DYNAMICS OF SEMI-LINEAR WAVE EQUATIONS. HOMEWORK, CHAPTER I

In all this exercise sheet we will consider the linear wave equation:

$$\partial_t^2 u - \Delta u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N. \tag{LW}$$

We will also consider the Cauchy problem associated to (LW), with initial data at $t = 0$:

$$\vec{u}_{\restriction t=0} = (u_0, u_1) \tag{ID}$$

Exercise 1. Let $f : \mathbb{R}^N \to \mathbb{R}$ ($N \geq 1$). Suppose f is radial (i.e. That it depends only on the variable $r = |x| = \sqrt{x_1^2 + x_2^2 + \ldots + x_N^2}$). Denote $f(x) = g(|x|)$, where $g : [0, \infty[\to \mathbb{R}$.

(1) Show that f is continuous on $\mathbb{R}^N$ if and only if g is continuous on $[0, \infty[$.
(2) Show that f is C^1 on $\mathbb{R}^N$ if and only if g is C^1 on $[0, \infty[$ and $g'(0) = 0$.
(3) Show that for any $k \geq 2$, f is C^k on $\mathbb{R}^N$ if and only if g is C^k on $\mathbb{R}^N$ and $g^{(j)}(0) = 0$ for all odd integers $j \leq k$.
(4) Assuming f is C^1, compute $\frac{\partial f}{\partial x_j}$ in terms of g', $j = 1, \ldots, N$. Compute $g'(r)$ in terms of ∇f.
(5) Assuming f is C^2 on $\mathbb{R}^N$, prove the formula

$$\Delta f(x) = g''(|x|) + \frac{N-1}{|x|} g'(|x|).$$

To lighten notation, we use the same notation (f) for functions f and g, and denote $g' = \frac{df}{dr}$, etc...

Exercise 2 (Explicit solutions of the radial wave equation in odd space dimension). Let $N \geq 3$ be an odd integer, written as $N = 2k + 1$. Let T_k be the operator defined by

$$T_k \phi = \left(r^{-1} \frac{d}{dr} \right)^{k-1} \left(r^{2k-1} \phi(r) \right).$$

(1) Show that

$$T_k \varphi = \sum_{j=0}^{k-1} c_j r^{j+1} \phi^{(j)} r,$$

for some $c_j \in \mathbb{R}$. Determine c_0 and c_{k-1}.
(2) Show that for any function $\varphi \in C^{k+1}([0, +\infty[)$,

$$\frac{d^2}{dr^2}(T_k \varphi) = \left(r^{-1} \frac{d}{dr} \right)^k (r^{2k} \varphi'(r)).$$

Hint: You can start by verifying that the formula is true when $\varphi(r) = r^m$ for any integer m.
(3) Consider a solution $u(t,x)$ of the linear wave equation in space dimension N, radial with respect to the space variable. Suppose u is C^{k+1} on $\mathbb{R}^{1+N}$. Show prove

$$(\partial_t^2 - \partial_r^2)(T_k u) = 0.$$

Deduce an expression of $T_k u$ in terms of u_0 and u_1.
(4) Express $u(t,r)$ in terms of u_0 and u_1 when $N = 5$. What regularity of u_0 and u_1 is required for u to be C^2 on $\mathbb{R}^{1+5}$?

Exercise 3. Let u be a solution of the wave equation (LW) in space dimension $N \geq 3$, radial with respect to the space variable. Recall that $\Delta u = \frac{d^2}{dr^2} + \frac{N-1}{r} \frac{d}{dr}$. Suppose $u \in C^2(\mathbb{R}^{1+N})$, with compactly supported initial data. Let

$$v(t,r) = \int_r^\infty \rho \partial_t u(t,\rho) d\rho.$$

Show that v defines a radial solution, of class C^2, to the wave equation in space dimension $N - 2$.

