose f is radial (i.e. That it depends only on the variable $r = |x| = \sqrt{x_1^2 + x_2^2 + \ldots + x_N^2}$). Denote $f(x) = g(|x|)$, where $g : [0,\infty[\to \mathbb{R}$.

1. Show that f is continuous on $\mathbb{R}^N$ if and only if g is continuous on $[0,\infty[$.
2. Show that f is C^1 on $\mathbb{R}^N$ if and only if g is C^1 on $[0,\infty[$ and $g'(0) = 0$.
3. Show that for any $k \geq 2$, f is C^k on $\mathbb{R}^N$ if and only if g is C^k on $\mathbb{R}^N$ and $g^{(j)}(0) = 0$ for all odd integers $j \leq k$.
4. Assuming f is C^1, compute $\frac{\partial f}{\partial x_j}$ in terms of g', $j = 1,\ldots,N$. Compute $g'(r)$ in terms of ∇f.
5. Assuming f is C^2 on $\mathbb{R}^N$, prove the formula

$$\Delta f(x) = g''(|x|) + \frac{N-1}{|x|} g'(|x|).$$

To lighten notation, we use the same notation (f) for functions f and g, and denote $g' = \frac{df}{dr}$, etc...

Exercice I.2 (Loss of regularity for the radial wave equation in dimension 1+3). Let $k \geq 0$ and $f \in C^k(\mathbb{R}^3)$ be a *radial* function. Define a function u on $\mathbb{R} \times (\mathbb{R}^3 \setminus \{0\})$, radial with respect to the space variable, by

$$u(t,x) = \frac{1}{2r}\Big((r+t)f(|r+t|) + (r-t)f(|r-t|)\Big),$$

where $r = |x|$. Note that this defines a function of class C^k on $\mathbb{R} \times \big(\mathbb{R}^3 \setminus \{0\}\big)$.

(1) Suppose that f is supported in the annulus $\{\frac{1}{2} \leq |x| \leq 2\}$ and is such that for $|\eta - 1| \leq 1/10$,

$$f(\eta) = \begin{cases} 2-\eta & \text{if } \eta > 1 \\ \eta & \text{if } \eta < 1 \end{cases}.$$

Calculate $\lim_{r\to 0} u(t,r)$ when $t = 1$, $t > 1$, and $t < 1$ (close to 1). Conclude that u cannot be extended to a continuous function on $\mathbb{R} \times \mathbb{R}^3$.

(2) Similarly, give an example of a C^2 function f such that u cannot be extended to a C^2 function on $\mathbb{R} \times \mathbb{R}^3$.

(3) Assume f is C^3. Show that u defines a C^2 function on $\mathbb{R} \times \mathbb{R}^3$.

(4) Let g be a C^2 radial function on $\mathbb{R}^3$. Show that

$$u(t,r) = \frac{1}{2r} \int_{r-t}^{r+t} \sigma g(|\sigma|) d\sigma,$$

extends to a C^2 function on $\mathbb{R}^3$.

Exercice I.3 (Explicit solutions of the radial wave equation in odd space dimension). Let $N \geq 3$ be an odd integer, written as $N = 2k+1$. Let T_k be the operator defined by

$$T_k \phi = \left(r^{-1} \frac{d}{dr} \right)^{k-1} \left(r^{2k-1} \phi(r) \right).$$

(1) Show that

$$T_k \varphi = \sum_{j=0}^{k-1} c_j r^{j+1} \phi^{(j)} r,$$

for some $c_j \in \mathbb{R}$. Determine c_0 and c_{k-1}.

(2) Show that for any function $\varphi \in C^{k+1}([0,+\infty[)$,

$$\frac{d^2}{dr^2}(T_k \varphi) = \left(r^{-1} \frac{d}{dr} \right)^k (r^{2k} \varphi'(r)).$$

Hint: You can start by verifying that the formula is true when $\varphi(r) = r^m$ for any integer m.

(3) Consider a solution $u(t,x)$ of the linear wave equation in space dimension N, radial with respect to the space variable. Suppose u is C^{k+1} on $\mathbb{R}^{1+N}$. Show prove

$$(\partial_t^2 - \partial_r^2)(T_k u) = 0.$$

Deduce an expression of $T_k u$ in terms of u_0 and u_1.

(4) Express $u(t,r)$ in terms of u_0 and u_1 when $N = 5$. What regularity of u_0 and u_1 is required for u to be C^2 on $\mathbb{R}^{1+5}$?

****

Exercice I.4. Let u be a solution of the wave equation (LW) in spce dimension $N \geq 3$, radial with respect to the space variable. Recall that $\Delta u = \frac{d^2}{dr^2} + \frac{N-1}{r}\frac{d}{dr}$. Suppose $u \in C^2(\mathbb{R}^{1+N})$, with compactly supported initial data. Let

$$v(t,r) = \int_r^\infty \rho \partial_t u(t,\rho) d\rho.$$

Show that v defines a radial solution, of class C^2, to the wave equation in space dimension $N-2$.

Exercice I.5. Let $f \in C^k(\mathbb{R}^3)$. Show that the function M_f, defined by (I.5.3), is also of class C^k.

Exercice I.6. Let $u \in C^2(\mathbb{R} \times \mathbb{R}^N)$ be a solution of (LW) with finite energy. Show

$$\forall \varepsilon > 0,\ \exists R > 0,\ \forall t \in \mathbb{R}, \quad \int_{|x|>R+|t|} e_u(t,x) dx \leq \varepsilon.$$

Exercice I.7 (Conservation of momentum). (1) Let u be a C^2 solution of (LW) on $\mathbb{R} \times \mathbb{R}^N$, and $j \in 1, \dots N$. Let $p_{j,u}(t,x) = \partial_{x_j} u(t,x) \partial_t u(t,x)$. Show

$$\frac{\partial p_{j,u}}{\partial t} = \frac{1}{2} \frac{\partial}{\partial x_j} \left((\partial_t u)^2 - |\nabla u|^2 \right) + \nabla \cdot V,$$

where V is a certain C^1 vector field to be specified.

(2) Assume that (u_0, u_1) has finite energy. Justify that

$$P_j(\vec{u}(t)) = \int_{\mathbb{R}^N} p_{j,u}(t,x)dx$$

is defined for all times. Show that this quantity is independent of time. You can start by considering a local version of the momentum

$$\int_{[-R,R]^N} p_{j,u}(t,x)dx \text{ or } \int_{\mathbb{R}^N} p_{j,u}(t,x)\varphi\left(\frac{x}{R}\right)dx$$

then let R tend to $+\infty$. Here φ denotes a C^2 function with compact support equal to 1 in a neighborhood of the origin.

EXERCICE I.8. (1) Let $u_1 \in C^2(\mathbb{R}^3)$ such that

$$\forall t \geq 0,\ \forall x \in \mathbb{R}^3, \quad u_1(x) \geq 0.$$

Assume $u_0 = 0$. Let u be the corresponding solution of (LW). Prove

$$\forall t \geq 0,\ \forall x \in \mathbb{R}^3, \quad u(t,x) \geq 0.$$

(2) Suppose now $N = 1$ or $N = 2$. Let u be the solution of (LW), (ID), with $(u_0, u_1) \in C^3 \times C^2$ (if $N = 2$) or $C^2 \times C^1$ (if $N = 1$).

Show that if $u_1 \geq 0$ and $u_0 = 0$ then $u(t,x)$ has the sign of t for all x and $t \neq 0$.

When $N = 1$, give a weaker sufficient condition on (u_0, u_1) such that:

$$\forall t \geq 0,\ \forall x \in \mathbb{R}, \quad u(t,x) \geq 0.$$

EXERCICE I.9. Assume $N = 1$ or $N = 2$. Let u be a solution of (I.1.2), with $u_0 = u_1 = 0$, and f of class C^1 (if $N = 1$) or C^2 (if $N = 2$). Express u in terms of f.

EXERCICE I.10. The *Minkowski spacetime* of dimension N is the space $\mathbb{R}^{1+N}$, equipped with the quadratic form of signature $(1, N)$:

$$g(X) = x_0^2 - \sum_{j=1}^{N} x_j^2 = t^2 - |x|^2 = {}^tXJX,$$

where ${}^tX$ is the transpose of X,

$$X = (x_0, x_1, \ldots, x_N),\ t = x_0,\ x = (x_1, \ldots, x_N),$$

and $J = [J_{\mu,\nu}]_{0\leq\mu\ \nu\leq N}$ is the matrix such that $J_{0,0} = 1$, $J_{\ell,\ell} = -1$ if $\ell \in 1, \ldots, N$, and $J_{\mu,\nu} = 0$ if $\mu \neq \nu$.

The Lorentz group $\mathrm{O}(1, N)$ is the group of real square matrices P of size $1 + N$ which leave the quadratic form g invariant, i.e., such that $g(PX) = g(X)$ for all X in $\mathbb{R}^{1+N}$. In other words, if P is a $(1 + N) \times (1 + N)$ matrix,

$$P \in \mathrm{O}(1, N) \iff {}^tPJP = J.$$

(1) Prove that a function v of class C^2 on $\mathbb{R}^{1+N}$ satisfies the wave equation (LW) if and only if $\mathrm{Tr}(Jv'') = 0$, where v'' is the Hessian matrix $\left[\partial_{x_\mu}\partial_{x_\nu}v\right]_{0\leq{\mu \atop \nu}\leq N}$.

(2) Let $P \in \mathrm{O}(1, N)$, $v \in C^2(\mathbb{R}^{1+N})$, and $w(X) = v(PX)$. Then

$$(\partial_t^2 - \Delta)v = 0 \iff (\partial_t^2 - \Delta)w = 0.$$

(3) Prove that the space rotations:

$$\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & R \end{bmatrix}, \quad R \in \mathrm{O}(N)$$

and the Lorentz boosts

$$\mathcal{R}\sigma = \begin{bmatrix} R_\sigma & \mathbf{0} \\ \mathbf{0} & I_{N-1}. \end{bmatrix}, \quad R_\sigma = \begin{bmatrix} \cosh(\sigma) & \sinh(\sigma) \\ \sinh(\sigma) & \cosh(\sigma) \end{bmatrix},$$

where I_{N-1} denotes the identity matrix $(N-1) \times (N-1)$ and $\sigma \in \mathbb{R}$ are Lorentz transformations. In these formulas, $\mathbf{0}$ always denotes the zero matrix of appropriate size.

EXERCICE I.11. In all Chapter I, we considered the Cauchy problem with initial conditions on a hyperplane in $\mathbb{R}^{1+N}$ of the form $\{t = t_0\}$. We now seek to solve the same problem by prescribing an initial condition on other hyperplanes. Therefore, we consider a hyperplane of the form

$$\Pi = \{X \in \mathbb{R}^{1+N}\ :\ {}^tAX = 0\}$$

where $A \in \mathbb{R}^{1+N} \setminus \{0\}$, $A = (a_0, a_1, \ldots, a_N) = (a_0, a)$.

(1) Prove that if $|a_0| > |a|$, there exists a transformation $P \in \mathrm{O}(1, N)$ such that

$$\Pi = P\left(\{(0,x),\ x \in \mathbb{R}^N\}\right).$$

Hint: use compositions of transformations defined in Question (3) *of Exercise I.10.*

(2) If the condition $|a_0| > |a|$ is satisfied, we can therefore reduce the Cauchy problem with an initial condition

$$u_{\restriction \Pi} = u_0, \quad A \cdot \nabla u_{\restriction \Pi} = u_1,$$

to a Cauchy problem with initial conditions at $t = 0$ as treated above. The hyperplane Π is called *timelike* when $A = (a_0, a)$ with $a_0 \in \mathbb{R}$, $A \in \mathbb{R}^N$, and $|a_0| > |a|$.

Prove that Π is timelike if and only if the restriction of the quadratic form g to Π is negatively defined.

(3) Under what condition on A does there exist $B = (b_0, b_1, \dots, b_N) \in \mathbb{R}^{N+1}$ such that the function

$$e^{A\cdot X + iB\cdot X}$$

is a solution of (LW)?

(4) Now assume that the hyperplane Π is not timelike. Let $Y \notin \Pi$. Construct a sequence of solutions $(u_n)_n$ of (LW) such that $u_n(X) = 0$ on Π, such that for any differential operator $D = \prod_{j=1}^N \partial_{x_1}^{\alpha_1} \dots \partial_{x_N}^{\alpha_N}$ (of arbitrarily large order), there exists $C > 0$ such that $|Du_n(X)| \leq Ce^{-n}$ on Π, but $|u_n(Y)| \to +\infty$ as $n \to \infty$.

CHAPTER II

The linear equation in Sobolev spaces

II.1. Reminders on the Fourier transform

Here, we recall the definition and basic properties of the Fourier transform on $\mathbb{R}^N$, in the most general framework possible, that of tempered distributions. We omit the proofs. For more details, one can consult, for example, the foundational writings of Laurent Schwartz [**25**], the course of Jean-Michel Bony [**4**], as well as [**2**, Section 1.2] for a quick introduction, and [**20**] for a more in-depth exposition (the first two references are in French).

We begin by introducing a notation: a *multi-index* is an element $\alpha = (\alpha_1, \dots, \alpha_N)$ of $\mathbb{N}^N$. The order of α is $|\alpha| = \sum_{j=1}^N \alpha_j$. The derivative with respect to α of a function f of class $C^{|\alpha|}$ on $\mathbb{R}^N$ is then defined by:

$$\partial_x^\alpha f = \prod_{j=1}^N \partial_{x_j}^{\alpha_j} f.$$

1.a. Fourier Transform on $\mathcal{S}$.

Definition II.1.1. The Schwartz space $\mathcal{S}(\mathbb{R}^N)$ is the space of functions f of class C^∞ on $\mathbb{R}^N$ such that for every $p \in \mathbb{N}$,

$$N_p(f) := \sup_{x \in \mathbb{R}^N \ |\alpha| \leq p} (1 + |x|)^p |\partial_x^\alpha f(x)| < \infty.$$

It can be observed that each N_p is a norm on $\mathcal{S}(\mathbb{R}^N)$, but N_p is not complete for any of these norms.

