Dynamics of semilinear wave equation
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CHAPTER I

Linear wave equation: classical theory

I.1. Presentation of the equation

The linear wave equation is the equation:

$$\text{(LW)} \qquad \partial_t^2 u - \Delta u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N,$$

where $N \geq 1$ is the spatial dimension (in this course, we will often assume $N = 3$), and

$$\Delta = \sum_{k=1}^{N} \frac{\partial^2}{\partial x_k^2}.$$

(We will use either the notations ∂_y or $\frac{\partial}{\partial y}$ for the derivative with respect to the variable $y \in \{t, x_1, \ldots, x_N\}$).

This is an evolution equation: we fix initial data at a certain time $t = t_0$, and are interested in the evolution of the equation over time t. Since the equation is of order 2, we actually fix an initial data for $\vec{u} = (u, \partial_t u)$:

$$\text{(I.1.1)} \qquad \vec{u}_{\restriction t=t_0} = (u_0, u_1)$$

where (u_0, u_1) is to be taken in a certain functional space.

We will consider in this course initial data *with real values.* The passage to complex or vector values is immediate for most properties of the equation (LW) (by working coordinate by coordinate), but can induce drastic changes in the nonlinear case, if the nonlinearity mixes the components.

Equation (LW) is invariant under several obvious space-time transformations. If u is a solution, it is also the case of

$$\mu u(t - t_0, \lambda(Rx - x_0)),$$

where $\mu \in \mathbb{R}$, $t_0 \in \mathbb{R}$, $\lambda > 0$, $R \in \mathcal{O}_N(R)$, $x_0 \in \mathbb{R}^N$[1]

As a consequence, we can limit ourselves, without loss of generality, to the case of an initial time $t_0 = 0$, i.e.

$$\text{(ID)} \qquad \vec{u}_{\restriction t=0} = (u_0, u_1)$$

Furthermore, the equation is invariant under time inversion: if u is solution, it is also the case of $t \mapsto u(-t, x)$. In particular, it is a reversible equation.

We will also consider the equation with a force:

$$\text{(I.1.2)} \qquad \partial_t^2 u - \Delta u = f,$$

(still with an initial condition of type (ID)), whose understanding will be crucial for the study of the nonlinear wave equation.

The Cauchy problem (LW), (ID) can be approached in at least 3 different ways:

- The classical approach which consists in finding an explicit formula to express the solution. It works when the initial data are sufficiently regular ($C^3 \times C^2$ in dimension 3 of space) and gives classical solutions (that is to say C^2 in (t,x) and satisfying (LW) in the sense of classical differentiation).
- The use of the Fourier transformation in space, which is very simple (once the Fourier transformation is known) and particularly effective in Sobolev spaces based on L^2 (which are natural spaces for the study of the equation by virtue of the conservation of energy). This method allows to obtain weak solutions with degrees of regularity lower than the previous one, and to use tools based on the Fourier transformation, which can be useful, for example, to demonstrate certain dispersive properties of the equation.
- The "functional analysis" approach, by the theory of semi-groups, which gives the same type of solutions as the previous method.

[1] Equation (LW) is in fact invariant under a larger group of linear transformations, the Lorentz group (cf §I.7 below)

In this chapter, we will detail the classical method, first by writing the explicit formula for solutions in dimension 1 of space, then in higher dimensions. We will study in the following chapter the equation in the energy space by the Fourier transformation. This chapter is partly based on Chapter 5 of the beautiful book by Folland on partial differential equations [**16**].

I.2. Explicit Formula in Dimension 1

In dimension 1, the equation (LW) can be written as:

$$(\partial_t^2 - \partial_x^2)u = 0, \tag{I.2.1}$$

which means $(\partial_t - \partial_x)(\partial_t + \partial_x)u = 0$. Thus, we make the change of variables $\eta = x + t$, $\xi = x - t$. Therefore, by setting $v(\eta, \xi) = u\left(\frac{\eta-\xi}{2}, \frac{\eta+\xi}{2}\right)$, or $u(t,x) = v(t+x, t-x)$, we have:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 v}{\partial \eta^2} + \frac{\partial^2 v}{\partial \xi^2} + 2\frac{\partial^2 v}{\partial \xi \partial \eta},$$

and

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial \eta^2} + \frac{\partial^2 v}{\partial \xi^2} - 2\frac{\partial^2 v}{\partial \xi \partial \eta},$$

which gives:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -4\frac{\partial^2 v}{\partial \eta \partial \xi}.$$

Thus, we obtain:

$$\text{(LW)} \iff \frac{\partial^2 v}{\partial \eta \partial \xi} = 0.$$

Let u be a C^2 solution of (I.2.1), (ID). Thus, $u_1 \in C^1(\mathbb{R})$ and $u_0 \in C^2(\mathbb{R})$.

The equality $\frac{\partial^2 v}{\partial \eta \partial \xi} = 0$ shows that $\frac{\partial v}{\partial \xi}$ is a (class C^1) function $w(\xi)$ independent of η. Integrating with respect to ξ for η fixed, we deduce:

$$v(\eta, \xi) = \underbrace{\int_0^{\xi} w(\sigma) d\sigma}_{\varphi(\xi)} + \psi(\eta),$$

for a certain function ψ, necessarily C^2 since v is of class C^2 and w of class C^1. Thus, we necessarily have:

$$v(\eta, \xi) = \varphi(\xi) + \psi(\eta), \quad \varphi, \psi \in C^2(\mathbb{R}^2),$$

or equivalently:

$$u(t,x) = \varphi(x-t) + \psi(x+t). \tag{I.2.2}$$

Using the initial condition (ID), a direct calculation gives:

$$\psi(\eta) = \frac{1}{2}\int_0^{\eta} u_1(\sigma) d\sigma + \frac{1}{2}u_0(\eta) + c,$$
$$\varphi(\xi) = -\frac{1}{2}\int_0^{\xi} u_1(y) dy + \frac{1}{2}u_0(\xi) - c,$$

where $c \in \mathbb{R}$ (the choice of this constant is irrelevant). Hence, we deduce:

$$u(t,x) = \frac{1}{2}(u_0(x+t) + u_0(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} u_1(y) dy. \tag{I.2.3}$$

