DYNAMICS OF SEMI-LINEAR WAVE EQUATIONS. 2024-2025. FINAL EXAM

The notations are those used in the course. The course handout is allowed. All results contained in the course may be used. "Solution" will always mean finite-energy solution, in the sense of Chapters 4 and 5 of the course.

We denote by $\mathbb{1}_A$ the characteristic function of a set A: $\mathbb{1}_A(X) = 1$ if $X \in A$, $\mathbb{1}_A(X) = 0$ if $X \notin A$. The space variable is always $x \in \mathbb{R}^3$. We use the notations $L^b = L^b(\mathbb{R}^3)$ and $L^aL^b = L^a(\mathbb{R}, L^b(\mathbb{R}^3))$.

Problem 1.

1.1. Let $a = \frac{56}{13}$, $b = \frac{56}{5}$. Check that for all interval I, for all $T \in I$, for all u, f such that

$$\partial_t^2 u - \Delta u = f \in L^1(I, L^2), \quad \vec{u}(T) = (u, \partial_t u)(T) \in \dot{\mathcal{H}}^1 = \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3),$$

one has

$$\|u\|_{L^{14}(I,L^7(\mathbb{R}^3))} + \|u\|_{L^a(I,L^b(\mathbb{R}^3))} \leq C\left(\|\vec{u}(T)\|_{\dot{\mathcal{H}}^1} + \|f\|_{L^1(I,L^2(\mathbb{R}^3))}\right).$$

Let u be a solution of

$$\partial_t^2 u - \Delta u = -u^5, \quad (u, \partial_t u)(0) = (u_0, u_1) \in \dot{\mathcal{H}}^1 = \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3). \tag{1}$$

Let E be the conserved energy of u, and T_+ the maximal time of existence of u. We assume $u \in L^{14}([0, T_+), L^7)$. We want to prove that u scatters to a linear solution.

1.2. We fix $T_0 \in [0, T_+)$ such that $\|u\|_{L^{14}([T_0,T_+),L^7)} \leq \varepsilon$, where $\varepsilon > 0$ is a small constant (depending on E) to be specified later. Prove that there exists $C > 0$ (independent of u) such that

$$\forall T \in [T_0, T_+), \quad \|u\|_{L^a([T_0,T),L^b)} \leq C\sqrt{E} + C\varepsilon\|u\|^4_{L^a([T_0,T),L^b)}.$$

1.3. Specifying $\varepsilon > 0$, prove that $u \in L^a([0, T_+), L^b)$ and $\|u\|_{L^a([0,T_+),L^b)} \leq 2C\sqrt{E}$, where C is the constant above.

1.4. Prove that $T_+ = +\infty$ and that there exists $(v_0, v_1) \in \dot{\mathcal{H}}^1$ such that

$$\lim_{t\to\infty} \left\|\vec{u}(t) - \vec{S}_L(t)(v_0, v_1)\right\|_{\dot{\mathcal{H}}^1} = 0. \tag{2}$$

1.5. Conversely, prove that for every solution u of (1), if $T_+(u) = +\infty$ and there exists $(v_0, v_1) \in \dot{\mathcal{H}}^1$ satisfying (2), then $u \in L^{14}([0, +\infty), L^7)$.

Problem 2. In this exercise we use the notations of Chapter III of the course, and in particular the notation Δ_j defined in Section III.4. One can of course use the results of Section III.4 without proof.

2.1. Let $p \in (2, \infty)$ and $j \in \mathbb{Z}$. Prove that

$$\forall f \in \mathcal{S}_0, \quad \left\|e^{it|D|}\Delta_j f\right\|_{L^p} \leq C(t, p, j)\|f\|_{L^{p'}}, \tag{3}$$

where the dependence of the constant $C(t, p, j)$ on p, t and j should be made explicit.

Let $T_j = e^{it|D|}\Delta_j$, considered as an operator from L^2 to $C^0(\mathbb{R}, L^2)$. In the sequel, we fix $q > 2$ and $r > 2$ such that $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$. We want to prove that T_j extends to an operator from L^2 to L^qL^r such that

$$\forall f \in L^2(\mathbb{R}^3), \quad \|T_j f\|_{L^qL^r} \leq \overline{C}\|f\|_{L^2}, \tag{4}$$

for some $\overline{C} = \overline{C}(j, r) > 0$.

2.2. Let T_j^* be the formal adjoint of T_j and $g \in C_c^\infty(\mathbb{R}^4)$. Compute $T_j^* g$.

2.3. Justify that it is sufficient to prove the boundedness of $T_jT_j^*$ as an operator from $\mathcal{X}$ to $\mathcal{Y}$, where the Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ should be specified.

2.4. Use (3) and the preceding question to prove that (4) holds. The dependence of the constant $\overline{C}$ with respect to r and j should be made explicit.

2.5. Prove

$$\forall f \in L^2(\mathbb{R}^3), \quad \|T_j f\|_{L^q L^r} \leq 10\overline{C}\|\Delta_j f\|_{L^2}, \tag{5}$$

2.6. Prove that there exists a constant $C > 0$ such that

$$\|e^{it|D|} f\|_{L^q L^r} \leq C\|f\|_{\dot{H}^s},$$

where $s = s(r)$ should be specified.