We equip $\mathcal{S}(\mathbb{R}^N)$ with the distance function

$$d(\varphi, \psi) = \sum_{p \geq 0} \frac{1}{2^p} \min\Big(N_p(\varphi - \psi), 1\Big). \tag{II.1.1}$$

Notice that $d(\varphi_n, \varphi)$ tends to 0 as n tends to infinity if and only if $N_p(\varphi_n - \varphi)$ tends to 0 for every p.

The metric space $(\mathcal{S}, d)$ is complete.[1]

The Fourier transform of an element φ of $\mathcal{S}$ is defined by the formula

$$\widehat{\varphi}(\xi) = \mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} \varphi(x) dx. \tag{II.1.2}$$

One easily checks that $\mathcal{F}$ is a continuous application from $\mathcal{S}$ into $\mathcal{S}$.

Fubini's theorem immediately implies the duality formula:

$$\int_{\mathbb{R}^N} \widehat{\varphi}(\xi)\psi(\xi) d\xi = \int_{\mathbb{R}^N} \varphi(x)\widehat{\psi}(x) dx, \tag{II.1.3}$$

for $\varphi, \psi \in \mathcal{S}(\mathbb{R}^N)$.

The Fourier transformation is a bijection of $\mathcal{S}$: by defining

$$\overline{\mathcal{F}}(\psi)(x) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{ix\cdot\xi} \psi(\xi) d\xi = \frac{1}{(2\pi)^N} \widehat{\psi}(-x), \tag{II.1.4}$$

we have the *Fourier inversion formula*: for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$,

$$\mathcal{F}\overline{\mathcal{F}}\varphi = \overline{\mathcal{F}}\mathcal{F}\varphi = \varphi. \tag{II.1.5}$$

[1] Such a vector space, equipped with a countable family of semi-norms, and which is complete as a metric space (where the distance function is defined as in (II.1.1)), is called a *Fréchet space*. It is a natural generalization of a Banach space when a unique norm is not sufficient to ensure completeness.

By combining the Fourier inversion formula (II.1.5) and the duality formula (II.1.3), we obtain the Plancherel theorem: for all φ, ψ in $\mathcal{S}$,

$$\int_{\mathbb{R}^N} \varphi(x)\overline{\psi}(x)dx = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} \widehat{\varphi}(\xi)\overline{\widehat{\psi}(\xi)}d\xi. \tag{II.1.6}$$

The Fourier transform exchanges multiplication by powers of x with differentiation. For all $\varphi \in \mathcal{S}(\mathbb{R}^N)$

$$\forall \alpha \in \mathbb{N}^N, \quad \mathcal{F}\partial_x^\alpha \varphi = i^{|\alpha|}\xi^\alpha \widehat{\varphi}(\xi), \quad \mathcal{F}(x^\alpha \varphi) = i^{|\alpha|}\partial_\xi^\alpha \widehat{\varphi}(\xi). \tag{II.1.7}$$

1.b. Fourier Transform of Tempered Distributions.

DEFINITION II.1.2. The space $\mathcal{S}'(\mathbb{R}^N)$ of *tempered distributions* is the topological dual of $\mathcal{S}(\mathbb{R}^N)$, i.e., the vector space of continuous linear forms on $\mathcal{S}$.

In the definition, continuity must be interpreted in the sense of the topology induced by the distance d defined by (II.1.1). Using the definition of this topology, one sees that a linear form f on $\mathcal{S}$ is an element of $\mathcal{S}'$ if and only if:

$$\exists p \in \mathbb{N}, \quad \forall \varphi \in \mathcal{S}, \quad |\langle f, \varphi\rangle| \leq CN_p(\varphi).$$

We equip $\mathcal{S}'$ with the topology of pointwise convergence: a sequence $(f_n)_n$ of elements of $\mathcal{S}'$ converges to f in $\mathcal{S}'$ if and only if

$$\forall \varphi \in \mathcal{S}, \quad \lim_{n\to\infty} \langle f_n, \varphi\rangle = \langle f, \varphi\rangle.$$

Several function spaces continuously embed into $\mathcal{S}'(\mathbb{R}^N)$ in the following manner. If f is a measurable, locally integrable function on f such that

$$\forall R > 0, \quad \int_{|x|\leq R} |f(x)|dx \leq C(1+R)^C$$

for some constant $C > 0$, we define an element L_f of $\mathcal{S}'(\mathbb{R}^N)$ by

$$\langle L_f, \varphi\rangle = \int_{\mathbb{R}^N} f(x)\varphi(x)dx.$$

The preceding application is injective, i.e., L_f is null if and only if f is null almost everywhere on $\mathbb{R}^N$. We then identify f with the linear form L_f, also denoted f. The preceding identification allows us to consider $\mathcal{S}$, Lebesgue spaces $L^p(\mathbb{R}^N)$ ($1 \leq p \leq \infty$), C_b^k (the space of C^k functions on $\mathbb{R}^N$ that are bounded along with all their derivatives up to order k) as subspaces of $\mathcal{S}'$.

Examples of tempered distributions that are not functions are given by the (improperly named) Dirac delta function at a, denoted δ_a and defined by $\langle \delta_a, \varphi\rangle = \varphi(a)$, as well as the surface measure σ on the sphere S^{N-1}, defined by:

$$\langle \sigma, \varphi\rangle = \int_{S^{N-1}} \varphi(y)d\sigma(y).$$

By duality, several actions can be defined on the elements of $\mathcal{S}'$.

Differentiation. Let $\alpha \in \mathbb{N}^N$ and $f \in \mathcal{S}'$. The *derivative* of f of order α is the element ∂_x^α of $\mathcal{S}'$ defined by:

$$\forall \varphi \in \mathcal{S}, \quad \langle \partial_x^\alpha f, \varphi\rangle = (-1)^{|\alpha|}\langle f, \partial_x^\alpha \varphi\rangle.$$

The integration by parts formula shows that if $f \in C_b^{|\alpha|}$, its derivative of order α in the sense of distributions coincides with its derivative in the classical sense.

Multiplication by a Function. We denote by $\mathcal{P} = \mathcal{P}(\mathbb{R}^N)$ the space of C^∞ functions with *slow growth*, i.e., such that

$$\forall \alpha, \quad \exists M, C > 0 \quad \forall x \in \mathbb{R}^N, \quad |\partial_x^\alpha g(x)| \leq C(1+|x|)^M. \tag{II.1.8}$$

It is easy to check that the multiplication by an element of $\mathcal{P}$ defines a continuous mapping from $\mathcal{S}$ into $\mathcal{S}$. We then define, for $f \in \mathcal{S}'$ and $g \in \mathcal{P}$, the product fg by:

$$\langle fg, \varphi\rangle = \langle f, g\varphi\rangle.$$

The product fg is an element of $\mathcal{S}'$. Fixing $g \in \mathcal{P}$, $f \mapsto fg$ is a continuous mapping from $\mathcal{S}'$ into $\mathcal{S}'$.

Fourier Transform. We define the Fourier transform of an element f of $\mathcal{S}'$ by

$$\forall \varphi \in \mathcal{S}, \quad \left\langle \widehat{f}, \varphi \right\rangle = \langle f, \widehat{\varphi} \rangle$$

The duality formula (II.1.3) shows that if $f \in \mathcal{S}$, its Fourier transform according to formula (II.1.2) and its Fourier transform in the sense of $\mathcal{S}'$ coincide.

We recall that $L^1(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N)$ are identified with subspaces of $\mathcal{S}'(\mathbb{R}^N)$. The Fourier transform on $\mathcal{S}'$ thus applies to elements of these two spaces. On $L^1(\mathbb{R}^N)$, we recover the Fourier transform in the classical sense.

Proposition II.1.3 (Fourier Transform in L^1). *Let $f \in L^1(\mathbb{R}^N)$, and $\widehat{f}$ be its Fourier transform in $\mathcal{S}'$. Then $\widehat{f}$ can be identified with the continuous function given by the formula:*

$$\widehat{f}(\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x)dx.$$

The second proposition immediately follows from the Plancherel theorem:

Proposition II.1.4 (Fourier Transform in L^2). *Let $f \in L^2(\mathbb{R}^N)$ then $\widehat{f} \in L^2(\mathbb{R}^N)$ and*

$$\|f\|_{L^2} = \frac{1}{(2\pi)^{N/2}} \|\widehat{f}\|_{L^2}.$$

Indeed, the Fourier inversion formula in $\mathcal{S}'$ (see below) implies that $f \mapsto \frac{1}{(2\pi)^{N/2}}\hat{f}$ is an isometry of $L^2(\mathbb{R}^N)$. The properties of the Fourier transform on $\mathcal{S}$ are transmitted by duality to the Fourier transform:

- We define the inverse Fourier transform $\overline{F}$ of an element f of $\mathcal{S}'$ by

$$\left\langle \overline{F} f, \varphi \right\rangle = \left\langle f, \overline{F}\varphi \right\rangle.$$

  Then we have the Fourier inversion formula:

$$\forall f \in \mathcal{S}', \quad \overline{\mathcal{F}}\mathcal{F} f = \mathcal{F}\overline{\mathcal{F}} f = f.$$

- Property (II.1.7) remains valid for $\varphi \in \mathcal{S}'$.

II.2. Sobolev Spaces

2.a. Definition. (cf [**2**, Section 1.3]) We mainly focus on Sobolev spaces on $\mathbb{R}^N$, of Hilbert type (i.e. based on L^2 spaces). In this section, we consider homogeneous Sobolev spaces $\dot{H}^\sigma$. We refer to the exercise sheet for classical Sobolev spaces H^σ.

The Hilbertian Sobolev spaces on $\mathbb{R}^N$ are easily defined using the Fourier transform:

Definition II.2.1. Let $\sigma \in \mathbb{R}$. The Sobolev space $\dot{H}^\sigma(\mathbb{R}^N)$ is the set of $f \in \mathcal{S}'(\mathbb{R}^N)$ such that $\widehat{f}$ can be identified with a function in $L^1(K)$ for every compact set K, such that the following quantity is finite:

$$\|f\|^2_{\dot{H}^\sigma} = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} |\xi|^{2\sigma} |\widehat{f}(\xi)|^2 d\xi.$$

The space $\dot{H}^\sigma$, equipped with the inner product:

$$(f, g)_{\dot{H}^\sigma} = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} |\xi|^{2\sigma} \widehat{f}(\xi)\overline{\widehat{g}(\xi)} d\xi$$

is a pre-Hilbert space.

Theorem II.2.2. *The space $\dot{H}^\sigma(\mathbb{R}^N)$ is complete if and only if $\sigma < N/2$. In this case, the vector space $\mathcal{S}_0$ of functions in $\mathcal{S}$ whose Fourier transform vanishes in a neighborhood of 0 is dense in $\dot{H}^\sigma(\mathbb{R}^N)$.*

Note that $\dot{H}^0$ is exactly the space L^2.

2.b. Sobolev Inequalities. We have the following Sobolev embedding on $\mathbb{R}^N$.

THEOREM II.2.3. *Let $\sigma \in]0, N/2[$, and $p \in (2,\infty)$ such that $\frac{1}{p} = \frac{1}{2} - \frac{\sigma}{N}$. Then $\dot{H}^\sigma(\mathbb{R}^N)$ is contained in L^p with continuous embedding.*

The result is well-known. We give a proof based on the Fourier transform, which yields a slightly stronger result that we will use later.

By the density result in Theorem II.2.2, it suffices to show that there exists a constant $C > 0$ such that

$$\forall f \in \mathcal{S}(\mathbb{R}^N), \quad \|f\|_{L^p(\mathbb{R}^N)} \leq C\|f\|_{\dot{H}^\sigma(\mathbb{R}^N)}. \tag{II.2.1}$$

Let $f \in \mathcal{S}$. We denote[2]

$$\|f\|^2_{\dot{B}^\sigma} = \sup_{k\in\mathbb{Z}} \frac{1}{(2\pi)^N}\int_{2^k\leq|x|\leq 2^{k+1}} |\xi|^{2\sigma}|\widehat{f}(\xi)|^2 d\xi,$$

and observe that $\|f\|_{\dot{B}^\sigma} \leq \|f\|_{\dot{H}^\sigma}$. We will prove the following result, which implies (II.2.1):

THEOREM II.2.4 (Improved Sobolev Inequality). *Let σ and p be as in the previous theorem. Then there exists a constant $C > 0$ such that*

$$\forall f \in \mathcal{S}(\mathbb{R}^N), \quad \|f\|^p_{L^p} \leq \|f\|^{p-2}_{\dot{B}^\sigma}\|f\|^2_{\dot{H}^\sigma}.$$

NOTATION II.2.5. Let φ be a function on $\mathbb{R}^N$. For $u \in \mathcal{S}'(\mathbb{R}^N)$, we denote

$$\varphi(D)u = \overline{\mathcal{F}}\left(\varphi(\xi)\widehat{u}(\xi)\right).$$

The operator $\varphi(D)$ is called Fourier multiplier (with symbol φ).

The tempered distribution $\varphi(D)u$ is not well-defined for all functions φ and $u \in \mathcal{S}'$: we need $\varphi\,\widehat{u}$ to define a tempered distribution. This is for example the case if $\varphi \in L^\infty$ and $u \in \dot{H}^\sigma$ (in this case $\varphi(D)u \in \dot{H}^\sigma$), or if $\varphi \in \mathcal{P}(\mathbb{R}^N)$ (the space of C^∞ functions with slow growth i.e. that satisfy (II.1.8)).

