DYNAMICS OF SEMI-LINEAR WAVE EQUATIONS. HOMEWORK, CHAPTER 5

In this exercise sheet, we consider the equations:

$$(\partial_t^2 - \Delta)u = \sigma u^5. \tag{W5}$$

and

$$(\partial_t^2 - \Delta)u_L = 0. \tag{LW}$$

in space dimension 3. A "solution" will always mean a finite-energy solution which satisfies the equation in the usual Duhamel sense.

Exercise 1. (1) Let u_L be a solution of (LW). Prove that

$$\lim_{t\to\infty} \|u_L(t)\|_{L^6} = 0.$$

Hint: start with the case where $\vec{u}_L(0)$ is smooth and compactly supported.

(2) Let f be a measurable function defined on $I \times \mathbb{R}^3$. Assume that the $L^\infty(I, L^6)$ and $L^4(I, L^{12})$ norms of f are finite. Prove that $f \in L^5(I, L^{10})$ and give a bound of the norm of f in this space in terms of the 2 preceding norms.

(3) Prove that there exists $\varepsilon > 0$ with the following property. For any solution u of (W5) with $T_+(u) = +\infty$ and

$$\limsup_{t\to\infty} \|u(t)\|_{L^6} \leq \varepsilon.$$

the solution u scatters to a linear solution.

Exercise 2 (Lorentz transformation). (1) Let u be a solution of (LW) (respectively (W5)). Prove that the momentum of u,

$$P(\vec{u})(t) = \int \nabla u(t,x)\partial_t u(t,x)dx,$$

is independent of t. What is the momentum of u when the initial data $\vec{u}_0$ of u is radial?

(2) Assume that $\vec{u}_0$ is smooth and compactly supported. Let $c > 0$ and

$$u_c(t,x) = u\left(\frac{t - cx_1}{\sqrt{1-c^2}}, \frac{x_1 - ct}{\sqrt{1-c^2}}, x_2, x_3\right).$$

Prove that u_c is a solution of (LW) (respectively (W5)). Compute the momentum and energy of u_c in terms of c, the momentum and the energy of u. The term "Energy" has to be interpreted as the conserved energy for the equation (LW) or (W5) satisfied by u.

Exercise 3. Let Q be a solution of $-\Delta Q = Q^5$ with $Q \in \dot{H}^1(\mathbb{R}^3)$. Let $F(x) := \mathcal{K}(Q)(x) = \frac{1}{|x|}Q(\frac{x}{|x|^2})$ (Kelvin transform of Q).

(1) Prove that $F \in \dot{H}^1$ and that $-\Delta F = F^5$.
(2) Compute $\mathcal{K}(\mathcal{K}Q)$.
(3) Compute $\mathcal{K}W$, where $W = (1 + |x|^2/3)^{-1/2}$.
(4) Assume that $Q(0) \neq 0$. Give the asymptotic behaviour of $F(x)$ as $|x| \to \infty$.