2.7. Deduce a Strichartz estimate for the wave equation $\partial_t^2 u - \Delta u = 0$.

Problem 3. We denote by $B_R(x_0)$ the open ball in $\mathbb{R}^3$ with center $x_0 \in \mathbb{R}^3$ and radius $R > 0$, and ${}^c B_R(x_0) = \mathbb{R}^3 \setminus B_R(x_0)$. We admit the following extension result: there exists a constant $M > 0$ such that for all $f \in \dot{H}^1(\mathbb{R}^3)$, there exist $\underline{f}$ and $\widetilde{f}$ in $\dot{H}^1(\mathbb{R}^3)$ such that

$$\forall x \in B_1(0),\ \underline{f}(x) = f(x) \quad \text{and} \quad \|\underline{f}\|_{\dot{H}^1(\mathbb{R}^3)} \leq M\left[\left\|\mathbb{1}_{B_1(0)}|\nabla f|\right\|_{L^2} + \|\mathbb{1}_{B_1(0)} f\|_{L^6}\right]$$

$$\forall x \in {}^c B_1(0),\ \widetilde{f}(x) = f(x) \quad \text{and} \quad \|\widetilde{f}\|_{\dot{H}^1(\mathbb{R}^3)} \leq M\left\|\mathbb{1}_{{}^c B_1(0)}|\nabla f(x)|\right\|_{L^2}.$$

3.1. Show that the previous extension property remains valid when replacing $B_1(0)$ with an arbitrary ball $B_R(x_0)$ in $\mathbb{R}^3$.

3.2. Show that there exist $\delta_0 > 0$ and $C_0 > 0$ such that for all $(u_0, u_1) \in \dot{\mathcal{H}}^1$, for all $x_0 \in \mathbb{R}^3$, $R > 0$ such that

$$\left\|\mathbb{1}_{B_R(x_0)}|\nabla u_0|\right\|_{L^2} + \|\mathbb{1}_{B_R(x_0)} u_0\|_{L^6} + \left\|\mathbb{1}_{B_R(x_0)} u_1\right\|_{L^2} = \varepsilon \leq \delta_0,$$

one has $\left\|\mathbb{1}_{\Gamma_R(x_0)} u\right\|_{L^5 L^{10}} \leq C_0 \varepsilon$, where u is the maximal solution of

$$\partial_t^2 u - \Delta u = u^5, \quad (u, \partial_t u)(0) = (u_0, u_1) \in \dot{\mathcal{H}}^1, \tag{6}$$

$I_{\max}$ its interval of existence, and $\Gamma_R(x_0) = \{(t,x) \in I_{\max} \times \mathbb{R}^3 \ :\ |x - x_0| \leq R - |t|\}$.

3.3. Similarly, show that there exist $\delta_1 > 0$, $C_1 > 0$ such that for all $(u_0, u_1) \in \dot{\mathcal{H}}^1$, for all $x_0 \in \mathbb{R}^3$, $R > 0$ such that $\left\|\mathbb{1}_{{}^c B_R(x_0)}|\nabla u_0|\right\|_{L^2} + \left\|\mathbb{1}_{{}^c B_R(x_0)} u_1\right\|_{L^2} = \varepsilon \leq \delta_1$, one has $\left\|\mathbb{1}_{\widetilde{\Gamma}_R(x_0)} u\right\|_{L^5(\mathbb{R}, L^{10}(\mathbb{R}^3))} \leq C_1 \varepsilon$, where $\widetilde{\Gamma}_R(x_0) = \{(t,x) \in I_{\max} \times \mathbb{R}^3 \ :\ |x - x_0| > R + |t|\}$.

3.4. Show that if u is a solution of (6), and $R > 0$ is such that

$$\sup_{x_0 \in \mathbb{R}^3} \left\|\mathbb{1}_{B_R(x_0)}|\nabla u_0|\right\|_{L^2} + \|\mathbb{1}_{B_R(x_0)} u_0\|_{L^6} + \left\|\mathbb{1}_{B_R(x_0)} u_1\right\|_{L^2} \leq \delta_0,$$

then $]-R, +R[\subset I_{\max}$.

Problem 4. Let u, v be two global solutions of (6), with radial initial data respectively (u_0, u_1), (v_0, v_1). We assume that for some $R_0 > 0$

$$\lim_{t \to \pm\infty} \int_{|x| > R_0 + |t|} |\nabla_{t,x}(u - v)(t,x)|^2 dx = 0 \quad \text{and} \quad \forall |x| > R_0,\ |u_0(x) - v_0(x)| \leq \frac{C}{|x|^2}.$$

Our goal is to prove that $(u_0, u_1) = (v_0, v_1)$.

Let $w = u - v$, $(w_0, w_1) = \vec{w}(0)$. Let $R_1 \geq R_0$ such that for $\mathfrak{u} \in \{u, v, w\}$, one has

$$\int_{|x| \geq R_1} \left[|\nabla \mathfrak{u}_0(x)|^2 + (\mathfrak{u}_1(x))^2\right] dx \leq \varepsilon^2,$$

where ε is a small constant.

4.1. Prove that for $\mathfrak{u} \in \{u, v, w\}$, one has

$$\left\|\mathbb{1}_{|x| > R_1 + |t|} \mathfrak{u}\right\|_{L^5 L^{10}} \leq C\varepsilon.$$

4.2. Prove that there exists $R > R_1$ such that $(w_0, w_1)(x) = 0$ for a.e x such that $|x| \geq R$.

4.3. Prove that $(w_0, w_1) = 0$ for a.e. x.