PROOF. We use a method introduced by Chemin and Xu in [**6**]. We fix a parameter $A > 0$ and decompose f into a *high-frequency* part $f_{>A}$ and a *low-frequency* part $f_{<A}$:

$$f_{>A} = \overline{\mathcal{F}}\left(\mathbb{1}_{|\xi|>A}\widehat{f}(\xi)\right) = \mathbb{1}_{|D|>A}f, \quad f_{<A} = \mathbb{1}_{|D|<A}f = 1 - f.$$

Let $k(A)$ be the largest integer such that $2^{k(A)} \leq A$. By using the Cauchy-Schwarz inequality, then the fact that $\sigma < N/2$, we obtain:

$$\begin{aligned}|f_{<A}(x)| = \frac{1}{(2\pi)^N}\left|\int_{|\xi|<A} e^{ix\cdot\xi}\widehat{f}(\xi)d\xi\right| &\leq \frac{1}{(2\pi)^N}\sum_{k\leq k(A)}\int_{2^k\leq|\xi|\leq 2^{k+1}}|\widehat{f}(\xi)|d\xi\\ &\leq \frac{1}{(2\pi)^N}\sum_{k\leq k(A)} 2^{k(N/2-\sigma)}\left(\int_{2^k\leq|\xi|\leq 2^{k+1}}|\xi|^{2\sigma}|\widehat{f}(\xi)|^2 d\xi\right)^{1/2} \leq C_N A^{N/2-\sigma}\|f\|_{\dot{B}^\sigma},\end{aligned}$$

where C_N depends only on the dimension N. Then we write (using Fubini's equality):

$$\|f\|^p_{L^p} = \int |f(x)|^p dx = \int_{\mathbb{R}^N} p\int_0^{|f(x)|}\lambda^{p-1}d\lambda dx = p\int_0^{+\infty}\lambda^{p-1}\left|\left\{x\in\mathbb{R}^N \ : \ |f(x)|\geq\lambda\right\}\right|d\lambda,$$

where $|S|$ denotes the Lebesgue measure of the measurable subset S of $\mathbb{R}^N$. Let $A(\lambda)$ be such that

$$C_N A(\lambda)^{\frac{N}{2}-\sigma}\|f\|_{\dot{B}^\sigma} = \lambda/2.$$

For any x in $\mathbb{R}^N$,

$$|f_{<A(\lambda)}(x)| \leq \frac{\lambda}{2}.$$

Thus $|f(x)| > \lambda \Longrightarrow |f_{>A(\lambda)}(x)| > \lambda/2$. Hence:

$$\|f\|^p_{L^p} \leq p\int_0^\infty \lambda^{p-1}\Big|\left\{x\in\mathbb{R}^N \ : \ |f_{>A(\lambda)}(x)| > \lambda/2\right\}\Big|d\lambda$$

By integrating $|f_{>A(\lambda)}|^2$ over the set $\left\{x\in\mathbb{R}^N \ : \ |f_{>A(\lambda)}(x)|>\lambda/2\right\}$, we get

$$\Big|\left\{x\in\mathbb{R}^N \ : \ |f_{>A(\lambda)}(x)|>\lambda/2\right\}\Big| \leq \frac{4}{\lambda^2}\|f_{>A(\lambda)}\|^2_{L^2}.$$

[2] This norm defines the Besov space $\dot{B}^\sigma_{2,\infty}$. See [**2**, Section 2.3] for the definition of general Besov spaces.

Combining with the Plancherel theorem, then Fubini's theorem, we obtain

$$\|f\|_{L^p}^p \leq \frac{4p}{(2\pi)^N}\int_0^\infty \lambda^{p-3}\int_{|\xi|>A(\lambda)}|\widehat{f}(\xi)|^2 d\xi d\lambda$$
$$= \frac{4p}{(2\pi)^N}\int_{\mathbb{R}^N}\left|\widehat{f}(\xi)\right|^2\int_0^{c(f,\xi)}\lambda^{p-3}d\lambda d\xi = C_{p,N}\int_{\mathbb{R}^N}\left|\widehat{f}(\xi)\right|^2 c(f,\xi)^{p-2}d\xi,$$

where $c(f,\xi)=2C_N\|f\|_{\dot{B}^\sigma}|\xi|^{\frac{N}{2}-s}$, and $C_{p,N}$ depends only on N and p. It can be easily verified that $(\frac{N}{2}-\sigma)(p-2)=2\sigma$, which proves the announced inequality. □

We will focus more particularly on the case $s=1$. According to the above, the Sobolev space $\dot{H}^1(\mathbb{R}^N)$, $N\geq 3$, is a Hilbert space, contained in $L^{\frac{2N}{N-2}}$, which can be defined as the closure of the space $\mathcal{S}(\mathbb{R}^N)$ (or $C_0^\infty(\mathbb{R}^N)$) for the $\dot{H}^1(\mathbb{R}^N)$-norm. We can characterize this norm with the first-order partial derivatives of f. Indeed,

$$\|f\|_{\dot{H}^1}^2 = \frac{1}{(2\pi)^N}\int |\xi|^2\left|\widehat{f}(\xi)\right|^2 d\xi = \sum_{j=1}^N\int\left|\xi_j\widehat{f}(\xi)\right|^2 d\xi,$$

which shows by Plancherel's theorem and formula (II.1.7)

$$\|f\|_{\dot{H}^1}^2 = \int |\nabla f(x)|^2 dx.$$

The attentive reader will have noticed that the space $\dot{H}^1(\mathbb{R}^N)$ is not the set of $\varphi\in\mathcal{S}'(\mathbb{R}^N)$ such that for all j, $\partial_{x_j}\varphi\in L^2(\mathbb{R}^N)$: indeed, nonzero constant functions are in this space, but not in $\dot{H}^1(\mathbb{R}^N)$ (the Fourier transform $\hat{c}$ of a nonzero constant function is the multiple of a Dirac function, which does not satisfies the assumption of local integrability in the definition of $\dot{H}^1$).

The density result of Theorem II.2.2 implies that $\dot{H}^1(\mathbb{R}^N)$ is the closure of $C_0^\infty(\mathbb{R}^N)$ for the norm $\|\cdot\|_{\dot{H}^1}^2$. An other characterization, using the Sobolev inequality, is given by

$$\dot{H}^1(\mathbb{R}^N) = \left\{f\in L^{\frac{2N}{N-2}}(\mathbb{R}^N),\ |\nabla f|\in L^2(\mathbb{R}^N)\right\}. \tag{II.2.2}$$

The proof of (II.2.2) is left to the reader.

II.3. The wave equation in the Schwartz space

Let $(u_0,u_1)\in\mathcal{S}(\mathbb{R}^N)$. We will write the solution u of (LW), (ID) using the Fourier transformation. We start with a formal calculation, assuming that $u(t)\in\mathcal{S}$ for all t (which we will prove later). We denote $\widehat{u}(t)$ as the Fourier transform of u with respect to the spatial variable, i.e.,

$$\widehat{u}(t,\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi}u(t,x)dx.$$

Thus, we have

$$\widehat{\Delta u}(t,\xi) = -|\xi|^2\widehat{u}(t,\xi),$$

and the wave equation (LW) is formally equivalent to the linear differential equation

$$\partial_t^2\widehat{u}(t,\xi)+|\xi|^2\widehat{u}(t,\xi)=0,$$

where the variable ξ is considered as a parameter. The solution to this equation, with initial conditions $(\widehat{u}(0),\partial_t\widehat{u}(0))=(u_0,u_1)$, yields

$$\widehat{u}(t,\xi) = \cos(t|\xi|)\widehat{u}_0(\xi)+\frac{\sin(t|\xi|)}{|\xi|}\widehat{u}_1(\xi),$$

or, with the previously introduced notation,

$$u(t) = \cos(t|D|)u_0+\frac{\sin(t|D|)}{|D|}u_1. \tag{II.3.1}$$

Theorem II.3.1. *Let* $(u_0,u_1)\in\mathcal{S}(\mathbb{R}^N)^2$. *Then* u *defined by* (II.3.1) *is an element of* $C^\infty(\mathbb{R}\times\mathbb{R}^N)$. *It is the unique* C^2 *solution of* (LW), (ID).

PROOF. Uniqueness follows from Theorem I.4.1. Hence, it suffices to prove that u, defined by (II.3.1), is C^∞ and satisfies (LW), (ID). We have

$$u(t,x) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{ix\cdot\xi}\left(\cos(t|\xi|)\widehat{u}_0(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\widehat{u_1}(\xi)\right)d\xi.$$

By writing

$$\frac{\sin(t|\xi|)}{|\xi|} = t\sum_{k\geq 0}\frac{(-1)^k(t|\xi|)^{2k}}{(2k+1)!},$$

we see that it is a C^∞ function of (t,ξ). Moreover, $\frac{\left|\partial_t^j\sin(t|\xi|)\right|}{|\xi|} \leq |t||\xi|^j$. Similarly, $(t,\xi)\mapsto\cos(t|\xi|)$ is C^∞ and $\left|\partial_t^j\cos(t|\xi|)\right| \leq |\xi|^j$. Using the fact that $\widehat{u}_0$ and $\widehat{u}_1$ are elements of $\mathcal{S}(\mathbb{R}^N)$, by the theorem of differentiation under the integral sign, we obtain that u is C^∞ and satisfies (LW). The Fourier inversion formula shows that u also satisfies the initial conditions (ID). □

II.4. The wave equation in Sobolev spaces

4.a. The equation in general homogeneous Sobolev spaces. Let $(u_0,u_1)\in\dot{H}^\sigma\times\dot{H}^{\sigma-1}$, $\sigma<N/2$. We define as before u by (II.3.1). We also define the formal derivative of u with respect to time:

$$u'(t,x) = \cos(t|D|)u_1 - |D|\sin(t|D|)u_0.$$

Then u and u' satisfy the following properties:

CLAIM II.4.1. *$u\in C^0(\mathbb{R},\dot{H}^\sigma)$, $u'\in C^0(\mathbb{R},\dot{H}^{\sigma-1})$, $u(0)=u_0$, $u'(0)=u_1$.*

PROOF. Using that $\widehat{u}_0\in L^2(|\xi|^{2\sigma}d\xi)$ and $\widehat{u}_1\in L^2(|\xi|^{2\sigma-2}d\xi)$, it is easy to see that

$$\widehat{u}\in C^0(\mathbb{R},L^2(|\xi|^{2\sigma}d\xi)),\quad \widehat{u'}\in C^0(\mathbb{R},L^2(|\xi|^{2\sigma-2}d\xi)), \tag{II.4.1}$$

which yields the announced continuity property. The facts that $u(0)=u_0$ and $u'(0)=u_1$ follow immediately from the definition. □

CLAIM II.4.2. *$\forall t,\quad \|(u(t),u'(t))\|_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}} = \|(u_0,u_1)\|_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}}$.*

PROOF.

$$\begin{aligned}\int_{\mathbb{R}^N}|\widehat{u}(t,\xi)|^2|\xi|^{2\sigma}d\xi + \int_{\mathbb{R}^N}\widehat{u'}(t,\xi)|\xi|^{2\sigma-2}d\xi\\
&= \int_{\mathbb{R}^N}\left|\cos(t|\xi|)\widehat{u}_0(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\widehat{u}_1(\xi)\right|^2|\xi|^{2\sigma}d\xi\\
&\quad + \int_{\mathbb{R}^N}\left|-|\xi|\sin(t|\xi|)\widehat{u}_0(\xi) + \cos(t|\xi|)\widehat{u}_1(\xi)\right|^2|\xi|^{2\sigma-2}d\xi\\
&= \int_{\mathbb{R}^N}\left(|\widehat{u}_0(\xi)|^2 + |\widehat{u}_1(\xi)|^2|\xi|^{-2}\right)|\xi|^{2\sigma}d\xi,\end{aligned}$$

which gives the desired property. □

CLAIM II.4.3. *Let $(u_{0,n},u_{1,n})\in(\mathcal{S}_0(\mathbb{R}^N))^2$ such that $(u_{0,n},u_{1,n})$ converges to (u_0,u_1) in $\dot{H}^\sigma\times\dot{H}^{\sigma-1}$. Let u_n be the solution of* (LW) *with data $(u_{0,n},u_{1,n})$. Then*

$$\lim_{n\to\infty}\sup_{t\in\mathbb{R}}\|u_n(t)-u(t)\|_{\dot{H}^\sigma} + \|\partial_t u_n(t)-u'(t)\|_{\dot{H}^{\sigma-1}} = 0.$$

PROOF. It follows immediately from the preceding claim, applied to $(u-u_n,u'-\partial_t u_n)$. □

CLAIM II.4.4. *One can identify u with a distribution on $\mathbb{R}\times\mathbb{R}^N$, and it satisfies the wave equation* (LW) *in the distributional sense. Furthermore $u'=\partial_t u$ in the sense of distribution.*

PROOF. We first give a "concrete" proof of these facts for the reader which is not familiar with the theory of distributions, assuming that σ is large enough so that the object considered are all functions on $\mathbb{R}\times\mathbb{R}^N$.

Let $\sigma\geq 0$. We let u_n be as in Claim II.4.3. Using that u_n is a C^∞ solution of (LW) and integrating by parts, we obtain

$$\iint u_n(t,x)(\partial_t^2-\Delta)\varphi\,dxdt = 0.$$

Using the Sobolev embedding $\dot{H}^\sigma \subset L^p$, $\frac{1}{p} = \frac{1}{2} - \frac{\sigma}{N}$, and the point (II.4.3), we see that

$$\lim_{n\to\infty} \|u - u_n\|_{L^p(K)} = 0,$$

for all compact K of $\mathbb{R}^N$. This implies

$$0 = \lim_{n\to\infty} \iint u_n(t,x)(\partial_t^2 - \Delta)\varphi dxdt = \lim_{n\to\infty} \iint u(t,x)(\partial_t^2 - \Delta)\varphi dxdt,$$

and thus

$$\forall \varphi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^N), \quad \iint u(\partial_t^2 - \Delta)\varphi dtdx = 0,$$

which is precisely the meaning of $\partial_t^2 u - \Delta u = 0$ in the distributional sense.

