DYNAMICS OF SEMI-LINEAR WAVE EQUATIONS. 2023-2024. FINAL EXAM

Exercice 1.

1.1. Let $t > 0$. Give the smallest subset $A(t)$ of $\mathbb{R}^2$ with the following property. If u is a solution of the linear wave equation (LW), (ID) in space dimension 2 such that $\operatorname{supp}(u_0, u_1) \subset \{x \in \mathbb{R}^2,\ |x| \leq 1\}$. Then $\operatorname{supp} u(t) \subset A(t)$.

1.2. Let $t > 0$. Give the smallest subset $B(t)$ of $\mathbb{R}^3$ with the following property. If u is a solution of the linear wave equation (LW), (ID) in space dimension 3, such that $\operatorname{supp}(u_0, u_1) \subset \{x \in \mathbb{R}^3,\ |x| \leq 1\}$. Then $\operatorname{supp} u(t) \subset B(t)$.

Exercice 2.

2.1. Let u be a solution of

$$\text{(LW)} \qquad \partial_t^2 u - \Delta u = 0,$$

$$\text{(ID)} \qquad \vec{u}_{\restriction t=0} = (u_0, u_1)$$

with $(t, x) \in \mathbb{R} \times \mathbb{R}^3$, such that $(u_0, u_1) \in \left(C_0^\infty(\mathbb{R}^3)\right)^2$. Prove that $\lim_{t\to\infty} \|u(t)\|_{L^6(\mathbb{R}^3)} = 0$.

2.2. Let u be a solution of (LW), (ID) with $(u_0, u_1) = \dot{\mathcal{H}}^1(\mathbb{R}^3) = \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. Prove again $\lim_{t\to\infty} \|u(t)\|_{L^6(\mathbb{R}^3)} = 0$.

2.3. We fix $\sigma \in \{\pm 1\}$ and consider a solution of

$$\text{(W5)} \qquad \partial_t^2 u - \Delta u = \sigma u^5,$$

where $x \in \mathbb{R}^3$, with initial data as in (ID) such that $(u_0, u_1) \in \dot{\mathcal{H}}^1(\mathbb{R}^3)$. Assume that u scatters to a linear solution forward in time. Prove that $\lim_{t\to\infty} \|u(t)\|_{L^6} = 0$.

Until the end of this exercise, we consider a solution u of (W5), (ID) such that $(u_0, u_1) \in \dot{\mathcal{H}}^1(\mathbb{R}^3)$, with maximal time of existence T_+ and such that

$$(1) \qquad \lim_{t\to T_+} \|u(t)\|_{L^6} = 0.$$

2.4. Prove that

$$\sup_{t\in[0,T_+[} \|\vec{u}(t)\|_{\dot{\mathcal{H}}^1} < \infty.$$

2.5. Let $\varepsilon > 0$. Prove that there exists $T \in [0, T_+[$ such that

$$\forall t \in (T, T_+), \quad \|u\|_{L^4([T,t],L^{12})} \leq C + \varepsilon \|u\|^4_{L^4([T,t],L^{12})}.$$

where $L^{12} = L^{12}(\mathbb{R}^3)$ and the constant $C > 0$ depends on u, but is of course independent of t and ε.

2.6. Prove that $u \in L^4([0, T_+[, L^{12})$.

2.7. Prove that u scatters in the future to a linear solution.

Exercice 3. In this exercise we use the notations of Chapter III of the course, and in particular the notation Δ_j defined in Section III.4. One can of course use the results of Section III.4 without proof. The space variable is in $\mathbb{R}^3$ in all the exercise. We will use the notations $L^b = L^b(\mathbb{R}^3)$ and $L^a L^b = L^a(\mathbb{R}, L^b(\mathbb{R}^3))$.

3.1. Let $p \in (2, \infty)$ and $j \in \mathbb{Z}$. Prove that

$$\forall f \in \mathcal{S}_0, \quad \left\| e^{it|D|} \Delta_j f \right\|_{L^p} \leq C(t, p, j) \|f\|_{L^{p'}},$$

where the dependence of the constant $C(t, p, j)$ on p, t and j should be made explicit.