Conversely, it is easy to verify that formula (I.2.3) gives a C^2 solution of (I.2.1), (ID). Therefore, we have shown:

PROPOSITION I.2.1. *Let $(u_0, u_1) \in C^2(\mathbb{R}) \times C^1(\mathbb{R})$. Then, there exists a unique solution $u \in C^2(\mathbb{R} \times \mathbb{R})$ of* (LW) *satisfying the initial condition* (ID). *This solution satisfies formula* (I.2.3).

On formula (I.2.2), we observe that a solution of the wave equation in dimension 1 is the sum of two waves: one, $\varphi(x-t)$, moving at speed 1 to the right (called a *progressive wave*), and the other $\psi(x+t)$, moving at the same speed to the left.[2]

[2] Note that the equations (LW), (I.2.1) have been normalized, so that the speed of propagation is exactly 1.

It is also possible to obtain a formula for the equation with the right-hand side (I.1.2). We leave this as an exercise to the reader. Further on, we will provide a general method giving the solution of the equation with the right-hand side in terms of the equation without the right-hand side.

We can see from formula (I.2.3) that $u(t,x)$ depends only on the values of (u_0,u_1) over $\Big[x-|t|, x+|t|\Big]$. This is a prime example of "finite speed of propagation" which holds in all spatial dimensions.

I.3. Integral on the Sphere and Divergence Theorem

We denote $S^{N-1}=\{x\in\mathbb{R}^N,\ |x|=1\}$, where $|\cdot|$ represents the Euclidean norm on $\mathbb{R}^N$:

$$|x|^2=\sum_{j=1}^N x_j^2.$$

More generally, S_R^{N-1} will denote the sphere of radius R: $\{x\in\mathbb{R}^N,\ |x|=R\}$.

We denote $d\sigma$ as the volume element on one of these spheres. Thus, the integral of a function $f\in\mathcal{L}^1(S_R^{N-1})$ (i.e., a function integrable on S_R^{N-1}) is written as

$$\int_{S_R^{N-1}} f(y)d\sigma(y).$$

In dimension 3, this integral can, for example, be calculated using spherical coordinates:

$$\int_{S_R^2} f(y)d\sigma(y)=R^2\int_0^{2\pi}\int_0^{\pi} f(R\sin\theta\cos\varphi, R\sin\theta\sin\varphi, R\sin\varphi)\sin(\theta)d\theta d\varphi.$$

We denote $B_R^N(x_0)$ as the ball centered at x_0 with radius R:

$$B_R^N(x_0)=\big\{x\in\mathbb{R}^N,\ |x-x_0|<R\big\}$$

and simply $B_R^N=B_R^N(0)$.

We will use the following formulas:

Scaling:

$$\int_{S_R^{N-1}} f(y)d\sigma(y)=R^{N-1}\int_{S^{N-1}} f(Ry)d\sigma(y)n \quad f\in\mathcal{L}^1(S_R^{N-1}).$$

Integral in radial coordinates: if $f\in\mathcal{L}^1(\{|x|\leq R\})$,

$$\int_{B_R^N} f(x)dx=\int_0^R\int_{S_r^{N-1}} f(y)d\sigma(y)dr=\int_0^R\int_{S^{N-1}} f(r\omega)d\sigma(\omega)r^{N-1}dr$$

Divergence theorem: if $F\in C^1(\overline{B_R},\mathbb{R}^N)$,

$$\int_{|x|\leq R}\nabla\cdot F(x)dx=\int_{S_R^{N-1}}\frac{y}{|y|}\cdot F(y)d\sigma(y),$$

where $\nabla\cdot F=\sum_{j=1}^N\partial_{x_j}F_j$ is the divergence of the vector field F.

I.4. Energy density. Uniqueness and finite speed of propagation

Before giving an explicit formula for the wave equation in dimension 3, we prove a uniqueness result valid in any dimension:

Theorem I.4.1. *Let $(t_0,x_0)\in\mathbb{R}^{1+N}$, $t_1>t_0$, $R>0$. We denote $\Gamma=\Big\{(t,x)\in\mathbb{R}\times\mathbb{R}^N\ :\ t_0\leq t\leq t_1,\ |x-x_0|\leq R-|t-t_0|\Big\}$. Let $u\in C^2(\Gamma)$ be a solution of* (LW) *on Γ. We suppose $(u,\partial_t u)(t_0,x)=0$ for all $x\in B_R(x_0)$. Then u is identically zero on Γ.*

The proof of the theorem is based on a monotonicity law that has its own interest.

We denote, for $(t,x)\in\Gamma$,

$$e_u(t,x)=\frac{1}{2}|\nabla u(t,x)|^2+\frac{1}{2}(\partial_t u(t,x))^2,$$

where $|\nabla u|^2=\sum_{j=1}^N(\partial_{x_j}u)^2$, and we consider, for $t_0\leq t\leq t_1$, the local energy

$$E_{\mathrm{loc}}(t)=\int_{B_{R-(t-t_0)}(x_0)} e_u(t,x)dx=\int_{|x-x_0|<R-(t-t_0)} e_u(t,x)dx.$$

Lemma I.4.2. *The function E_{loc} is decreasing on $[t_0, t_1]$.*

The lemma immediately implies Theorem I.4.1. Indeed, if $\vec{u}(t_0)$ vanishes on $B(x_0, R)$, then $E_{\text{loc}}(t_0) = 0$, and thus $E_{\text{loc}}(t) = 0$ for all $t \in [t_0, t_1]$, showing that u is zero on Γ.