Let $\sigma \geq 1$. The equality

$$\partial_t u_n = -|D|\sin(t|D|)u_{0,n} + \cos(t|D|)u_{1,n}.$$

holds by differentiation below the integral sign. By integration by parts,

$$\forall \varphi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^N), \quad \iint \partial_t u_n \varphi dtdx = -\iint u_n \partial_t \varphi dtdx,$$

Letting $n \to \infty$, we obtain

$$\forall \varphi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^N), \quad \iint u'\varphi dtdx = -\iint u\partial_t \varphi dtdx,$$

which means that $u' = \partial_t u$ in the distributional sense.

The proof for general σ is essentially the same, and can be skipped by the reader who is not familiar with distributions.

If $\varphi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^N)$ (the space of smooth functions with compact support on $\mathbb{R}\times\mathbb{R}^N$), one defines the action of u on φ by

$$\langle u, \varphi\rangle = \int_{-\infty}^{+\infty} \langle u(t), \varphi(t)\rangle_{\mathcal{S}',\mathcal{S}}\, dt,$$

where $\varphi(t)$ is the function $t \mapsto \varphi(t,\cdot)$. It is a straightforward exercise to prove that u is well-defined and that is is a distribution on $\mathbb{R}\times\mathbb{R}^N$. The facts that u satisfies the wave equation in the distributional sense and that $u'(t) = \partial_t u(t)$ follow immediately from Claim II.4.3, that implies that $\lim u_n = u$ in the distributional sense, where u_n is a in Claim II.4.3. This last fact is an immediate consequence of Claim II.4.3. □

From now on, we will use the formula (II.1.2) as the definition of the solution u of (LW), (ID) with $(u_0, u_1) \in \dot{H}^\sigma \times \dot{H}^{\sigma-1}$. The preceding claims show that such a u is a limit of smooth, classical solutions of (LW),(ID), and that it satisfies (LW) in a weak sense. Also, we have

$$\partial_t u = -|D|\sin(t|D|)u_0 + \cos(t|D|)u_1$$

in the sense of distribution. In the sequel, we will always use the notation $\partial_t u$ to denote this quantity.

4.b. The wave equation in the energy space. Of particular interest for us is the case $s = 1$. We will call "finite energy solutions" the weak solutions with initial data $\dot{H}^1 \times L^2$ given by the preceding subsection in the case $s = 1$, $N \geq 3$. We will focus on the case $N = 3$. We note that if $(u_0, u_1) \in (C^3 \times C^2)(\mathbb{R}^3) \cap \left(\dot{H}^1 \times L^2\right)(\mathbb{R}^3)$, we have two ways of defining the solution u: by integrals on spheres, as in Theorem I.5.2, and using the Fourier transform, i.e. by formula (II.3.1). Let us prove that these two definitions coincide:

Proposition II.4.5. *Let $u \in C^2(\mathbb{R}\times\mathbb{R}^3)$ be a solution of* (LW), (ID). *Assume furthermore $u_0 = u(0) \in \dot{H}^1$, $u_1 = \partial_t u(0) \in L^2$. Then*

$$u(t) = \cos(t|D|)u_0 + \frac{\sin(t|D|)}{|D|}u_1, \quad \partial_t u(t) = -|D|\sin(t|D|)u_0 + \cos(t|D|)u_1.$$

Proof. Let $(u_{0,n}, u_{1,n}) \in \left(\mathcal{S}(\mathbb{R}^N)\right)^2$ with

$$\lim_{n\to\infty} \|u_{0,n} - u_0\|_{\dot{H}^1} + \|u_{1,n} - u_1\|_{L^2} = 0.$$

Let u_n be the corresponding solution of (LW) given by (II.3.1) (note that by uniqueness it is also the solution given by Theorem I.5.2). Since $u - u_n$ is a C^2, finite energy solution of (LW), Theorem I.6.1 yields

$$\forall t, \quad \|u(t) - u_n(t)\|_{\dot{H}^1}^2 + \|\partial_t u(t) - \partial_t u_n(t)\|_{L^2}^2 = \|u_0 - u_{0,n}\|_{\dot{H}^1}^2 + \|u_1 - u_{1,n}\|_{L^2}^2,$$

which tends to 0 as n goes to infinity. This proves the result, since $u_n(t)$ converges to $\cos(t|D|)u_0 + \frac{\sin(t|D|)}{|D|}u_1$ in $\dot{H}^1(\mathbb{R}^3)$ and $\partial_t u_n(t)$ converges to $-|D|\sin(t|D|)u_0 + \cos(t|D|)u_1$ in L^2 by Claim II.4.3. □

Using the approximation of finite energy solutions by solutions with initial data in $\mathcal{S}$, we can transfer several results of Chapter I to general finite energy solutions. This is the case of the decay of energy on past wave cones, which imply finite speed of propagation. If u is a finite energy solution (in any dimension $N \geq 3$) and $R > 0$, $x_0 \in \mathbb{R}^N$, $t_0 \in \mathbb{R}$, we denote by

$$E_{\mathrm{loc}}(t) = \int_{|x-x_0|<R-|t-t_0|} e_u(t,x)dx.$$

Then

Theorem II.4.6. *$E_{\mathrm{loc}}(t)$ is nonincreasing for $t \geq t_0$.*

Proof. From Theorem I.4.1, this quantity is nonincreasing when $(u_0, u_1) \in \mathcal{S}$. Considering the approximation given by Claim II.4.3, we obviously have, as a consequence of this claim,

$$\forall t, \quad \lim_{n\to\infty} \int_{|x-x_0|<R-|t-t_0|} e_{u_n}(t,x)dx = \int_{|x-x_0|<R-|t-t_0|} e_u(t,x)dx.$$

This gives the desired monotonicity property. □

We note that for general finite energy solution the integration by parts used in the proof of Theorem I.4.1 is no longer valid (since the boundary terms are not always well-defined).

4.c. Equation with a source term. We next consider the wave equation with a source term (I.1.2). By linearity, it is sufficient to study the equation with zero initial data:

$$\partial_t^2 u - \Delta u = f, \quad \vec{u}_{\restriction t=0} = (0,0). \tag{II.4.2}$$

Proposition II.4.7. *Assume $f \in C^0\left(\mathbb{R}, \mathcal{S}(\mathbb{R}^N)\right)$. Then u defined by*

$$u(t) = \int_0^t \frac{\sin\left((t-s)|D|\right)}{|D|} f(s)ds \tag{II.4.3}$$

is the unique solution of (II.4.2).

Proof. The uniqueness follows as usual by Theorem I.4.1. It is thus sufficient to check that u defined by (II.4.3) is of class C^2, and is a solution of (II.4.2). We consider F the function defined on $\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N$ by

$$F(t,s,x) = \left(\frac{\sin\left((t-s)|D|\right)}{|D|} f(s)\right)(x).$$

Thus

$$F(t,s,x) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{ix\cdot\xi} \frac{\sin\left((t-s)|\xi|\right)}{|\xi|} \widehat{f}(s,\xi)d\xi$$

Using that $\widehat{f} \in C^0\Big(\mathbb{R}, \mathcal{S}(\mathbb{R}^N)\Big)$, it is easy to check that F is continuous and C^∞ with respect to the variable (t,x), and that one can differentiate below the integral sign. The result follows since by integration by parts in the ξ variable,

$$\Delta F(t,s,x) = -\frac{1}{(2\pi)^N} \int |\xi|^2 e^{ix\cdot\xi} \frac{\sin\left((t-s)|\xi|\right)}{|\xi|} \widehat{f}(s,\xi)d\xi$$

□

We note that Duhamel formula (II.4.3) is still valid when $f \in L^1([-T,+T], \dot{H}^{\sigma-1})$ for all T, where σ is a fixed real number (assumed to be $< N/2$ for simplicity), and that it yields a function $u \in C^0(\mathbb{R}, \dot{H}^\sigma)$ with $\partial_t u \in C^0(\mathbb{R}, \dot{H}^{\sigma-1})$,

$$\partial_t u = \int_0^t \cos\left((t-s)|D|\right) f(s)ds, \tag{II.4.4}$$

in the sense of distribution, and such that

$$\|\vec{u}(t)\|_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}} \leq \int_0^t \|f(s)\|_{\dot{H}^{\sigma-1}}ds. \tag{II.4.5}$$

Note that (II.4.5) is exactly the energy inequality proved in Chapter I when $\sigma = 1$.

We can approximate such an f by a sequence of functions (f_n) with

$$f_n \in C^0(\mathbb{R}, \mathcal{S}), \quad \forall t, \quad \lim_{n\to\infty} \int_{-T}^{+T} \|f(s) - f_n(s)\|_{\dot{H}^{\sigma-1}} ds = 0.$$

The corresponding solutions u_n defined by

$$u_n(t) = \int_0^t \frac{\sin\big((t-s)|D|\big)}{|D|} f_n(s) ds$$

are C^2 solutions of (II.4.2) and satisfy

$$\sup_{-T\leq t\leq T} \|\vec{u}_n(t) - \vec{u}(t)\|_{\dot{H}^\sigma \times \dot{H}^{\sigma-1}} \underset{n\to\infty}{\longrightarrow} 0. \tag{II.4.6}$$

As in the case of the free wave equation (LW) with nonzero initial data, this proves that u satisfies (LW) in the sense of distribution. In this situation, we will take the formula (II.4.3) as a definition of the solution u of (LW).

Exercice II.1. Assume that $\sigma = 1$. Let f defined on $\mathbb{R} \times \mathbb{R}^N$, such that $f \in L^1([-T, +T, L^2(\mathbb{R}^N))$. Prove that there exists a sequence of functions $f_n \in C_0^\infty(\mathbb{R} \times \mathbb{R}^N)$ such that

$$\forall T > 0, \quad \lim_{n\to\infty} \|f_n - f\|_{L^1([-T,+T],L^2(\mathbb{R}^N))} = 0.$$

Exercice II.2. Let u be a C^2 solution of (LW) for some $f \in C^0(\mathbb{R}\times\mathbb{R}^N)$. Assume that $f \in L^1([-T, +T], L^2(\mathbb{R}^N))$ for all $T > 0$. Show that u satisfies (II.4.3).

CHAPTER III

Strichartz inequalities

III.1. Introduction

In view of Plancherel theorem and the Fourier representation formulas for the wave equation, it is natural to study the wave equation in $L^2(\mathbb{R}^N)$ or in L^2 based spaces such as the Sobolev spaces $\dot{H}^s$ considered in the preceding chapter. However, this is not sufficient for the study of nonlinear wave equations. Indeed since $\||f|^p\|_{L^2(\mathbb{R}^N)} = \|f\|_{L^{2p}}^{2p}$, the appearance of Lebesgue spaces L^q with $q \neq 2$ is unavoidable. A first way to deal with this issue is to use Sobolev inequalities. For example, if one wants to consider solutions in the energy spaces for the equation

$$\partial_t^2 u - \Delta u = u^3, \quad x \in \mathbb{R}^3, \tag{III.1.1}$$

the energy inequality will yields terms of the form[1] $\|u^3\|_{L^1([0,T],L^2)} = \|u\|_{L^3([0,T],L^6)}^3 \lesssim T\|u\|_{L^\infty([0,T],\dot{H}^1)}$, which is sufficient to prove the existence and uniqueness of finite energy solutions for (III.1.1). However this strategy will not work for higher order nonlinearities, and in particular the quintic one which we will focus on in several chapters of this course. In this chapter I will introduce the celebrated *Strichartz inequalities*, that use the dispersive properties of the wave equation to improve over Sobolev type inequalities. This type of inequalities was introduced by Robert Strichartz in an article published in 1977 [**27**], and generalized later by several authors. See e.g. [**19**] or the book [**26**].

The original inequalities of Strichartz were formulated in terms of Lebesgue spaces $L^q(\mathbb{R}\times\mathbb{R}^N)$ on the whole space time $\mathbb{R}\times\mathbb{R}^N$. Having in minds applications to nonlinear wave equations, it is useful to consider more general spaces where the Lebesgue exponents in space and times are distinct. If I is an interval, we will define $L^p(I, L^q(\mathbb{R}^N))$ as the set of integrable function $f : I \mapsto L^q(\mathbb{R}^N)$ such that

$$\|u\|_{L^p(\mathbb{R},L^q(\mathbb{R}^N))} = \left\| \|u(\cdot)\|_{L^q(\mathbb{R}^N)} \right\|_{L^p(\mathbb{R})} = \left(\int_{\mathbb{R}} \|u(t)\|_{L^q}^p dt \right)^{1/p}. \tag{III.1.2}$$

if finite (with the usual modification if $p = \infty$). The notion of integrable functions with values in a Banach space can be rigorously defined by the theory of Bochner's integration, see e.g. section 1.2 in the book [**5**]. An element of $L^p(I, L^q(\mathbb{R}^N))$ can be identified with a (class) of measurable function on $I\times\mathbb{R}^N$. With the identification, we can use the density of $C_0^\infty(\mathbb{R}^N)$ in $L^q(\mathbb{R}^N)$, $q<\infty$, to prove that $C_0^\infty(I\times\mathbb{R}^N)$ is dense in $L^p(I, L^q)$ if q and p are finite. Using this fact, we will mainly work on L^pL^q norms of smooth functions, for which the definition of (III.1.2) is clear.

We will often write $L^p(I, L^q)$ instead of $L^p(I, L^q(\mathbb{R}^N))$ to lighten notations. When $I = \mathbb{R}$, we will also use the notation L^pL^q.