3.2. Prove
$$\forall f \in \mathcal{S}_0, \quad \left\| e^{it|D|}\Delta_j f \right\|_{L^p} \leq 10C(t,p,j)\|\Delta_j f\|_{L^{p'}}.$$

3.3. Prove
$$\forall f \in \mathcal{S}_0, \quad \left\| e^{it|D|} f \right\|_{L^p} \lesssim \frac{1}{|t|^{1-\frac{2}{p}}} \left\| |D|^{2-\frac{4}{p}} f \right\|_{L^{p'}}.$$

3.4. Let f be a function of space and time and
$$u(t) = \int_0^t \frac{\sin((t-s)|D|)}{|D|} f(s)ds.$$
Prove (for a parameter $a \in (1,\infty)$ depending on p to be specified) that if $f \in L^{a'}L^{p'}$ then $u \in L^a L^p$ and
$$\|u\|_{L^a L^p(\mathbb{R}^3)} \lesssim \left\| |D|^{1-\frac{4}{p}} f \right\|_{L^{a'}L^{p'}}.$$

3.5. Let now $u(t) = e^{it|D|}|D|^{-\gamma}u_0$, where $u_0 \in L^2(\mathbb{R}^3)$, and γ is a real parameter. Prove that for a good choice of γ (depending on p and to be explicited), one has $u \in L^a L^p$ and
$$\|u\|_{L^a L^p} \lesssim \|u_0\|_{L^2},$$
where a is as in the preceding question.

3.6. Let $u(t) = \cos(t|D|)u_0 + \frac{\sin(t|D|)}{|D|}u_1$, where $(u_0,u_1) \in \dot{\mathcal{H}}^\gamma$. Prove that $u \in L^a L^p$ and
$$\|u\|_{L^a L^p} \lesssim \|(u_0,u_1)\|_{\dot{\mathcal{H}}^\gamma},$$
where a and γ are as in the preceding questions.

Exercice 4. We fix $N \geq 3$ and denote by $\dot{\mathcal{H}}^1_R$ the subspace of $(\dot{H}^1 \times L^2)(\{x \in \mathbb{R}^N,\ |x| > R\})$ made up of radial functions. This is a Hilbert space, with the norm $\|\cdot\|_R$ defined by:
$$\|(u_0,u_1)\|_R^2 = \int_R^\infty \left((\partial_r u_0(r))^2 + u_1^2(r)\right) r^{N-1}dr.$$

4.1. At what condition on $a \in \mathbb{R}$ is the function $r \mapsto (r^{-a},0)$ an element of $\dot{\mathcal{H}}^1_R$? Same question for the function $r \mapsto (0, r^{-a})$.

4.2. Assume that $N = 3$. Let V_R be the subvector space of $\dot{\mathcal{H}}^1_R$ spanned by $(r^{-1},0)$. Let u be a radial solution of (LW) with initial data $(u_0,u_1) \in \dot{\mathcal{H}}^1$. Prove that
$$\lim_{t\to\pm\infty} \int_R^\infty \left((\partial_r u(t,r))^2 + (\partial_t u(t,r))^2\right) r^2 dt = c \left\|\Pi^\perp_{V_R}(u_0,u_1)\right\|_R^2,$$
where $\Pi^\perp_{V_R}$ is the orthogonal projection, in $\dot{\mathcal{H}}^1_R$, on the orthogonal of V_R, and $c > 0$ is a constant to be specified.

In the three next questions, we assume $N = 5$.

4.3. Let u be a solution of (LW) with initial data (u_0,u_1).
$$v(t,r) = \int_r^\infty \rho\partial_t u(t,\rho)d\rho.$$
Prove that v is well defined for $r > R + |t|$, and that it is a solution of (LW) in space dimension 3 for $r > R + |t|$.

4.4. Let V_R be the subspace of $\dot{\mathcal{H}}^1_R(\mathbb{R}^5)$ spanned by $(1/r^3,0)$ and $(0,1/r^3)$. What is the value of $\Delta(r^{-3})$ for $r > 0$? Using the preceding questions, prove that
$$\lim_{t\to\pm\infty} \int_R^\infty \left((\partial_r u(t,r))^2 + (\partial_t u(t,r))^2\right) r^4 dt = c \left\|\Pi^\perp_{V_R}(u_0,u_1)\right\|_R^2.$$

4.5. What are the R-nonradiative solutions of (LW) in space dimension 5?

4.6. Try to generalize to higher odd dimensions.