Proof of Lemma I.4.2. We notice that

$$\frac{\partial e}{\partial t} = \sum_{j=1}^{N} \left(\partial_{x_j} u \partial_t \partial_{x_j} u + \partial_{x_j}^2 u \partial_t u \right) = \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \left(\partial_{x_j} u \partial_t u \right) = \nabla \cdot \left(\partial_t u \nabla u \right). \tag{I.4.1}$$

Without loss of generality, we can assume for simplification of notations that $x_0 = 0$ and $t_0 = 0$. By the integration formula in radial coordinates,

$$E_{\text{loc}}(t) = \int_0^{R-t} s^{N-1} \int_{S^{N-1}} e_u(t, s\omega) d\sigma(\omega) ds.$$

By differentiation under the integral sign, we get that E_{loc} is differentiable and

$$E'_{\text{loc}}(t) = -(R-t)^{N-1} \int_{S^{N-1}} e_u(t, (R-t)\omega) d\sigma(\omega) + \int_{B^N_{R-t}} \frac{\partial e_u}{\partial t}(t, x) dx.$$

By formula (I.4.1), then the divergence formula

$$\int_{B^N_{R-t}} \frac{\partial e_u}{\partial t}(t, x) dx = \int_{B^N_{R-t}} \nabla \cdot \left(\partial_t u \nabla u \right)(t, x) dx = \int_{S^{N-1}_{R-t}} \frac{y}{|y|} \nabla u \partial_t u(t, y) d\sigma(y).$$

We thus have

$$\begin{aligned} E'_{\text{loc}}(t) = &- \int_{S^{N-1}_{R-t}} \left(\frac{1}{2} |\nabla u|^2 + \frac{1}{2} (\partial_t u)^2 + \frac{y}{|y|} \nabla u \partial_t u(t, y) \right) d\sigma(y) \\ &\leq -\frac{1}{2} \int_{S^{N-1}_{R-t}} \left(\frac{y}{|y|} \nabla u + \partial_t u(t, y) \right)^2 d\sigma(y). \end{aligned}$$

□

I.5. Explicit formulas.

We now consider higher space dimensions. In dimension $N = 3$, we will show that for any initial data $(u_0, u_1) \in C^2 \times C^3$, there exists a unique solution $u \in C^2(\mathbb{R}^{1+3})$ of (LW), (ID), and provide an explicit formula for this solution. We will also provide a formula in dimension $N = 2$. We refer the reader to [**16**, Chapter 5B] for expressions of solutions when $N \geq 4$.

5.a. The radial case in dimension 3. When the initial conditions depend only on the variable $r = |x|$, the explicit formula is very simple.

We start by observing that if f depends only on the variable r, then the function f is C^2 as a function on $\mathbb{R}^3$ if and only if it is C^2 as a function of the variable r on $[0, \infty[$, and satisfies $\frac{df}{dr}(0) = 0$. Moreover,

$$\Delta f = \frac{d^2 f}{dr^2} + \frac{2}{r} \frac{df}{dr}$$

(cf Exercise I.1). We notice that we can rewrite this formula as

$$r \Delta f = \frac{d^2}{dr^2}(rf).$$

Now let u be a C^2 solution of (LW), (ID) with initial conditions (u_0, u_1) that are radial. We assume that for all t, $u(t)$ is radial. We will show a posteriori that this assumption is satisfied. The previous formula gives

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2} \right)(ru) = 0.$$

The function $(t, r) \mapsto ru(t, r)$ is thus a solution of the wave equation in dimension 1, on $\mathbb{R}_t \times]0, \infty[$. To obtain a function on $\mathbb{R}^2$, we extend $ru(t, r)$ to an odd function:

$$v(t, y) = yu(t, |y|).$$

One can verify (using Exercise I.1) that v is of class C^2 on $\mathbb{R}^2$, and that

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2} \right) v = 0.$$

Formula (I.2.3) then gives:

$$v(t,y)=\frac{1}{2}(v_0(y+t)+v_0(y-t))+\frac{1}{2}\int_{y-t}^{y+t}v_1(\sigma)d\sigma,$$

where $(v_0,v_1)=\vec{v}_{\restriction t=0}$, thus

$$u(t,r)=\frac{1}{2r}\Big((r+t)u_0(|r+t|)+(r-t)u_0(|r-t|)\Big)+\frac{1}{2r}\int_{r-t}^{r+t}\sigma u_1(|\sigma|)d\sigma. \tag{I.5.1}$$

Notice that when $t>0$ (to fix ideas),

$$\int_{r-t}^{r+t}\sigma u_1(|\sigma|)d\sigma=\int_{|r-t|}^{r+t}\sigma u_1(|\sigma|)d\sigma.$$

The finite speed of propagation is satisfied: the solution $u(t,r)$ depends only on the initial condition (u_0,u_1) on the ball centered at r with radius $|t|$.