We will use the generalized Hölder inequality in these spaces:

Proposition III.1.1. *Let p, q, p_1, q_1, p_2, q_2 in $[1,\infty]$ with*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Let $f \in L^{p_1}L^{q_1}$ and $g \in L^{p_2}L^{q_2}$. Then $fg \in L^pL^q$ and

$$\|fg\|_{L^pL^q} \leq \|f\|_{L^{p_1}L^{q_1}}\|g\|_{L^{p_2}L^{q_2}}.$$

The proof of Proposition III.1.1, using the standard Hölder inequality, is left as an exercise to the reader. We will also use the following consequence of Hölder inequality:

Exercice III.1. Let $\theta \in [0,1]$, p, q, p_1, q_1, p_2, q_2 in $[1,\infty]$ with

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

[1] See below for the notations $L^p(I, L^q)$

Let $f \in L^{p_1}L^{q_1} \cap L^{p_2}L^{q_2}$. Prove that $f \in L^pL^q$ and

$$\|f\|_{L^pL^q} \leq \|f\|^{\theta}_{L^{p_1}L^{q_1}} \|f\|^{1-\theta}_{L^{p_2}L^{q_2}}.$$

III.2. Statement of the estimate

The Strichartz inequalities in space dimension 3 with initial data in the energy space read as follows:

THEOREM III.2.1. *Let* $(u_0, u_1) \in (\dot{H}^1 \times L^2)(\mathbb{R}^3)$ *and* $f \in L^1(\mathbb{R} \times L^2(\mathbb{R}^3))$. *Let*

$$u(t) = \cos(t|D|)u_0 + \frac{\sin(t|D|)u_1}{|D|} + \int_0^t \frac{\sin\left((t-s)|D|\right)}{|D|} f(s)ds. \tag{III.2.1}$$

Then for any (p,q) *with* $p > 2$,

$$\frac{1}{p} + \frac{3}{q} = \frac{1}{2}, \tag{III.2.2}$$

one has $u \in L^p(\mathbb{R}, L^q(\mathbb{R}^3))$ *and*

$$\|u\|_{L^p(\mathbb{R},L^q)} \leq C\left(\|(u_0,u_1)\|_{\dot{H}^1\times L^2} + \|f\|_{L^1(\mathbb{R},L^2)}\right).$$

for a constant $C > 0$ *depending only on* p.

REMARK III.2.2. If I is an interval with $0 \in I$, $f \in L^1(I, L^2(\mathbb{R}^N))$, and u satisfies (III.2.1) for $t \in I$, then $u \in L^p(I, L^q(\mathbb{R}^3))$ and

$$\|u\|_{L^p(I,L^q)} \leq C\left(\|(u_0,u_1)\|_{\dot{H}^1\times L^2} + \|f\|_{L^1(I,L^2)}\right). \tag{III.2.3}$$

This follows immediately from the Theorem, extending f by $f(t) = 0$ if $t \notin I$.

REMARK III.2.3. We recall that in the setting of Theorem III.2.1, we also have $\vec{u} \in C^0\left(\mathbb{R}, \dot{H}^1 \times L^2\right)$, and the energy inequality

$$\|\vec{u}(T)\|_{\dot{H}^1\times L^2} \leq \|\vec{u}(0)\|_{\dot{H}^1\times L^2} + \|f\|_{L^1([0,T],L^2)},$$

for any $T > 0$, which can be easily checked using the space Fourier transform of formula (III.2.1)

We have focused on solutions with initial data $\dot{H}^1 \times L^2$ in space dimension 3, in view of application to the quintic wave equation in space dimension 3. Analogs of Theorem III.2.1 exist in all space dimensions $N \geq 2$, with more general assumptions on the initial data (u_0, u_1) and the right hand-side f. The condition (III.2.2) is necessary by the scaling of the equation. For solutions in space dimension N with initial data in $\dot{H}^\sigma \times \dot{H}^{\sigma-1}$, it becomes

$$\frac{1}{p} + \frac{N}{q} = \frac{N}{2} - \sigma.$$

Let us mention that there is in general another condition on p and q. This condition does not appear in Theorem III.2.1 as it is implied by the scaling condition (III.2.2).

Of particular interest is the case $\sigma = 1/2$ in space dimension 3, which was considered by R. Strichartz in his article [**27**], and which is useful to solve the cubic wave equation. We state this inequality and will leave some of the details of the proof to the reader:

THEOREM III.2.4. *Let* u *be defined by* (III.2.1) *with*

$$(u_0, u_1) \in \dot{H}^{1/2}(\mathbb{R}^3) \times \dot{H}^{-1/2}(\mathbb{R}^3), \quad f \in L^{4/3}(\mathbb{R} \times \mathbb{R}^3).$$

Then $u \in L^4(\mathbb{R} \times \mathbb{R}^3)$, $\vec{u} \in C^0\left(\mathbb{R}, \dot{H}^{1/2} \times \dot{H}^{-1/2}(\mathbb{R}^3)\right)$ *and*

$$\sup_{t\in\mathbb{R}} \|\vec{u}(t)\|_{\dot{H}^{\frac{1}{2}}\times \dot{H}^{-\frac{1}{2}}} + \|u\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \leq C\left(\|f\|_{L^{4/3}(\mathbb{R}\times\mathbb{R}^3)} + \|(u_0,u_1)\|_{\dot{H}^{\frac{1}{2}}\times H^{-\frac{1}{2}}}\right).$$

In the sequel of this chapter we will prove Theorem III.2.1 for $p \geq 4$, which will be sufficient for our applications to the nonlinear equations below.

We will use the following notations. If A and B are positive quantities, we will write $A \lesssim B$ when there exists a constant C, independent of the parameters, such that $A \leq CB$, and $A \equiv B$ when $A \lesssim B$ and $B \lesssim A$.

By the energy inequality and Sobolev embedding, we have for all t.

$$\|u(t)\|_{L^6} \lesssim \|u(t)\|_{\dot{H}^1} \lesssim \|(u_0,u_1)\|_{\dot{H}^1\times L^2} + \|f\|_{L^1(\mathbb{R},L^2)},$$

which solves the case $p = \infty$, $q = 6$. Next, we notice that by Hölder inequality, if p and q satisfy (III.2.2) with $p \in (4, \infty)$, we have

$$\|u\|_{L^pL^q} \lesssim \|u\|^{1-\theta}_{L^\infty L^6} \|u\|^{\theta}_{L^4L^{12}} \tag{III.2.4}$$

where $\theta = \frac{4}{p}$. Thus the inequality (III.2.3) for this pair (p, q) will follows from the same equality for $p = 4$, $q = 12$. We are just reduced to prove the estimate (III.2.3) for $p = 4$, $q = 12$. By density, we can assume $(u_0, u_1) \in \left(C_0^\infty(\mathbb{R}^3)\right)^2$, $f \in C_0^\infty(\mathbb{R} \times \mathbb{R}^3)$.

The inequality will follow from the dispersion inequality (Theorem I.5.5) proved in Chapter I.

To deduce the Strichartz inequality from the dispersion inequality a few tools from harmonic analysis are needed. These tools, that include Hardy-Littlewood-Sobolev inequality, dyadic decomposition, Littlewood-Paley theory and interpolation of Lebesgue spaces, are recalled in Section III.3. In Section III.4, we prove the Strichartz inequality for the "half-wave equation", which is an order 1 equation related to the wave equation. Section III.5 is devoted to the end of the proof of Theorem III.2.1.

III.3. Some tools from harmonic analysis

We first recall an interpolation Theorem for a linear operator between L^p space.

Theorem III.3.1 (RieszThorin interpolation Theorem). *Let (X, μ), (Y, ν) be measure spaces. Let*

$$\theta \in]0, 1[, \quad (p_0, p_1, q_0, q_1, p, q) \in [1, \infty]^6$$

with

$$\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}, \quad \frac{1}{q} = \frac{\theta}{q_0} + \frac{1-\theta}{q_1}. \tag{III.3.1}$$

Let A be a linear operator defined on $L^{p_0}(X) + L^{p_1}(X)$ which is bounded from $L^{p_0}(X)$ to $L^{q_0}(Y)$ and from $L^{p_1}(X)$ to $L^{q_1}(Y)$. Then A is a bounded linear operator from $L^p(X)$ to $L^q(Y)$, and

$$\|A\|_{L^p(X)\to L^q(Y)} \leq \|A\|^{\theta}_{L^{p_0}(X)\to L^{q_0}(Y)} \|A\|^{1-\theta}_{L^{p_1}(X)\to L^{q_1}(Y)}.$$

In the theorem, $\|A\|_{E\to F}$ denotes the operator norm of the bounded operator $A : E \to F$, where E and F are Banach spaces.

We next recall Young's inequality for the convolution

Theorem III.3.2. *Let $f \in L^q(\mathbb{R}^N)$, $g \in L^r(\mathbb{R}^N)$ with $1/q + 1/r \geq 1$, and p defined by $\frac{1}{p} + 1 = \frac{1}{q} + \frac{1}{r}$. Then*

$$f * g(x) = \int f(x-y)g(y)dy$$

is defined for almost every $x \in \mathbb{R}^N$ and

$$\|f * g\|_{L^p} \leq \|f\|_{L^q}\|g\|_{L^r}, \tag{III.3.2}$$

Exercice III.2. Prove Young's inequality. Hint: start with the cases $(q, r) = (1, 1)$, $(q, r) = (\infty, 1)$, $(q, r) = (1, \infty)$ and use the interpolation theorem III.3.1.

When $N = 1$ and $\theta \in]0, 1[$, the function $t \mapsto 1/t^\theta$, is not in $L^{1/\theta}$ due to a logarithmic divergence at 0 and ∞. The Hardy-Littlewood-Sobolev inequality says that this function behaves as a $L^{1/\theta}$ function from the point of view of convolution. We will use this inequality in the particular case $\theta = 1/2$, $p = 4/3$, $q = 4$. We refer e.g. to [**2**, Theorem 1.7] for the proof.

Theorem III.3.3 (Hardy Littlewood Sobolev). *Let $\theta \in]0, 1[$, $(p, q) \in]1, \infty[^2$ satisfy*

$$\frac{1}{p} + \theta = 1 + \frac{1}{q}.$$

Let $f \in L^p(\mathbb{R}^N)$. Let, for $t \in \mathbb{R}$,

$$g(t) = \int_{\mathbb{R}} f(s)\frac{1}{|t-s|^\theta}ds. \tag{III.3.3}$$

Then the integral defining g converges for almost every t, and

$$\|g\|_{L^q(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}.$$

We next give a few elements of Littlewood-Paley theory, which is a useful tool to study L^p spaces with $p \neq 2$ by Fourier transformation. What follows is by no mean a complete account on Littlewood-Paley theory: we will just state the needed results, and will give only some of the proofs. We refer to [**2**, Chapter 2] for a complete introduction to the subject.

We start with some inequalities on frequency localized function.

Theorem III.3.4 (Berstein-type estimates). *Let $\psi \in C_0^\infty(\mathbb{R}^N)$. Then if $1 \leq q \leq p \leq \infty$*

$$\forall f \in \mathcal{S}(\mathbb{R}^N), \quad \forall \lambda > 0, \quad \|\psi(\lambda D) f\|_{L^p} \lesssim \lambda^{\left(\frac{N}{p}-\frac{N}{q}\right)} \Big\| f \Big\|_{L^q} \tag{III.3.4}$$

Assume furthermore $\psi(\xi) = 0$ for ξ close to 0. Then, if $s \in \mathbb{R}$ and $p \in [1, \infty]$,

$$\forall f \in \mathcal{S}(\mathbb{R}^N), \quad \forall \lambda > 0, \quad \Big\| |D|^s \psi(\lambda D) f \Big\|_{L^p} \approx \lambda^{-s} \Big\| \psi(\lambda D) f \Big\|_{L^p}. \tag{III.3.5}$$

Moreover, if $s \in \mathbb{N}$,

$$\forall f \in \mathcal{S}(\mathbb{R}^N), \quad \forall \lambda > 0, \quad \sup_{|\alpha|=s} \Big\| \partial_x^\alpha \left(\psi(\lambda D) f\right) \Big\|_{L^p} \approx \lambda^{-s} \Big\| \psi(\lambda D) f \Big\|_{L^p}. \tag{III.3.6}$$

In the theorem, the implicit constants might depend on ψ, but of course not on f and $\lambda > 0$.

Proof. *Step 1.*

We first prove (III.3.4) for $\lambda = 1$. We have

$$\psi(D) u = (\overline{\mathcal{F}}\psi) * u, \tag{III.3.7}$$

where $f * g$ is the convolution of f and g. This is a classical property of the Fourier transform, which can be checked by an explicit computation of $\mathcal{F}(\psi(D)u)$. Note that $\overline{\mathcal{F}}\psi \in \mathcal{S} \subset \bigcap_{1 \leq p \leq \infty} L^p$. Using Young's inequality we obtain that (III.3.4) holds for $\lambda = 1$, i.e. that there exists $C > 0$ such that

$$\forall f \in \mathcal{S}(\mathbb{R}^N), \quad \|\psi(D) f\|_{L^p} \leq \|f\|_{L^q}.$$

Step 2: rescaling. Denote by $T_\lambda u(x) = u(\lambda x)$. By a simple change of variable, one can prove

$$\Psi(D)(T_\lambda u) = T_\lambda \left(\psi(\lambda D) u\right)$$

Thus by Step 1,

$$\|T_\lambda \left(\psi(\lambda D) u\right)\|_{L^p} \lesssim \|T_\lambda u\|_{L^q}.$$

Since $\|T_\lambda f\|_{L^p} = \frac{1}{\lambda^{N/p}} \|f\|_{L^p}$, we obtain (III.3.4) for any $\lambda > 0$.
Step 3: proof of (III.3.5).