Exercise 4. Let $f \in C^k(\mathbb{R}^3)$. Show that the function M_f defined by

$$(M_f)(t,x) = \frac{1}{4\pi}\int_{S^2} f(x+ty)d\sigma(y) = \frac{1}{4\pi t^2}\int_{S^2_{|t|}} f(x+z)d\sigma(z). \tag{1}$$

is also of class C^k.

Exercise 5. Let $u \in C^2(\mathbb{R}\times\mathbb{R}^N)$ be a solution of (LW) with finite energy. Show

$$\forall \varepsilon>0,\ \exists R>0,\ \forall t\in\mathbb{R},\quad \int_{|x|>R+|t|} e_u(t,x)dx \leq \varepsilon.$$

Exercise 6 (Conservation of momentum).

(1) Let u be a C^2 solution of (LW) on $\mathbb{R}\times\mathbb{R}^N$, and $j \in 1,\dots N$. Let $p_{j,u}(t,x) = \partial_{x_j}u(t,x)\partial_t u(t,x)$. Show

$$\frac{\partial p_{j,u}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x_j}\left((\partial_t u)^2 - |\nabla u|^2\right) + \nabla\cdot V,$$

where V is a certain C^1 vector field to be specified.

(2) Assume that (u_0,u_1) has finite energy. Justify that

$$P_j(\vec{u}(t)) = \int_{\mathbb{R}^N} p_{j,u}(t,x)dx$$

is defined for all times. Show that this quantity is independent of time. You can start by considering a local version of the momentum

$$\int_{[-R,R]^N} p_{j,u}(t,x)dx \text{ or } \int_{\mathbb{R}^N} p_{j,u}(t,x)\varphi\left(\frac{x}{R}\right)dx$$

then let R tend to $+\infty$. Here φ denotes a C^2 function with compact support equal to 1 in a neighborhood of the origin.

Exercise 7.

(1) Let $u_1 \in C^2(\mathbb{R}^3)$ such that

$$\forall t\geq 0,\ \forall x\in\mathbb{R}^3,\quad u_1(x)\geq 0.$$

Assume $u_0 = 0$. Let u be the corresponding solution of (LW). Prove

$$\forall t\geq 0,\ \forall x\in\mathbb{R}^3,\quad u(t,x)\geq 0.$$

(2) Suppose now $N=1$ or $N=2$. Let u be the solution of (LW), (ID), with $(u_0,u_1)\in C^3\times C^2$ (if $N=2$) or $C^2\times C^1$ (if $N=1$).
Show that if $u_1\geq 0$ and $u_0=0$ then $u(t,x)$ has the sign of t for all x and $t\neq 0$.
When $N=1$, give a weaker sufficient condition on (u_0,u_1) such that:

$$\forall t\geq 0,\ \forall x\in\mathbb{R},\quad u(t,x)\geq 0.$$

Exercise 8. Assume $N=1$ or $N=2$. Let u be a solution of

$$\partial_t^2 u - \Delta u = f, \tag{2}$$

with initial data $(u(0),\partial_t u(0)) = (0,0)$, and f of class C^1 (if $N=1$) or C^2 (if $N=2$). Express u in terms of f.

Exercise 9. The *Minkowski spacetime* of dimension N is the space $\mathbb{R}^{1+N}$, equipped with the quadratic form of signature $(1,N)$:

$$g(X) = x_0^2 - \sum_{j=1}^N x_j^2 = t^2 - |x|^2 = {}^tXJX,$$

where ${}^tX$ is the transpose of X,

$$X = (x_0,x_1,\dots,x_N),\ t=x_0,\ x=(x_1,\dots,x_N),$$

and $J = [J_{\mu,\nu}]_{0\leq\mu\ \nu\leq N}$ is the matrix such that $J_{0,0}=1$, $J_{\ell,\ell}=-1$ if $\ell\in 1,\dots,N$, and $J_{\mu,\nu}=0$ if $\mu\neq\nu$.