The formula (I.5.1) defines a function $u(t,r)$ of class C^2 outside the origin $x=0$, as soon as the initial conditions (u_0,u_1) have the expected regularity $C^2\times C^1$. However, there is a subtle phenomenon of loss of regularity at the origin of the solution u compared to the initial data: there exist data $(u_0,u_1)\in C^2\times C^1$ such that u, defined by formula (I.5.1), cannot be extended by a C^2 function up to $r=0$. Indeed, it can be checked that (at fixed t),

$$\lim_{r\to 0}u(t,r)=u_0(t)+tu_0'(t)+tu_1(t), \tag{I.5.2}$$

which shows that if (u_0,u_1) are $C^k\times C^{k-1}$ functions, then $u(t,0)$ is only C^{k-1} in general (see also Exercise I.2). We can interpret this phenomenon physically as follows: a singularity on the circle $r=r_0$ at the initial time 0 that travels at speed 1 towards the origin will concentrate at the origin at time $t=r_0$, causing a stronger singularity.

The limit (I.5.2) suggests a maximal loss of regularity of a derivative with respect to the initial data, which is indeed the case:

Proposition I.5.1. *Let $(u_0,u_1)\in(C^3\times C^2)(\mathbb{R}^3)$ be radial functions. Then formula* (I.2.3) *extended by $u(t,0)=u_0(t)+tu_0'(t)+tu_1(t)$, defines a C^2 function on $\mathbb{R}\times\mathbb{R}^3$, radial with respect to the variable x, and satisfying* (LW)*,* (ID)*.*

The Proposition I.5.1 is left as an exercise to the reader.

The formula (I.5.1) is remarkably simple. In higher space dimensions, we also have an explicit formula for radial solutions, which becomes more complicated as the dimension increases (see Exercise I.3). The loss of regularity observed in dimension 3 (and absent in dimension 1) increases with dimension, as the reader can verify.

There is no simple formula in the radial case in even dimensions.

We also have explicit formulas (of course more complicated) without radiality assumptions, in all dimensions. We will explicitly state these formulas when $N=3$, then $N=2$.

5.b. General solutions in dimension 3: averaging over spheres. If $f\in C^0(\mathbb{R}^3)$, we define

$$(M_f)(t,x)=\frac{1}{4\pi}\int_{S^2}f(x+ty)d\sigma(y)=\frac{1}{4\pi t^2}\int_{S^2_{|t|}}f(x+z)d\sigma(z). \tag{I.5.3}$$

the average of f over the sphere of radius $|t|$ and center x. The function M_f inherits the regularity of f (cf exercise I.5).

Theorem I.5.2. *Let $(u_0,u_1)\in C^3(\mathbb{R}^3)\times C^2(\mathbb{R}^3)$. Then the unique C^2 solution of the wave equation* (LW) *with initial conditions* (ID) *is given by*

$$u(t,x)=tM_{u_1}(t,x)+\frac{\partial}{\partial t}(tM_{u_0}(t,x)).$$

Proof. We start by verifying that $tM_{u_1}(t,x)$ is the solution of the wave equation (LW), with initial condition $(0,u_1)$. By the theorem of differentiation under the integral sign, if $g\in C^2(\mathbb{R}^3)$,

$$\frac{\partial}{\partial t}\left(M_g(t,x)\right)=\frac{1}{4\pi}\int_{S^2}(y\cdot\nabla g)(x+ty)d\sigma(y).$$

Using the divergence formula,

$$\int_{S^2}(y\cdot\nabla g)(x+ty)d\sigma(y)=t\int_{|y|\leq 1}\left(\nabla\cdot(\nabla g)\right)(x+ty)dy$$
$$=t\int_{|y|\leq 1}(\Delta g)(x+ty)dy\ =\frac{1}{t^2}\int_0^t\int_{S^2}(\Delta g)(x+sy)s^2d\sigma(y)ds.$$

Thus:

$$\frac{\partial}{\partial t}\left(tM_{u_1}(t,x)\right)=M_{u_1}(t,x)+\frac{1}{t}\int_0^t\int_{S^2}(\Delta u_1)(x+sy)d\sigma(y)s^2ds.$$

and therefore

$$\frac{\partial^2}{\partial t^2}\left(tM_{u_1}(t,x)\right)=\frac{1}{4\pi t^2}\int_0^t\int_{S^2}(\Delta u_1)(x+sy)d\sigma(y)s^2ds$$
$$-\frac{1}{4\pi t^2}\int_0^t\int_{S^2}(\Delta u_1)(x+sy)d\sigma(y)s^2ds+\frac{t}{4\pi}\int_{S^2}(\Delta u_1)(x+ty)d\sigma(y)\ =\Delta\left(tM_{u_1}(t,x)\right).$$

This shows that tM_{u_1} satisfies the wave equation (LW). Furthermore, since $M_{u_1}(0,x)=u_1(0,x)$, the initial condition at $t=0$ is indeed $(0,u_1)$.

Now let $v(t,x)=tM_{u_0}(t,x)$. Then, by the same reasoning, v is a solution of the wave equation (LW) with initial condition $(0,u_0)$. We deduce that $\partial_t v$ is a solution of the wave equation with initial condition $(u_0,0)$, which concludes the proof. □

Notice that we can rewrite the formula of the theorem as:

$$u(t,x)=tM_{u_1}(t,x)+M_{u_0}(t,x)+tM_{y\cdot\nabla u_0}(t,x). \tag{I.5.4}$$

We now give three consequences of the previous formula.

Corollary I.5.3 (Strong Huygens' principle). *The solution $u(t,x)$ depends only on the values of u_0, ∇u_0, and u_1 on the sphere centered at x and of radius $|t|$.*

Remark I.5.4. The strong Huygens' principle is a stronger version of the speed of propagation, which states that $u(t,x)$ depends only on the values of (u_0,u_1) on the *ball* centered at x and of radius $|t|$. This principle remains valid in any odd dimension ≥ 3 (the number of derivatives of u_0 and u_1 in the statement increases with the dimension). In even dimension, solutions only satisfy the finite speed of propagation: see §5.c. In dimension 1, as shown by formula (I.2.3), only solutions even in time (with initial condition of the form $(u_0,0)$) satisfy the strong Huygens' principle.