Let $\chi \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$, such that $\chi(\xi) = 1$ if $\xi \in \operatorname{supp}(\psi)$. Then

$$|D|^s \psi(\lambda D) u = |D|^s \chi(\lambda D) \psi(\lambda D) u = \frac{1}{\lambda^s} \Xi(\lambda D) \psi(\lambda D) u,$$

where $\Xi(\xi) = |\xi|^s \chi(\xi)$. Using (III.3.4) with $p = q$, we obtain

$$\big\| |D|^s \psi(\lambda D) u \big\|_{L^p} \lesssim \frac{1}{\lambda^s} \big\| \psi(\lambda D) u \big\|_{L^p}. \tag{III.3.8}$$

Using (III.3.8), with s replaced by $-s$ and u replaced by $|D|^s \chi(\lambda D) u$, we obtain

$$\big\| \psi(\lambda D) u \big\|_{L^p} = \big\| |D|^{-s} \psi(\lambda D) |D|^s u \big\|_{L^p} \lesssim \lambda^s \big\| \psi(\lambda D) |D|^s u \big\|_{L^p}.$$

This concludes the proof of (III.3.5).
Step 4: proof of (III.3.6). First, we have

$$\Big\| \psi(\lambda D) \partial_x^\alpha f \Big\|_{L^p} = \Big\| \partial_x^\alpha \chi(\lambda D) \psi(\lambda D) f \Big\|_{L^p} = \frac{1}{|\lambda|^{|\alpha|}} \Big\| \Xi_\alpha(\lambda D) \psi(\lambda D) f \Big\|_{L^p}, \tag{III.3.9}$$

where χ is as above and $\Xi_\alpha(\xi) = (i\xi)^\alpha \chi(\xi)$. The estimate $\lesssim$ in (III.3.6) then follows from (III.3.4) with $q = p$.

Next, if s is even, we have $|D|^s = (-\Delta)^{s/2}$, which shows that (III.3.5) implies the other estimate in (III.3.6).

If s is odd, we write

$$\Big\| \psi(\lambda D) |D|^s f \Big\| = \|\psi(\lambda D) |D|^{s+1} \frac{1}{|D|} f\|_{L^p} \lesssim \sup_{|\alpha|=s+1} \big\| \partial_x^\alpha |D|^{-1} \psi(\lambda D) f \big\|_{L^p}$$

$$\approx \frac{1}{\lambda} \sup_{|\alpha|=s+1} \|\partial_x^\alpha \psi(\lambda D) f\|_{L^p},$$

and we conclude with (III.3.9) that the inequality $\gtrsim$ in (III.3.6) holds in this case also. □

The Littlewood-Paley theory is based on a dyadic decomposition of a distribution $f \in \mathcal{S}'(\mathbb{R}^N)$. We fix once and for all a radial function $\varphi \in C_0^\infty(\mathbb{R}^N)$ with $\varphi(\xi) = 1$ if $|\xi| \leq 1/2$, and $\varphi(x) = 0$ if $|x| \geq 1$. We let

$$\Theta_j(\xi) = \varphi\left(\frac{\xi}{2^{j+1}}\right) - \varphi\left(\frac{\xi}{2^j}\right) = \Theta\left(\frac{\xi}{2^j}\right), \quad \Theta(\xi) = \varphi(\xi/2) - \varphi(\xi).$$

We have

$$\operatorname{supp} \Theta_j \subset \left\{2^{j-1} \leq |\xi| \leq 2^{j+1}\right\}, \quad \sum_{j=-\infty}^{+\infty} \Theta_j(\xi) = 1,\ (\xi \neq 0),$$

where the sum is, for any fixed ξ, a finite sum. We denote

$$\Delta_j f = \Theta_j(D),$$

so that (at least formarly) $f = \sum_{j\in\mathbb{Z}} \Theta_j(D) f$ (*Dyadic decomposition of f in frequencies*). If $f \in \mathcal{S}_0$, it is easy to prove that this sum converges in $\mathcal{S}$.

We have the inequality

$$\forall \xi \neq 0, \quad \frac{1}{2} \leq \sum_{j\in\mathbb{Z}} \Theta_j^2(\xi) \leq 1. \tag{III.3.10}$$

Exercice III.3. Prove (III.3.10). *Hint:* Let

$$A(\xi) = \sum_{j \text{ odd}} \Theta_j(\xi), \quad B(\xi) = \sum_{j \text{ even}} \Theta_j(\xi).$$

Check that if $\xi \neq 0$,

$$A(\xi) + B(\xi) = 1, \quad A^2(\xi) = \sum_{j \text{ odd}} \Theta_j^2(\xi), \quad B^2(\xi) = \sum_{j \text{ even}} \Theta_j^2(\xi).$$

Combining with Plancherel identity, it follows that if $f \in \mathcal{S}(\mathbb{R}^N)$,

$$\|f\|^2_{L^2(\mathbb{R}^N)} \approx \sum_{j\in\mathbb{Z}} \|\Delta_j f\|^2_{L^2(\mathbb{R}^N)}, \tag{III.3.11}$$

and more generally,

$$\|f\|^2_{\dot{H}^s} \approx \sum_{j\in\mathbb{Z}} \|\Delta_j |D|^s f\|^2_{L^2} \approx \sum_{j\in\mathbb{Z}} (2^{2j})^s \|\Delta_j f\|^2_{L^2}. \tag{III.3.12}$$

The situation is more complicated for $p \neq 2$. Nevertheless, we have the following estimates:

Theorem III.3.5. *For all $p \in (1, 2]$, for any $f \in \mathcal{S}$*

$$\sum_{j\in\mathbb{Z}} \|\Delta_j f\|^2_{L^p} \lesssim \|f\|^2_{L^p} \tag{III.3.13}$$

For all $p \in [2, \infty)$, for any $f \in L^p$,

$$\|f\|^2_{L^p} \lesssim \sum_{j\in\mathbb{Z}} \|\Delta_j f\|^2_{L^p}. \tag{III.3.14}$$

We omit the proof refering the interested reader to [**2**, Theorem 2.40].

Exercice III.4. Prove:

- For all $p \in [1, 2]$, for any $f \in \mathcal{S}$

$$\|f\|^p_{L^p} \lesssim \sum_{j\in\mathbb{Z}} \|\Delta_j f\|^p_{L^p} \tag{III.3.15}$$

- For all $p \in [2, \infty]$, for any $f \in L^p$,

$$\sum_{j\in\mathcal{Z}} \|\Delta_j f\|^p_{L^p} \lesssim \|f\|^p_{L^p} \tag{III.3.16}$$

(where the sum has to be interpreted as $\sup_j \|\Delta_j f\|_{L^\infty}$ when $p = \infty$).

Hint: Start with the cases $p = 1$ and $p = 2$ for (III.3.15) and $p = \infty$ and $p = 2$ for (III.3.16), then use an interpolation argument.

The two estimates of Exercise III.4 complete the estimates of Theorem III.3.5. The proofs are simpler than the proof of Theorem III.3.5, but are not detailed here since we will not need these estimates below.

Note that there is no perfect equivalence between the norm $\|f\|_{L^p}$ and a norm defined as a ℓ^q norm of the sequence $\left(\|\Delta_j f\|_{L^p}\right)_j$ if $p \neq 2$.

Let us mention that the quantities

$$\|f\|^q_{\dot{B}^0_{p,q}} = \sum_{j\in\mathbb{Z}} \|\Delta_j f\|^q_{L^p} \tag{III.3.17}$$

appearing in (III.3.13), (III.3.14), (III.3.15) and (III.3.16) defines the norm of the so-called Besov space $\dot{B}^0_{p,q}$. See Sections 2.3, 2.4 and 2.5 of [**2**] for more details on Besov spaces.

III.4. A Strichartz inequality for the half wave equation

It is sometimes useful to decompose the wave equation in two first-order equations in the time-variable. This is particularly the case when dealing with Fourier analysis tools. We thus introduce the half-wave equations

$$\partial_t u + i|D|u = 0, \quad \partial_t u - i|D|u = 0,$$

and their solutions (given in term of Fourier representations) $e^{-it|D|}\varphi$ and $e^{it|D|}\varphi$. Note that the solution to the usual wave equation (LW), (ID) is given by

$$2u(t) = e^{it|D|}u_0 + e^{-it|D|}u_0 + \frac{e^{it|D|}}{i|D|}u_1 - \frac{e^{-it|D|}}{i|D|}u_1$$

Note also that if $v(t) = e^{it|D|}\varphi$, then $e^{-it|D|}u_0 = v(-t)$, thus it is sufficient to consider only the solution $e^{it|D|}\varphi$. The function $e^{it|\xi|}$ is not smooth at $\xi = 0$, so that $e^{it|D|}$ does not map $\mathcal{S}(\mathbb{R}^N)$ to $\mathcal{S}(\mathbb{R}^N)$. However it maps $\mathcal{S}_0(\mathbb{R}^N)$ to $\mathcal{S}_0(\mathbb{R}^N)$ (where as before $\mathcal{S}_0(\mathbb{R}^N)$ is the space of functions φ in $\mathcal{S}(\mathbb{R}^N)$ such that $\hat{\varphi}$ is identically 0 in a neighborhood of the origin).

In this Section, we will prove

Proposition III.4.1. *There exists $C > 0$ such that*

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^N), \quad \left\|\frac{e^{i\cdot|D|}}{|D|}\varphi\right\|_{L^4(\mathbb{R},L^{12})} \lesssim \|\varphi\|_{L^2}, \tag{III.4.1}$$

where as usual $e^{i\cdot|D|}\varphi$ denotes $(t,x) \mapsto \left(e^{it|D|}\varphi\right)(x)$.

Proof. *Step 1: frequency-localized dispersion estimate.*

We will use the Littlewood-Paley decomposition of φ, $\varphi = \sum_{j\in\mathbb{Z}} \Delta_j\varphi$. In this step we prove the following frequency localized version of the dispersion inequality for the wave equation

$$\forall j, \quad \left\|\frac{e^{it|D|}}{|D|}\Delta_j\varphi\right\|_{L^\infty} \lesssim \frac{2^j}{t}\|\Delta_j\varphi\|_{L^1}. \tag{III.4.2}$$

We let $\varphi_j = \Delta_j\varphi$. By the dispersion inequality for the full wave equation and Theorem III.3.4, we have

$$\left\|\frac{\sin(t|D|)}{|D|}\varphi_j\right\|_{L^\infty} \lesssim \frac{1}{|t|}\|\varphi_j\|_{\dot{W}^{1,1}} \approx \frac{2^j}{|t|}\|\varphi_j\|_{L^1}$$

and

$$\left\|\frac{\cos(t|D|)}{|D|}\varphi_j\right\|_{L^\infty} \approx \frac{1}{2^j}\left\|\cos(t|D|)\varphi_j\right\|_{L^\infty} \lesssim \frac{1}{2^j|t|}\left\|\varphi_j\right\|_{\dot{W}^{2,1}} \approx \frac{2^j}{|t|}\|\varphi_j\|_{L^1}.$$

Step 2. A $L^4/L^{4/3}$ dispersion inequality

We next introduce $\widetilde{\Delta}_j f = \Delta_{j-1}f + \Delta_j f + \Delta_{j+1}f$. Noting that $\Theta_{j-1} + \Theta_j + \Theta_{j+1} = 1$ on the support of Θ_j, we see that $\widetilde{\Delta}_j\Delta_j f = \Delta_j f$. For fixed $t > 0$ and j, consider the operator $e^{it|D|}|D|^{-1}\widetilde{\Delta}_j$. By Step 1, it is a bounded operator from L^1 to L^∞, with operator norm $\lesssim 2^j/t$. By Plancherel and Theorem III.3.4, it is bounded from L^2 to L^2 with operator norm $\lesssim 2^{-j}$. Using the interpolation Theorem III.3.1, we obtain that $e^{it|D|}|D|^{-1}\widetilde{\Delta}_j$ is a bounded operator from $L^{4/3}$ to L^4 with operator norm $\lesssim t^{-1/2}$. Using that $\widetilde{\Delta}_j\Delta_j = \Delta_j$, we deduce

$$\left\|e^{it|D|}\frac{1}{|D|}\Delta_j\varphi\right\|_{L^4} \lesssim \frac{1}{|t|^{1/2}}\|\Delta_j\varphi\|_{L^{4/3}}.$$

Taking the square and summing up, we deduce (using Theorem III.3.5)

$$\left\|e^{it|D|}\frac{1}{|D|}\varphi\right\|_{L^4} \lesssim \frac{1}{|t|^{1/2}}\|\varphi\|_{L^{4/3}}. \tag{III.4.3}$$

Step 3. Strichartz inequality.

Next, we consider the operator T defined by

$$(T\varphi)(t,x) = \left(e^{it|D|}|D|^{-1/2}\varphi\right)(x)$$

In this step we prove that T extends to a bounded operator from $L^2(\mathbb{R}^3)$ to $L^4(\mathbb{R}\times\mathbb{R}^3)$, with an operator norm that is independent of j, i.e.

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^3), \quad \left\| e^{it|D|}|D|^{-1/2}\varphi \right\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|\varphi\|_{L^2}. \tag{III.4.4}$$

We will use a so-called TT^* argument to reduce the proof of (III.4.4) to the proof of the boundedness of an operator acting on functions on $\mathbb{R}\times\mathbb{R}^3$.

The inequality (III.4.4) is equivalent to the following statement:

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^3), \quad \forall g \in C_0^\infty(\mathbb{R}\times\mathbb{R}^3), \quad \left| \iint (T\varphi)\overline{g} dx dt \right| \lesssim \|\varphi\|_{L^2(\mathbb{R}^3)} \|g\|_{L^{4/3}(\mathbb{R}\times\mathbb{R}^3)}.$$

Using Plancherel equality in the space variable for every $t\in\mathbb{R}$, we obtain

$$\iint (T\varphi)\overline{g} dx dt = \int \varphi(x)(T^* g)(x) dx,$$

where the (formal) adjoint T^* of T is defined by

$$T^* g(x) = \int_{\mathbb{R}} e^{-it|D|}|D|^{-1/2} g(t) dt.$$

We are thus reduced to prove

$$\forall g \in C_0^\infty(\mathbb{R}\times\mathbb{R}^3), \quad \|T^* g\|_{L^2(\mathbb{R}^3)} \lesssim \|g\|_{L^{4/3}(\mathbb{R}\times\mathbb{R}^3)}. \tag{III.4.5}$$

We have

$$\|T^* g\|_{L^2}^2 = \int_{\mathbb{R}^3} T^* g \overline{T^* g} dx = \iint_{\mathbb{R}\times\mathbb{R}^3} TT^* g \overline{g} dx dt, \tag{III.4.6}$$

and (III.4.5) would follow from the inequality

$$\|TT^* g\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|g\|_{L^{4/3}(\mathbb{R}\times\mathbb{R}^3)}. \tag{III.4.7}$$

We have

$$TT^* g(t,x) = \int_{\mathbb{R}} e^{i(t-s)|D|}|D|^{-1} g(s) ds.$$

Using the $L^4/L^{4/3}$ dispersion inequality of Step 2, we obtain at fixed t,

$$\|(TT^* g)(t)\|_{L^4(\mathbb{R}^3)} \lesssim \int_{\mathbb{R}} \frac{1}{|t-s|^{1/2}} \|g(s)\|_{L^{4/3}(\mathbb{R}^3)} ds$$

By Hardy Littlewood Sobolev inequality, we deduce

$$\|TT^* g\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|g\|_{L^{4/3}(\mathbb{R}\times\mathbb{R}^3)},$$

which yields (III.4.7) and thus concludes the proof of (III.4.4).

