DYNAMICS OF SEMI-LINEAR WAVE EQUATIONS 2025.
EXERCISE SHEET, CHAPTER 4

In this exercise sheet, we consider the equations:

$$(\partial_t^2 - \Delta)u = \sigma u^5. \tag{W5}$$

Exercise 1. Assume $\sigma = 1$.

(1) Prove that there exists a solution of the ODE: $Y'' = Y^5$ defined on $]-1,+1[$ and such that
$$\lim_{t\to\pm 1} |Y(t)| = +\infty$$
(2) Prove that for all a, b with $a < b$, there exists a solution of (W5) with maximal interval of existence (a, b).

Exercise 2. Assume $\sigma = 1$. Let u be a solution of (W5) such that $u_0 = 0$ and $u_1 \in L^2 \cap C^2$ is nonnegative. Prove that $u(t, x) \geq 0$ for all x and for all $t \geq 0$ in the domain of existence of u. Hint: use exercise 7 of the exercise sheet of Chapter 4, where it is shown that $\frac{\sin(t|D|)}{|D|} f \geq 0$ whenever $f \in C^2(\mathbb{R}^3)$ and $f \geq 0$.

Exercise 3. (1) Let $R > 0$ and u, v be two solutions of (W5) defined on $[0, \infty[\times\mathbb{R}^3$. Assume that $\vec{u}(0, x) = \vec{v}(0, x)$ for $|x| > R$. Prove that $\vec{u}(t, x) = \vec{v}(t, x)$ for $t \geq 0$, $|x| > R + |t|$.

(2) Prove that there exists a small $\delta_1 > 0$ with the following property: if u is a solution of (W5) with initial data $\vec{u}_0 = (u_0, u_1) \in \dot{\mathcal{H}}^1$ at $t = 0$ with maximal time of existence $T_+ = +\infty$ and such that
$$\int_{|x|>R} |\nabla u_0|^2 + (u_1)^2 dx = \delta^2 \leq \delta_1^2,$$
Then
$$\int_{|x|>R+|t|} |\nabla u(t)|^2 + (\partial_t u(t))^2 dx \leq 2\delta^2 \tag{1}$$

Exercise 4. (Exercise IV.1 of the course) Prove that the definition of finite energy solutions in the course does not depend on the choice of the initial time. In other words, if u is a solution of (W5) on an interval I such that $t_0 \in I$, with initial data $\vec{u}(t_0) = (u_0, u_1) \in \dot{\mathcal{H}}^1$, and $t_1 \in I$, then
$$\forall t \in I, \quad u(t) = \cos\big((t-t_1)|D|\big)u(t_1) + \frac{\sin\big((t-t_1)|D|\big)}{|D|}\partial_t u(t_1) + \int_{t_1}^t \frac{\sin\big((t-s)|D|\big)}{|D|} u^5(s) ds.$$

Exercise 5. (Exercice IV.6 of the course): prove Theorem IV.7.2 of the course (assuming $t_0 = 0$, $x_0 = 0$ to lighten notations). Let $R > 0$, and $\Gamma = \Big\{(t, x) \in \mathbb{R} \times \mathbb{R}^N \ : \ t_0 \leq t \leq t_1,\ |x| \leq R - |t|\Big\}$ Let $u, v \in C^2(\Gamma)$ be two classical solutions of (W5) on Γ. We suppose $(u, \partial_t u)(0, x) = (v, \partial_t v)(0, x)$ for all x with $|x| < R$. We want to prove $u(t, x) = v(t, x)$ for all $(t, x) \in \Gamma$.

$$V(t) = \frac{1}{2}\int_{|x|<R-t} (u(t,x) - v(t,x))^2 dx + \frac{1}{2}\int_{|x|<R-t} (\partial_t u(t,x) - \partial_t v(t,x))^2 dx + \frac{1}{2}\sum_{j=1}^{3}\int_{|x|<R-t} (\partial_{x_j} u(t,x) - \partial_{x_j} v(t,x))^2 dx.$$

(1) Prove that $V'(t) \leq CV(t)$ for $t \in [0, t_1]$.
(2) Prove that $V(t) = 0$ for all $t \in [0, t_1]$