Corollary I.5.5 (Dispersion). *Let $(u_0,u_1)\in(C^3\times C^2)(\mathbb{R}^3)$, with compact support included in the ball $\overline{B}(0,R)$. Then $|t|-R\leq|x|\leq t+R$ on the support of x and*

$$|u(t,x)|\lesssim\frac{C}{|t|}.$$

Proof. The assertion on the support follows from the strong Huygens' principle (Corollary I.5.3). The second assertion is a consequence of formula (I.5.4). Indeed, we have: $M_{u_1}(t,x)=\frac{1}{4\pi t^2}\int_{S_t^2}u_1(x+y)dy$. The integrand is zero outside the set

$$\{y\in S_t^2\ :\ x+y\in\operatorname{supp}(u)\},$$

whose measure is uniformly bounded independently of t and x. Thus we have

$$|tM_{u_1}(t,x)|\leq\frac{C}{|t|},$$

where the constant C depends only on $\sup_x|u_1(x)|$ and R. The same reasoning allows to bound the other terms. □

Finally, we state a positivity property of the wave equation in space dimension 3. This property also holds if $N=1,2$, but is false if $N\geq 4$.

Corollary I.5.6 (Positivity). *Let $u_1\in C^2(\mathbb{R}^3)$ such that*

$$\forall t\geq 0,\ \forall x\in\mathbb{R}^3,\quad u_1(x)\geq 0.$$

Assume $u_0=0$. Let u be the solution of (LW)*. Then*

$$\forall t\geq 0,\ \forall x\in\mathbb{R}^3,\quad u(t,x)\geq 0.$$

PROOF. This follows immediately from formula (I.5.4). □

We will give a more precise version of Corollary I.5.5 in Chapter III: See Theorem III.3.1 there.

5.c. Dimension $1+2$. A solution u of equation (LW) with $N=2$ is also a solution of the same equation with $N=3$, constant with respect to the 3rd spatial coordinate. From Theorem I.5.2, one can derive an expression of u from the initial data. This strategy is called "descent method".

THEOREM I.5.7. *Let $(u_0,u_1)\in(C^3\times C^2)(\mathbb{R}^2)$. Then equation* (LW) *has a unique C^2 solution on $\mathbb{R}\times\mathbb{R}^2$, given by the formula*

$$u(t,x)=\frac{1}{2\pi}\left[\frac{\partial}{\partial_t}\left(t\int_{|y|\leq 1}\frac{u_0(x+ty)}{\sqrt{1-|y|^2}}dy\right)+t\int_{|y|\leq 1}\frac{u_1(x+ty)}{\sqrt{1-|y|^2}}dy\right]. \tag{I.5.5}$$

PROOF. Uniqueness follows from Theorem I.4.1. Moreover, as in the proof of Theorem I.5.2, the formula for even solutions in time (with initial condition $(u_0,0)$) can be easily deduced from the formula for odd solutions in time (with initial condition $(0,u_1)$). So we only consider this second case.

Let u be a C^2 solution of (LW) on $\mathbb{R}\times\mathbb{R}^2$, with initial data $(u,\partial_t u)(0)=(0,u_1)$, where $u_1\in C^2(\mathbb{R}^2)$. By Theorem I.5.2, considering u as a solution on $\mathbb{R}\times\mathbb{R}^3$, we obtain:

$$u(t,x_1,x_2)=\frac{t}{4\pi}\int_{S^2}\tilde{u}_1((x_1,x_2,0)+ty)d\sigma(y)dy,$$

where by definition $\tilde{u}_1(x_1,x_2,x_3)=u_1(x_1,x_2)$. Passing to spherical coordinates, we get

$$\begin{aligned}\int_{S^2}&\tilde{u}_1((x_1,x_2,0)+ty)d\sigma(y)\\ &=\int_0^{2\pi}\int_0^{\pi}u_1(x_1+t\sin\theta\cos\varphi,x_2+t\sin\theta\sin\varphi)\sin\theta d\theta d\varphi\\ &=2\int_0^{2\pi}\int_0^{\pi/2}u_1(x_1+t\sin\theta\cos\varphi,x_2+t\sin\theta\sin\varphi)\sin\theta d\theta d\varphi.\end{aligned}$$

The announced formula then follows from the change of variable $y_1=t\sin\theta\cos\varphi$, $y_2=t\sin\theta\sin\varphi$. □

It can be seen from the formula in Theorem I.5.7 that the strong Huygens principle is not verified in dimension $1+2$: the solution $u(t,x)$ depends on the values of the initial condition over the entire ball $B_{|t|}(x)$, not just on the sphere $x\;:\;|x|=|t|$.

I.6. Conservation Laws

The energy of a solution u on $\mathbb{R}\times\mathbb{R}^N$ is defined as:

$$E(\vec{u}(t))=\int_{\mathbb{R}^N}e_u(t,x)dx=\frac{1}{2}\int_{\mathbb{R}^N}\Big((\partial_t u(t,x))^2+|\nabla u(t)|^2\Big)dx.$$

This is the global version of the local energy considered in §I.4. The energy of a solution is conserved over time.