Step 4. The L^4L^{12} Strichartz inequality. We next conclude the proof of Proposition III.4.1 by proving that for $\varphi\in\mathcal{S}$,

$$\left\| e^{it|D|}\varphi \right\|_{L^4(\mathbb{R},L^{12}(\mathbb{R}^3))} \lesssim \|\varphi\|_{\dot{H}^1}. \tag{III.4.8}$$

The inequality (III.4.8) follows from (III.4.4) (applied to $|D|\varphi$) and the Sobolev inequality

$$\forall f \in \mathcal{S}, \quad \|f\|_{L^{12}(\mathbb{R}^3)} \lesssim \left\| |D|^{1/2} f \right\|_{L^4(\mathbb{R}^3)}. \tag{III.4.9}$$

To illustrate the tools introduce in the preceding section, we give a proof that does not use (III.4.9), but rather Theorems III.3.4 and III.3.5. By the preceding step, applied to $\Delta_j|D|\varphi$, we have

$$2^j \left\| e^{i\cdot|D|}\Delta_j\varphi \right\|^2_{L^4(\mathbb{R}\times\mathbb{R}^3)} \leq \|\Delta_j|D|\varphi\|^2_{L^2}. \tag{III.4.10}$$

By Theorem III.3.4 (Bernstein inequalities), at fixed t,

$$\|e^{it|D|}\Delta_j\varphi\|_{L^{12}} \lesssim 2^{j/2} \left\| e^{it|D|}\Delta_j\varphi \right\|_{L^4(\mathbb{R}^3)}.$$

Taking the L^4 norm in time, then summing up the squares, we obtain

$$\sum_j \|e^{i\cdot|D|}\Delta_j\varphi\|^2_{L^4L^{12}} \lesssim \sum_j 2^j \left\|e^{it|D|}\Delta_j\varphi\right\|^2_{L^4(\mathbb{R}\times\mathbb{R}^3)} \lesssim \sum_j \|\Delta_j|D|\varphi\|^2_{L^2}, \tag{III.4.11}$$

where we have used (III.4.10) to obtain the last inequality. The right-hand side of (III.4.11) is $\approx \|\varphi\|^2_{\dot{H}^1}$ by Plancherel equality (see (III.3.11)). We must prove that the left-hand side dominates $\left\|e^{it|D|}\varphi\right\|_{L^4L^{12}}$. Let $u = e^{it|D|}\varphi$ and $u_j = \Delta_j u$. By Minkowski inequality (i.e. the triangle inequality for the $L^2(\mathbb{R})$ norm), we see that

$$\sum_{j\in\mathbb{Z}} \|u_j\|^2_{L^4L^{12}} = \sum_{j\in\mathbb{Z}} \left\|\|u_j(t)\|^2_{L^{12}(\mathbb{R}^3)}\right\|_{L^2(\mathbb{R})} \geq \left\|\sum_{j\in\mathbb{Z}} \|u_j(t)\|^2_{L^{12}}\right\|_{L^2(\mathbb{R})}$$

By Theorem III.3.5, at fixed t,

$$\|u(t)\|^2_{L^{12}} \lesssim \sum_{j\in\mathbb{Z}} \|u_j(t)\|^2_{L^{12}}.$$

This shows

$$\sum_{j\in\mathbb{Z}} \|u_j\|^2_{L^4L^{12}} \gtrsim \left\|\|u(t)\|^2_{L^{12}(\mathbb{R}^3)}\right\|_{L^2(\mathbb{R})} = \|u\|^{1/2}_{L^4L^{12}},$$

which together with (III.4.11) concludes the proof of Proposition III.4.1.

□

Remark III.4.2. An alternative, somehow simpler approach is to sum up over j the frequency localized dispersion inequality of Step 2 of the preceding proof. Using Theorem III.3.5, one obtains a $L^4/L^{4/3}$ dispersion inequality for the half-wave equation:

$$\left\|e^{it|D|}|D|^{-1}\varphi\right\|_{L^4} \lesssim \frac{1}{|t|^{1/2}}\|\varphi\|_{L^{4/3}}.$$

It is then possible to forget about frequency cut-off and run the preceding arguments to obtain Strichartz inequalities for the half-wave equation directly.

III.5. Proof of the Strichartz estimate for the full wave equation

We are now ready to prove Theorem III.2.1. We can treat separately the terms

$$u_L(t) = \cos(t|D|)u_0 + \frac{\sin(t|D|)u_1}{|D|}$$

and

$$(Bf)(t) = \int_0^t \frac{\sin\left((t-s)|D|\right)}{|D|} f(s)ds. \tag{III.5.1}$$

Using that $\cos(t|D|) = \frac{1}{2}\left(e^{it|D|} + e^{-it|D|}\right)$, $\sin(t|D|) = \frac{1}{2i}\left(e^{it|D|} - e^{-it|D|}\right)$, we obtain immediately from Proposition III.4.1

$$\|u_L\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|u_0\|_{\dot{H}^1} + \|u_1\|_{L^2}.$$

The other term is more delicate. We first consider

$$u_a(t) = \int_0^\infty \frac{e^{i(t-s)|D|}}{|D|} f(s)ds = e^{it|D|}F, \quad F = \int_0^\infty \frac{e^{-is|D|}}{|D|} f(s)ds$$

and

$$u_b(t) = \int_0^\infty \frac{e^{-i(t-s)|D|}}{|D|} f(s)ds$$

Using that $e^{-is|D|}/|D|$ is a bounded operator from L^2 to $\dot{H}^1$, we obtain that $F \in \dot{H}^1$ with

$$\|F\|_{\dot{H}^1} \lesssim \|f\|_{L^1(\mathbb{R},L^2(\mathbb{R}^3))}.$$

By the Strichartz estimate for the half-wave equation, Proposition III.4.1, we deduce

$$\|u_a\|_{L^4(\mathbb{R},L^{12}(\mathbb{R}^3))} \lesssim \|f\|_{L^1(\mathbb{R},L^2(\mathbb{R}^3))}.$$

Similarly

$$\|u_b\|_{L^4(\mathbb{R},L^{12}(\mathbb{R}^3))} \lesssim \|f\|_{L^1(\mathbb{R},L^2(\mathbb{R}^3))}.$$

Combining, we obtain

$$\|Af\|_{L^4(\mathbb{R},L^{12}(\mathbb{R}^3))} \lesssim \|f\|_{L^1(\mathbb{R},L^2(\mathbb{R}^3))}, \tag{III.5.2}$$

where A is the operator defined by

$$Af(t) = \int_0^\infty \frac{\sin\big((t-s)|D|\big)}{|D|} f(s)ds.$$

Note that Af is analogous to Bf defined above, the only difference between the two being that the integral defining Af is on $[0,\infty)$, whereas the integral defining Bf is on $[0,t[$. An important functional analysis result, due to Michael Christ and Alexander Kiselev [**7**], shows that the boundedness of A implies the boundedness of B. We state this result in a version that was proposed by Christopher Sogge:

LEMMA III.5.1. *Let X and Y be Banach spaces. Let $1 \leq p < q \leq \infty$. Let K a continuous function from $\mathbb{R}^2$ to the space of bounded linear operators from X to Y. Let*

$$(Af)(t) = \int_{-\infty}^{\infty} K(t,\tau)f(\tau)d\tau,$$

and assume that A is a bounded operator from $L^p(\mathbb{R},X)$ to $L^q(\mathbb{R},Y)$, with operator norm C. Define the operator B by

$$(Bf)(t) = \int_{-\infty}^{t} K(t,\tau)f(\tau)d\tau.$$

Then B extends to a bounded operator from $L^p(\mathbb{R},X)$ to $L^q(\mathbb{R},Y)$, with operator norm $\leq \frac{2C\theta^2}{1-\theta}$, where $\theta = 2^{\frac{1}{q}-\frac{1}{p}}$.

Applying Christ and Kiselev Lemma to

$$K(t,\tau) = \mathbb{1}_{\tau>0}\frac{\sin\big((t-\tau)|D|\big)}{|D|}\chi(\varepsilon|D|), \tag{III.5.3}$$

where $\chi \in C_0^\infty(\mathbb{R}^3)$ is equal to 1 close to 0, one obtains

$$\forall \varepsilon > 0, \quad \forall f \in L^1(\mathbb{R},L^2), \quad \|\chi(\varepsilon D)Bf\|_{L^4L^{12}} \lesssim \|f\|_{L^1L^2},$$

where Bf is as in (III.5.1). Letting $\varepsilon \to 0$ we obtain the desired result.

EXERCICE III.5. Justify this last argument.

CHAPTER IV

Cauchy theory for the non-linear equation

In this chapter we will consider the nonlinear wave equation with a power-like nonlinearity

$$\partial_t u^2 - \Delta u = \sigma u^p, \tag{NLW}$$

on $I \times \mathbb{R}^N$, where N is an interval, where the power p is an integer ≥ 2 and σ is nonzero real parameter. Considering the unknown λu instead of u for a suitable choice of $\lambda > 0$, we see that we can assume

$$\sigma \in \{\pm 1\}.$$

We will briefly consider the general case, then restrict to the quintic case $p = 5$ in space dimension 3. We will also comment on the cubic case $p = 3$, in the same space dimension.

IV.1. Scaling invariance. Critical Sobolev space

Let u be a (nonzero) C^2 solution of (NLW) on $(a, b) \times \mathbb{R}^N$, where $a < b$. Let $u_\lambda(t, x) = \lambda^\alpha u(\lambda t, \lambda x)$, where $\lambda > 0$ and $\alpha = \alpha(p, N)$ will be specified later. We have

$$\partial_t^2 u_\lambda - \Delta u_\lambda = \lambda^{\alpha+2-\alpha p} \sigma u_\lambda^p.$$

Thus, if $\alpha = \frac{2}{p-1}$, we see that u_λ is a solution of (NLW) on $\left(\frac{a}{\lambda}, \frac{b}{\lambda}\right) \times \mathbb{R}^N$. We will assume that α has this particular value in the sequel, denoting

$$u_\lambda(t, x) = \lambda^{\frac{2}{p-1}} u(\lambda t, \lambda x).$$

Let

$$\dot{\mathcal{H}}^s = \dot{H}^s(\mathbb{R}^N) \times \dot{H}^{s-1}(\mathbb{R}^N).$$

The critical Sobolev exponent is by definition the unique s such that

$$\|\vec{u}_\lambda(0)\|_{\dot{\mathcal{H}}^{s_c}} = \|\vec{u}(0)\|_{\dot{\mathcal{H}}^{s_c}}.$$

Since by explicit computation

$$\|\vec{u}_\lambda(0)\|_{\dot{\mathcal{H}}^s} = \lambda^{\frac{2}{p-1}+s-N/2} \|\vec{u}(0)\|_{\dot{\mathcal{H}}^s}. \tag{IV.1.1}$$

We see that

$$s_c = \frac{N}{2} - \frac{2}{p-1}.$$

We observe that s_c grows with p, and is always strictly smaller than $N/2$.

Consider a solution u of (NLW) defined on a finite interval $[0, T[$. The corresponding solution u_λ is defined on $[0, T/\lambda[$. Growing λ has the effect of decreasing the time of existence. If $s > s_c$, the $\dot{\mathcal{H}}^s$ norm of $\vec{u}_\lambda(0)$ becomes larger with λ. If $s < s_c$ it becomes smaller. Thus in the case where $s < s_c$, the effect of scaling is to simultaneously decrease the norm of the initial data in $\dot{\mathcal{H}}^s$, $s < s_c$ and shrinking its interval of existence. This is contrary to the intuition that for smaller solutions, the effect of the nonlinearity is weaker, and the solution should behave in a linear way (and in particular has a long time of existence). This leads to an informal conjecture that s_c is a threshold for local well-posedness. It turns out that this conjecture is true for the wave equation: the equation (NLW) is locally well-posed[1] in $\dot{\mathcal{H}}^s$ for $s \geq s_c$, and ill-posed if $\dot{\mathcal{H}}^s$ for $s < s_c$.

We will focus on the quintic case $p = 5$ in space dimension $N = 3$:

$$(\partial_t^2 - \Delta) u = \sigma u^5. \tag{W5}$$

In this case the critical Sobolev case is $\dot{\mathcal{H}}^1$, and the equation is called "energy critical". We will also sometimes consider the cubic equation

$$(\partial_t^2 - \Delta) u = \sigma u^3, \tag{W3}$$

[1] By "well-posed in X", we mean that there is existence and uniqueness of solutions with initial data in X and a reasonable stability theory. We will not give a more rigorous definition of local well-posedness. See e.g. Definition 3.4, Remark 3.5 of T. Tao's book [**29**]

in dimension $1+3$, for which $s_c = 1/2$. As usual, we will take initial data, say at $t = t_0$:

$$(u, \partial_t u)_{t=t_0} = (u_0, u_1). \tag{ID}$$

In all the sequel, we fix $N = 3$.