The Lorentz group $\mathrm{O}(1,N)$ is the group of real square matrices P of size $1+N$ which leave the quadratic form g invariant, i.e., such that $g(PX) = g(X)$ for all X in $\mathbb{R}^{1+N}$. In other words, if P is a $(1+N)\times(1+N)$ matrix,

$$P \in \mathrm{O}(1,N) \iff {}^tPJP = J.$$

(1) Prove that a function v of class C^2 on $\mathbb{R}^{1+N}$ satisfies the wave equation (LW) if and only if $\mathrm{Tr}(Jv'') = 0$, where v'' is the Hessian matrix $\left[\partial_{x_\mu}\partial_{x_\nu} v\right]_{0\leq \mu,\nu \leq N}$.

(2) Let $P \in \mathrm{O}(1,N)$, $v \in C^2(\mathbb{R}^{1+N})$, and $w(X) = v(PX)$. Then

$$(\partial_t^2 - \Delta)v = 0 \iff (\partial_t^2 - \Delta)w = 0.$$

(3) Prove that the space rotations:

$$\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & R \end{bmatrix}, \quad R \in \mathrm{O}(N)$$

and the Lorentz boosts

$$\mathcal{R}\sigma = \begin{bmatrix} R_\sigma & \mathbf{0} \\ \mathbf{0} & I_{N-1}. \end{bmatrix}, \quad R_\sigma = \begin{bmatrix} \cosh(\sigma) & \sinh(\sigma) \\ \sinh(\sigma) & \cosh(\sigma) \end{bmatrix},$$

where I_{N-1} denotes the identity matrix $(N-1)\times(N-1)$ and $\sigma \in \mathbb{R}$ are Lorentz transformations. In these formulas, $\mathbf{0}$ always denotes the zero matrix of appropriate size.

Exercise 10. In all Chapter I, we considered the Cauchy problem with initial conditions on a hyperplane in $\mathbb{R}^{1+N}$ of the form $\{t = t_0\}$. We now seek to solve the same problem by prescribing an initial condition on other hyperplanes. Therefore, we consider a hyperplane of the form

$$\Pi = \{X \in \mathbb{R}^{1+N} \ : \ {}^tAX = 0\}$$

where $A \in \mathbb{R}^{1+N} \setminus \{0\}$, $A = (a_0, a_1, \dots, a_N) = (a_0, a)$.

(1) Prove that if $|a_0| > |a|$, there exists a transformation $P \in \mathrm{O}(1,N)$ such that

$$\Pi = P\left(\{(0,x),\ x \in \mathbb{R}^N\}\right).$$

Hint: use compositions of transformations defined in Question (3) *of Exercise 9.*

(2) If the condition $|a_0| > |a|$ is satisfied, we can therefore reduce the Cauchy problem with an initial condition

$$u_{\restriction\Pi} = u_0, \quad A\cdot\nabla u_{\restriction\Pi} = u_1,$$

to a Cauchy problem with initial conditions at $t = 0$ as treated above. The hyperplane Π is called *timelike* when $A = (a_0, a)$ with $a_0 \in \mathbb{R}$, $A \in \mathbb{R}^N$, and $|a_0| > |a|$.

Prove that Π is timelike if and only if the restriction of the quadratic form g to Π is negatively defined.

(3) Under what condition on A does there exist $B = (b_0, b_1, \dots, b_N) \in \mathbb{R}^{N+1}$ such that the function

$$e^{A\cdot X + iB\cdot X}$$

is a solution of (LW)?

(4) Now assume that the hyperplane Π is not timelike. Let $Y \notin \Pi$. Construct a sequence of solutions $(u_n)_n$ of (LW) such that $u_n(X) = 0$ on Π, such that for any differential operator $D = \prod_{j=1}^N \partial_{x_1}^{\alpha_1} \dots \partial_{x_N}^{\alpha_N}$ (of arbitrarily large order), there exists $C > 0$ such that $|Du_n(X)| \leq Ce^{-n}$ on Π, but $|u_n(Y)| \to +\infty$ as $n \to \infty$.