THEOREM I.6.1. *Let $u\in C^2(\mathbb{R}^{1+N})$ be a solution of* (LW), (ID). *Assume (u_0,u_1) has finite energy. Then for any t, $E(\vec{u}(t))$ is finite and $E(\vec{u}(t))=E(u_0,u_1)$.*

PROOF. One might be tempted to write

$$\frac{d}{dt}(E(\vec{u}(t)))=\int\partial_t e_u(t,x)dx=\int\nabla\cdot(\partial_t u\nabla u)dx=0,$$

but the last equality, obtained by integration by parts ignoring the "boundary" term (i.e., when $|x|\to\infty$) is purely formal. To justify the preceding calculation, we can use the decay of the local energy (Lemma I.4.2). For $R>0$, we define:

$$E_{<R}(\vec{u}(t))=\int_{|x|<R}e_u(t,x)dx.$$

Notice that this quantity is finite as soon as $u\in C^1(\mathbb{R}^{1+N})$. Fix $t>0$. By Lemma I.4.2, for any $R>t$,

$$E_{<R-t}(\vec{u}(t))\leq E_{<R}(\vec{u}(0))\leq E(u_0,u_1).$$

As we let R tend to $+\infty$, we obtain that $E(\vec{u}(t))$ is finite, and

$$E(\vec{u}(t))\leq E(u_0,u_1).$$

Reversing the direction of time, we also obtain the inequality

$$E(u_0, u_1) \leq E(\vec{u}(t)).$$

We have shown that the energy is conserved for $t \geq 0$. By applying this result to the solution $(t, x) \mapsto u(-t, x)$, we obtain energy conservation for $t \leq 0$, which concludes the proof. $\square$

There exists another (vectorial) conserved quantity, the momentum, defined as

$$P(\vec{u}(t)) = \int \partial_t u(t, x) \nabla u(t, x) dx \in \mathbb{R}^N.$$

PROPOSITION I.6.2. *Let $u \in C^2(\mathbb{R}^{1+N})$ be a solution of* (LW) *with finite energy. Then*

$$\forall t \in \mathbb{R}, \quad P(\vec{u}(t)) = P(u_0, u_1).$$

The proof of this proposition is left as an exercise (see Exercise I.7).

I.7. Lorentz Transformations. Time-like Hyperplanes

The *Minkowski spacetime* of dimension N is the space $\mathbb{R}^{1+N}$, equipped with the quadratic form of signature $(1, N)$:

$$g(X) = x_0^2 - \sum_{j=1}^{N} x_j^2 = t^2 - |x|^2 = {}^t X J X,$$

where ${}^t X$ is the transpose of X,

$$X = (x_0, x_1, \dots, x_N),\ t = x_0,\ x = (x_1, \dots, x_N),$$

and $J = [J_{\mu,\nu}]_{0 \leq \mu\ \nu \leq N}$ is the matrix such that $J_{0,0} = 1$, $J_{\ell,\ell} = -1$ if $\ell \in 1, \dots, N$, and $J_{\mu,\nu} = 0$ if $\mu \neq \nu$.

The Lorentz group $\mathrm{O}(1, N)$ is the group of real square matrices P of size $1 + N$ which leave the quadratic form g invariant, i.e., such that $g(PX) = g(X)$ for all X in $\mathbb{R}^{1+N}$. In other words, if P is a $(1 + N) \times (1 + N)$ matrix,

$$P \in \mathrm{O}(1, N) \iff {}^t P J P = J.$$

LEMMA I.7.1. *Let $P \in \mathrm{O}(1, N)$, $v \in C^2(\mathbb{R}^{1+N})$, and $w(X) = v(PX)$. Then*

$$(\partial_t^2 - \Delta) v = 0 \iff (\partial_t^2 - \Delta) w = 0.$$

PROOF. It can be noted that a function v of class C^2 on $\mathbb{R}^{1+N}$ satisfies the wave equation (LW) if and only if $\mathrm{Tr}(Jv'') = 0$, where v'' is the Hessian matrix $\left[\partial_{x_\mu} \partial_{x_\nu} v\right]_{0 \leq \substack{\mu \\ \nu} \leq N}$.

An explicit calculation yields $w''(X) = {}^t P v''(Px) P$, and thus

$$\mathrm{Tr}(J w''(X)) = \mathrm{Tr}(J\, {}^t P v''(PX) P) = \mathrm{Tr}(P J\, {}^t P v''(PX)) = \mathrm{Tr}(v''(PX)),$$

which proves the claimed result. $\square$

Two important examples of elements in $\mathrm{O}(1, N)$ are given by space rotations:

$$\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & R \end{bmatrix}, \quad R \in \mathrm{O}(N) \tag{I.7.1}$$

and Lorentz boosts, such as:

$$\mathcal{R}\sigma = \begin{bmatrix} R_\sigma & \mathbf{0} \\ \mathbf{0} & I_{N-1}. \end{bmatrix}, \quad R_\sigma = \begin{bmatrix} \cosh(\sigma) & \sinh(\sigma) \\ \sinh(\sigma) & \cosh(\sigma) \end{bmatrix}, \tag{I.7.2}$$

where I_{N-1} denotes the identity matrix $(N - 1) \times (N - 1)$ and $\sigma \in \mathbb{R}$. In these formulas, $\mathbf{0}$ always denotes a matrix of appropriate size.

In the preceding sections, we considered the Cauchy problem with initial conditions on a hyperplane in $\mathbb{R}^{1+N}$ of the form $\{t = t_0\}$. We now seek to solve the same problem by prescribing an initial condition on other hyperplanes. Therefore, we consider a hyperplane of the form

$$\Pi = \{X \in \mathbb{R}^{1+N}\ :\ {}^t A X = 0\} \tag{I.7.3}$$

where $A \in \mathbb{R}^{1+N} \setminus \{0\}$, $A = (a_0, a_1, \dots, a_N) = (a_0, a)$.