IV.2. Definition of solutions

As for the linear wave equation, the notion of classical (C^2) solution is too restrictive for the equation (W5), and we will define the following weaker notion of solution, based on Duhamel's formulation of the equation:

Definition IV.2.1. A *finite energy solution* of (W5), (ID) on an interval I with $t_0 \in I$ is a function $u \in L^5_{\mathrm{loc}}(I, L^{10})$ such that $\forall t \in I$,

$$u(t) = \cos\big((t-t_0)|D|\big)u_0 + \frac{\sin\big((t-t_0)|D|\big)}{|D|}u_1 + \int_{t_0}^t \frac{\sin\big((t-s)|D|\big)}{|D|}u^5(s)ds, \tag{IV.2.1}$$

where $(u_0, u_1) \in \dot{\mathcal{H}}^1$.

In the definition, by $u \in L^5_{\mathrm{loc}}(I, L^{10}(\mathbb{R}^3))$, we mean that $u \in L^5(J, L^{10})$ for any compact interval $J \subset I$.

Note that if u is a finite-energy solution in the above sense, one has $u^5 \in L^1_{\mathrm{loc}}(I, L^2(\mathbb{R}^3))$, and thus by energy estimates (see Remark III.2.3),

$$\vec{u} \in C^0(I, \dot{\mathcal{H}}^1).$$

Also, by Chapter II, u satisfies the equation (W5) in the sense of distribution on $I \times \mathbb{R}^3$.

The solutions given by the Duhamel formula as in Definition IV.2.1 are called "strong" solutions in the book of Terence Tao [**29**], by opposition to the weaker notion of distributional solutions (that do not impose continuity in time) and the stronger notion of classical solutions (that are C^2 and satisfy the equation in a classical sense). Note however that this terminology is not universal. For example the solutions of Definition IV.2.1 are called … "weak" solutions in the book [**26**] of Christopher Sogge.

We refer to Section 3.2 of [**29**] "What is a solution?", for a discussion on different types of solutions.

In the sequel, by "solution to (W5)" we will always mean (unless specified otherwise) a solution in the sense of Definition IV.2.1.

Exercice IV.1. Check that the definition of finite energy solutions above does not depend on the choice of the initial time. In other words, if u is a solution of (W5) on I and $t_1 \in I$, then for all $t \in I$,

$$u(t) = \cos\big((t-t_1)|D|\big)u(t_1) + \frac{\sin\big((t-t_1)|D|\big)}{|D|}\partial_t u(t_1) + \int_{t_1}^t \frac{\sin\big((t-s)|D|\big)}{|D|}u^5(s)ds.$$

IV.3. Existence and uniqueness

3.a. A local statement. We introduce the following notations:

$$S_L(t)\vec{u}_0 = \cos(t|D|)\,u_0 + \frac{\sin(t|D|)}{|D|}u_1, \quad \vec{S}_L(t)\vec{u}_0 = \left(S_L(t)\vec{u}_0, \partial_t S_L(t)\vec{u}_0\right),$$

where $\vec{u}_0 = (u_0, u_1)$. We start with the following local statement:

Theorem IV.3.1. *There exists $\delta_0 > 0$ with the following property. Let I be an interval with $t_0 \in I$. Let $\vec{u}_0 \in \dot{\mathcal{H}}^1$. Assume*

$$\|S_L(\cdot - t_0)\vec{u}_0\|_{L^5(I, L^{10})} = \delta \leq \delta_0. \tag{IV.3.1}$$

Then there exists a unique solution u of (W5), (ID) *on I. Furthermore*

$$\sup_{t\in I}\left\|\vec{u}(t) - \vec{S}_L(t-t_0)\vec{u}_0\right\|_{\dot{\mathcal{H}}^1} + \|u - S_L(\cdot - t_0)\vec{u}_0\|_{L^5(I,L^{10})} \lesssim \delta^5. \tag{IV.3.2}$$

In the Theorem, $S_L(\cdot - t_0)\vec{u}_0$ denotes the map $t \mapsto S_L(t-t_0)\vec{u}_0$.

Theorem IV.3.1 has two important consequences:

Local well-posedness: Note that $(5, 10)$ is a $\dot{\mathcal{H}}^1$-admissible couple in dimension 3 (it satisfies (III.2.2)). By Theorem III.2.1, if $\vec{u}_0 \in \dot{\mathcal{H}}^1$, then $S_L(\cdot)\vec{u}_0 \in L^5(\mathbb{R}, L^{10}(\mathbb{R}^3))$. Thus if $T > 0$ is small enough, then

$$\|\vec{u}_0\|_{L^5([-T,+T],L^{10})} \leq \delta_0,$$

and Theorem IV.3.1 implies that there exists a solution to (W5), (ID) on $[-T, +T]$.

Small data global well-posedness: If $\vec{u}_0 \in \dot{\mathcal{H}}^1$ and $\|u_0\|_{\dot{H}^1} \leq \delta_0/C_S$, where C_S is the constant in the Strichartz inequality (III.2.3) with $p=5$, $q=10$, then $\|S_L(\cdot)\vec{u}_0\|_{L^5(\mathbb{R},L^{10})} \leq \delta_0$, and one can use Theorem IV.3.1 with $I=\mathbb{R}$. This shows that the corresponding solution u is globally defined, and that $u \in L^5(\mathbb{R}, L^{10})$.

Proof of Theorem IV.3.1. Assume without generality that $t_0 = 0$. We use the Banach fixed point theorem, proving that the operator A, defined by

$$Av\,(t) = S_L(t)\vec{u}_0 + Bv\,(t), \quad Bv\,(t) = \int_0^t \frac{\sin\big((t-s)|D|\big)}{|D|} v^5(s)ds, \tag{IV.3.3}$$

is a contraction on X defined by

$$X = \left\{ v \in L^5(I, L^{10}),\ \|v\|_{L^5(I,L^{10})} \leq 2\delta_0 \right\}.$$

We first prove that A maps X into X. Indeed, If $v \in X$, then by Theorem III.2.1 (see Remark III.2.2),

$$\|Bv(t)\|_{L^5(I,L^{10})} \leq C_S \left\|v^5\right\|_{L^1(I,L^2)} \leq C_S \|v\|^5_{L^5(I,L^{10})} \leq C_S\delta_0^5 \leq \delta_0,$$

assuming $\delta_0 \leq C_S^{-1/4}$. Thus $Av \in X$.

We next prove that A is a contraction on X. Let $v, w \in X$. Using $w^5 - v^5 = (w-v)(w^4 + w^3v + w^2v^2 + wv^3 + v^4)$ and Young's inequality $ab \leq a^p/p + b^q/q$, $1/p + 1/q = 1$, one obtains

$$|v^5 - w^5| \leq \frac{5}{2}\,|v-w|\,(v^4 + w^4).$$

Combining with Hölder's inequality, we obtain

$$\left\|v^5 - w^5\right\|_{L^1(I,L^2)} \leq \frac{5}{2}\|v-w\|_{L^5(I,L^{10})}\left(\|v\|^4_{L^5(I,L^{10})} + \|w\|^4_{L^5(I,L^{10})}\right). \tag{IV.3.4}$$

By Strichartz estimates

$$\|Av - Aw\|_{L^5(I,L^{10})} = \|Bv - Bw\|_{L^5(I,L^{10})} \leq C_S\left\|v^5 - w^5\right\|_{L^1(I,L^2)} \leq 5C_S\|v-w\|_{L^5(I,L^{10})}\delta_0^4.$$

If δ_0 is small enough ($\delta_0 = (10C_S)^{-1/4}$ works), one has

$$\|Av - Aw\|_{L^5(I,L^{10})} \leq \frac{1}{2}\|v-w\|_{L^5(I,L^{10})}.$$

This shows that A is a contraction on X.

Let u be the only fixed point of A in X. Since $u = Au$ and $u \in L^5(I, L^{10})$ we see that u is a solution of (W5) on I.[2] Using

$$u - S_L(\cdot)\vec{u}_0 = Bu,$$

and $\|Bu\|_{L^5(I,L^{10})} \leq \delta^5$, and Strichartz inequality, we obtain (IV.3.2). It remains to prove the uniqueness statement. From the contraction argument, we see that u is the unique solution of (W5) such that $\|u\|_{L^5(I,L^{10})} \leq \delta_0$. We prove a stronger statement, Lemma IV.3.2 below, that will conclude the proof. □

Lemma IV.3.2. *Let u, v be two solutions of* (W5) *on an interval I with $t_0 \in I$. Assume $\vec{u}(t_0) = \vec{v}(t_0)$. Then $u = v$.*

Proof. Assume again $t_0 = 0$ to simplify notations. Let $\delta_0 > 0$ be as in Theorem IV.3.1. We let $K = [a,b]$ be a compact subinterval of I such that $t_0 \in K$. We will prove that $u(t) = v(t)$ for $t \in K$. Since K is compact, we have by Definition IV.2.1,

$$u \in L^5(K, L^{10}), \quad v \in L^5(K, L^{10}).$$

We can thus divide K into p subintervals $[\tau_j, \tau_{j+1}]$, $0 \leq j \leq p-1$, with $\tau_0 < \tau_1 < \ldots < \tau_p$, such that

$$\forall j \in \{0, \ldots, J-1\}, \quad \max\left(\|u\|_{L^5([\tau_j,\tau_{j+1}],L^{10})}, \|v\|_{L^5([\tau_j,\tau_{j+1}],L^{10})}\right) \leq \delta_0.$$

Let j_0 be an index such that $0 \in [\tau_{j_0}, \tau_{j_0+1}]$. By the proof of Theorem III.2.1, with $I = [\tau_{j_0}, \tau_{j_0+1}]$, noting that u and v are in X, we obtain $u(t) = v(t)$ for $t \in [\tau_{j_0}, \tau_{j_0+1}]$. This implies

$$\vec{u}(\tau_{j_0}) = \vec{v}(\tau_{j_0}) \text{ and } \vec{u}(\tau_{j_0+1}) = \vec{v}(\tau_{j_0+1}).$$

We can then iterate the preceding arguments on the intervals $[\tau_j, \tau_{j+1}]$, $j = j_0+1$, $j = j_0+2$ until $j = J-1$, and $j = j_0-1$, $j = j_0-2$ until $j = 0$ to obtain that $u(t) = v(t)$ for $t \in K$, concluding the proof. □

[2] Recall that "solution" is to be taken in the sense of Definition IV.2.1.

3.b. Maximal solution. Using the above local existence theorem, we can now glue the solutions above to construct a maximal solution of (W5).

Corollary IV.3.3. *Let $\vec{u}_0 \in \dot{\mathcal{H}}^1$ and $t_0 \in \mathbb{R}$. Then there is a unique maximal solution of* (W5), (ID). *Denoting by $I_{\max} = (T_-, T_+)$ its interval of existence, we have the following* blow-up criteria*:*

$$T_+ < \infty \Longrightarrow u \notin L^5([t_0, T_+[, L^{10}), \quad T_- > -\infty \Longrightarrow u \notin L^5(]T_-, t_0], L^{10}). \tag{IV.3.5}$$

The phrase "maximal solution" in the theorem means that if v is another solution of (W5), (ID) defined on an interval I with $t_0 \in I$, then $I \subset I_{\max}$ and $u(t) = v(t)$ for all $t \in I$.

Proof. Let $\mathcal{J}$ be the set of all open intervals I such that $t_0 \in I$, and there exists a solution v of (W5), (ID) on I. Let

$$I_{\max} = \bigcup_{I \in \mathcal{H}} I.$$

By Theorem IV.3.1, $\mathcal{J}$ is nonempty. Thus $I_{\max}$ is an open interval containing t_0. If $t \in I_{\max}$, there exists an interval I and a solution v of (W5), (ID) on I. By the uniqueness Lemma IV.3.2, the value $v(t)$ does not depend on the choice of I. We denote by $u(t)$ this common value. Let K be a compact subinterval of $I_{\max}$. We next prove:

$$u \in L^5(K, L^{10}). \tag{IV.3.6}$$

Indeed, for all $t \in K$, there exist an open interval $I \in \mathcal{J}$ such that $t \in I$ and u is a solution of (W5) on I. This implies in particular that $u \in L^5([t - \varepsilon, t + \varepsilon], L^{10})$ if $\varepsilon = \varepsilon(t)$ is small enough. Using the compactness of K, we can cover K by a finite numbers of interval $]t - \varepsilon(t), t + \varepsilon(t)[$, and thus we obtain (IV.3.6).

If $t \in I_{\max}$, by the definition of $I_{\max}$ and the uniqueness Lemma IV.3.2, we have that

$$u(t) = S_L(t)\vec{u}_0 + \int_0^t \frac{\sin\big((t-s)|D|\big)}{|D|} u^5(s)ds,$$

which concludes the proof that u is a solution of (W5), (ID) on $I_{\max}$. The maximality of u is a direct consequence of the definition of $I_{\max}$ and Lemma IV.3.2. □

Let us mention that it is not possible to improve the blow-up criterion to

$$T_+ < \infty \Longrightarrow \limsup_{t \to \infty} \|\vec{u}(t)\|_{\dot{\mathcal{H}}^1} = +\infty.$$

Indeed, it was proved by Krieger, Schlag and Tataru [**24**] that there exist solutions of (W5) with $\sigma = 1$, with finite time of existence T_+ and such that

$$\limsup_{t \to T_+} \|\vec{u}(t)\|_{\dot{\mathcal{H}}^1} < \infty.$$

Exercice IV.2. Consider the cubic nonlinear wave equation (W3), (ID) with initial data (u_0, u_1) in the critical space $\dot{\mathcal{H}}^{1/2}$, in space dimension 3. Define a concept of "solution" for this equation analogous to the one of Definition IV.2.1. Prove the analogs of Theorem IV.3.1 and Corollary IV.3.3. *Hint*: use the $L^4(I \times \mathbb{R}^3)$ norm instead of the $L^5(I, L^{10}(\mathbb{R}^3))$ norm, and the Strichartz inequality of Theorem III.2.4.

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