

## DYNAMICS OF SEMI-LINEAR WAVE EQUATIONS. HOMEWORK, CHAPTER II

*Exercise 1.* Let  $N \geq 1$ ,  $\varphi \in \mathcal{S}(\mathbb{R}^N)$ , which is not identically 0. Let

$$\varphi_\varepsilon(x) = e^{ix_1/\varepsilon} \varphi(x).$$

Show that for  $\sigma > 0$ ,  $\lim_{\varepsilon \rightarrow 0} \|\varphi_\varepsilon\|_{\dot{H}^\sigma} = +\infty$ . Show that the sequences  $(\|u_n\|_{L^p})_n$  and  $(\|u_n\|_{\dot{B}^\sigma})_n$  are bounded for all  $\sigma > 0$  and  $p \in [1, \infty]$ .

*Exercise 2.* Let  $\sigma \in \mathbb{R}$ . We define the (inhomogeneous) Sobolev space  $H^\sigma(\mathbb{R}^N)$  by the space of  $u \in \mathcal{S}'(\mathbb{R}^N)$  such that  $\hat{u}$  is in  $L^1(K)$  for all compact  $K$  and that the following quantity is finite

$$\|f\|_{H^\sigma}^2 = \int_{\mathbb{R}^N} (1 + |\xi|^2)^\sigma |\hat{f}(\xi)|^2 d\xi.$$

- (1) Prove that  $H^\sigma$  is an Hilbert space.
- (2) Assume that  $\sigma$  is a positive integer. Prove that  $f$  is in  $H^\sigma$  if and only if for all  $\alpha \in \mathbb{N}^N$  with  $|\alpha| \leq \sigma$ ,  $\partial_x^\alpha f \in L^2(\mathbb{R}^N)$ . Prove that the norm on  $H^\sigma$  is equivalent to the norm defined by

$$\|f\|^2 = \sum_{|\alpha| \leq \sigma} \|\partial_x^\alpha f\|_{L^2}^2.$$

In the case where  $\sigma$  is even, prove that this norm is also equivalent to the norm defined by

$$\|f\|^2 = \|f\|_{L^2}^2 + \|\Delta^{\sigma/2} f\|_{L^2}^2.$$

- (3) Prove that for all  $(u_0, u_1) \in H^\sigma \times H^{\sigma-1}$  the formula

$$(1) \quad u(t) = \cos(t|D|)u_0 + \frac{\sin(t|D|)}{|D|}u_1$$

defines  $u \in C^0(\mathbb{R}, H^\sigma)$  such that  $\partial_t u \in C^0(\mathbb{R}, H^{\sigma-1})$  that satisfies the linear wave equation in the sense of distribution.

*Exercise 3.* Let  $u_1 \in L^2(\mathbb{R}^N)$  and consider

$$u(t) = \frac{\sin(t|D|)}{|D|}u_1,$$

the solution of the wave equation with initial data  $(0, u_1)$ .

- (1) Prove that for all  $t$ ,  $u(t) \in L^2(\mathbb{R}^N)$ , and  $\|u(t)\|_{L^2} \leq t\|u_1\|_{L^2}$ .
- (2) Give a sufficient condition on  $u_1$  so that

$$(*) \quad \sup_{t \in \mathbb{R}} \|u(t)\|_{L^2} < \infty$$

- (3) Give example of functions  $u_1 \in L^2$  such that  $(*)$  does not hold.

*Exercise 4 (Conservation laws).* Let  $(u_0, u_1) \in \mathcal{S}'(\mathbb{R}^N) \times \mathcal{S}'(\mathbb{R}^N)$ . Assume that  $\widehat{u_0}$  and  $\widehat{u_1}$  are locally integrable functions. Let

$$u(t, x) = \cos(t|D|)u_0 + \frac{\sin(t|D|)}{|D|}u_1.$$

Let  $\varphi \in C^\infty(\mathbb{R}^N)$  such that  $(|\widehat{u_0}(\xi)|^2 |\xi|^2 + |\widehat{u_1}(\xi)|^2) \varphi(\xi)$  is integrable on  $\mathbb{R}^N$ . Prove that

$$\int \left[ |\xi|^2 |\widehat{u}(t, \xi)|^2 + \left| \partial_t \widehat{u}(t, \xi) \right|^2 \right] \varphi(\xi) d\xi$$

is well-defined and independent of  $t$ .

*Exercise 5.* Consider the function defined on  $\mathbb{R}^3$  by

$$(2) \quad S(x) = \frac{\sin(|x|)}{|x|}.$$

Compute the Fourier transform of  $S$ . Hint: compare the two formulas for the wave equation in space dimension 3 obtained in Chapter I and Chapter II.