We have:

THEOREM I.7.2. *Suppose $|a_0| > |a|$. Then there exists a transformation $P \in \mathrm{O}(1, N)$ such that*

$$\Pi = P\left(\{(0, x),\ x \in \mathbb{R}^N\}\right).$$

The proof of this theorem is left as an exercise. See Exercise I.10.

If the condition of the preceding theorem is satisfied, we can therefore reduce the Cauchy problem with an initial condition

$$u_{\restriction\Pi} = u_0, \quad A \cdot \nabla u_{\restriction\Pi} = u_1,$$

to a Cauchy problem with initial conditions at $t = 0$ as treated above.

Definition I.7.3. The hyperplane Π is called *timelike* when $A = (a_0, a)$ with $a_0 \in \mathbb{R}$, $A \in \mathbb{R}^N$, and $|a_0| > |a|$.

It can be shown that Π is time-like if and only if the restriction of the quadratic form g to Π is negatively defined.

I.8. Equation with a source term

We now consider the equation with a source term (I.1.2). We will express this solution in terms of the propagator of the free equation (LW). For $(u_0, u_1) \in C^3 \times C^2(\mathbb{R}^3)$, let $S_L(t)(u_0, u_1)$ denote the solution of (LW) with initial data (u_0, u_1) at $t = 0$. We denote $S(t)u_1 = S_L(t)(0, u_1)$, so that

$$S_L(t)(u_0, u_1) = \frac{\partial}{\partial t}\left(S(t)u_0\right) + S(t)u_1.$$

For $u_1 \in C^2$, we recall that

$$(S(t)u_1)(x) = tM_{u_1}(t, x) = t\int_{S^2} u_1(x + ty)d\sigma(y).$$

Theorem I.8.1 (Duhamel's Formula). *Let $(u_0, u_1) \in (C^2 \times C^3)(\mathbb{R}^3)$ and $f \in C^2(\mathbb{R} \times \mathbb{R}^3)$. Then the equation* (I.1.2)*,* (ID) *has a unique C^2 solution, given by the formula:*

$$u(t) = S_L(t)(u_0, u_1) + \int_0^t S(t - s)f(s)ds.$$

Remark I.8.2. The Duhamel term $\int_0^t S(t - s)f(s)ds$ can be explicited, see (I.8.1).

Proof of Theorem I.8.1. Uniqueness follows immediately from Theorem I.4.1, since the difference of 2 solutions of (I.1.2) with the same source term f is a solution of (LW). For existence, taking into account Theorem I.5.2, it is sufficient to check that the function

$$U : (t, x) \mapsto \int_0^t S(t - s)f(s)ds$$

is C^2 and satisfies equation (I.1.2) with zero initial conditions.

We have:

$$U(t, x) = \frac{1}{4\pi}\int_0^t (t - s)\int_{S^2} f(s, x + (t - s)y)d\sigma(y)ds, \tag{I.8.1}$$

and the fact that U is C^2 follows from the theorem on differentiation under the integral sign.

Furthermore, using that $S(0)g = 0$ for any function g,

$$\frac{\partial U}{\partial t} = \int_0^t \frac{\partial}{\partial t}\Big(S(t - s)f(s)\Big)ds.$$

Upon further differentiation, we obtain

$$\begin{aligned}\frac{\partial^2 U}{\partial t^2} &= \frac{\partial}{\partial t}\Big(S(t - s)f(s)\Big)_{\restriction s=t} + \int_0^t \frac{\partial^2}{\partial t^2}\Big(S(t - s)f(s)\Big)ds \\ &= f(t) + \int_0^t \Delta\Big(S(t - s)f(s)\Big)ds = f(t) + \Delta U.\end{aligned}$$

where we used that $\frac{\partial}{\partial t}(S(t)g)_{\restriction t=0} = g$ for any function g of class C^2. $\square$

Remark I.8.3. Duhamel's formula is certainly not specific to dimension 3, as shown by the calculation leading to this formula, which is completely independent of dimension. The reader is invited to explicitly rewrite the solution of equation (I.1.2) when $N = 1$ and $N = 2$.

From Duhamel's formula, we deduce the energy inequality:

PROPOSITION I.8.4. *Let u be a C^2 solution of* (I.1.2) *with $N = 3$ with initial data (u_0, u_1), such that $f \in C^2(\mathbb{R}^{1+3})$. Suppose furthermore that (u_0, u_1) has finite energy, and for all $T > 0$,*

$$\int_{[-T,+T]} \sqrt{\int_{\mathbb{R}^3} |f(t,x)|^2 dx} dt < \infty.$$

Then for all $t > 0$,

$$\sqrt{E(\mathbf{u}(t))} \leq \sqrt{E(u_0, u_1)} + \int_0^t \sqrt{\int_{\mathbb{R}^3} |f(s,x)|^2 dx} ds.$$

PROOF. From Duhamel's formula and the conservation of energy for the free equation (LW), it suffices to verify that for all $T > 0$,

$$\sqrt{E\left(\int_0^t S(t-s)f(s)ds, \partial_t \int_0^t S(t-s)f(s)ds\right)} \leq \int_0^t \sqrt{\int_{\mathbb{R}^3} |f(s,x)|^2 dx} ds.$$

By conservation of energy (Theorem I.6.1), we have

$$E\left(S(t-s)f(s)ds, \partial_t \int_0^t S(t-s)f(s)ds\right) = \|f(s)\|_{L^2}^2.$$

This implies (using that $\sqrt{E}$ is a norm)

$$\sqrt{E\left(\int_0^t S(t-s)f(s)ds, \partial_t \int_0^t S(t-s)f(s)ds\right)} \leq \int_0^t |f(s)|_{L^2} ds,$$

completing the proof. □

I.9. Exercises

EXERCICE I.1. Let $f : \mathbb{R}^N \to \mathbb{R}$ $(N \geq 1)$. Suppose f is radial, meaning it depends only on the variable $r = |x| = \sqrt{x_1^2 + x_2^2 + \ldots + x_N^2}$. Denote $f(x) = g(|x|)$, where $g : [0, \infty[\to \mathbb{R}$.

(1) Show that f is continuous on $\mathbb{R}^N$ if and only if g is continuous on $[0, \infty[$.
(2) Show that f is C^1 on $\mathbb{R}^N$ if and only if g is C^1 on $[0, \infty[$ and $g'(0) = 0$.
(3) Show that for any $k \geq 2$, f is C^k on $\mathbb{R}^N$ if and only if g is C^k on $\mathbb{R}^N$ and $g^{(j)}(0) = 0$ for all odd integers $j \leq k$.
(4) Assuming f is C^1, determine $\frac{\partial f}{\partial x_j}$ in terms of g', $j = 1, \ldots, N$. Determine $g'(r)$ in terms of ∇f.
(5) Assuming f is C^2 on $\mathbb{R}^N$, prove the formula

$$\Delta f(x) = g''(|x|) + \frac{N-1}{|x|} g'(|x|).$$

In practice, we use the same notation (f) for functions f and g, and denote $g' = \frac{df}{dr}$, etc...

EXERCICE I.2. Let $k \geq 0$ and $f \in C^0(\mathbb{R}^3)$ be a *radial* function. Define a function u on $\mathbb{R} \times (\mathbb{R}^3 \setminus \{0\})$, radial with respect to the space variable, by

$$u(t,r) = \frac{1}{2r}\left((r+t)f(|r+t|) + (r-t)f(|r-t|)\right).$$

It is noted that u defines a function of class C^k on $\mathbb{R} \times \left(\mathbb{R}^3 \setminus \{0\}\right)$.

(1) Suppose that f is supported in the annulus $\{\frac{1}{2} \leq |x| \leq 2\}$ and is such that for $|\eta - 1| \leq 1/10$,

$$f(\eta) = \begin{cases} 2 - \eta & \text{if } \eta > 1 \\ \eta & \text{if } \eta < 1 \end{cases}.$$

Calculate $\lim_{r \to 0} u(t,r)$ when $t = 1$, $t > 1$, and $t < 1$ (close to 1). Conclude that u cannot be extended to a continuous function on $\mathbb{R} \times \mathbb{R}^3$.
(2) Similarly, give an example of a C^2 function f such that u cannot be extended to a C^2 function on $\mathbb{R} \times \mathbb{R}^3$.
(3) Assume f is C^3. Show that u defines a C^2 function on $\mathbb{R} \times \mathbb{R}^3$.

(4) Let g be a radial function on $\mathbb{R}^3$, C^2. Show that

$$u(t,r)=\frac{1}{2r}\int_{r-t}^{r+t}\sigma g(|\sigma|)d\sigma,$$

extends to a C^2 function on $\mathbb{R}^3$.

EXERCICE I.3 (Solution of the radial wave equation in odd dimension). Let $N\geq 3$ be an odd integer, written as $N=2k+1$. Let T_k be the operator defined by

$$T_k\phi=\left(r^{-1}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\phi(r)\right).$$

(1) Show that

$$T_k\varphi=\sum_{j=0}^{k-1}c_jr^{j+1}\phi^{(j)}r,$$

for some $c_j\in\mathbb{R}$. Determine c_0 and c_{k-1}.

(2) Show that for any function $\varphi\in C^{k+1}([0,+\infty[)$,

$$\frac{d^2}{dr^2}(T_k\varphi)=\left(r^{-1}\frac{d}{dr}\right)^k(r^{2k}\varphi'(r)).$$

Hint: You can start by verifying that the formula is true when $\varphi(r)=r^m$ for any integer m.

(3) Consider a solution $u(t,x)$ of the linear wave equation in space dimension N, radial with respect to the space variable. Suppose u is C^{k+1} on $\mathbb{R}^{1+N}$. Show

$$(\partial_t^2-\partial_r^2)(T_ku)=0.$$

Deduce an expression of T_ku in terms of u_0 and u_1.

(4) Express $u(t,r)$ in terms of u_0 and u_1 when $N=5$. What regularity of u_0 and u_1 is required for u to be C^2 on $\mathbb{R}^{1+5}$?

EXERCICE I.4. Let u be a solution of the wave equation (LW) in spatial dimension $N\geq 3$, radial with respect to the space variable. Recall that $\Delta u=\frac{d^2}{dr^2}+\frac{N-1}{r}\frac{d}{dr}$. Suppose $u\in C^2(\mathbb{R}^{1+N})$, with compactly supported initial data. Let

$$v(t,r)=\int_r^\infty\rho\partial_tu(t,\rho)d\rho.$$

Show that v defines a radial solution, of class C^2, to the wave equation in spatial dimension $N-2$.

EXERCICE I.5. Let $f\in C^k(\mathbb{R}^3)$. Show that the function M_f, defined by (I.5.3), is also of class C^k.

EXERCICE I.6. Let $u\in C^2(\mathbb{R}\times\mathbb{R}^N)$ be a solution of (LW) with finite energy. Show

$$\forall\varepsilon>0,\ \exists R>0,\ \forall t\in\mathbb{R},\quad\int_{|x|>R+|t|}e_u(t,x)dx\leq\varepsilon.$$

EXERCICE I.7 (Conservation of momentum). (1) Let u be a C^2 solution of (LW) on $\mathbb{R}\times\mathbb{R}^N$, and $j\in 1,\ldots N$. Let $p_{j,u}(t,x)=\partial_{x_j}u(t,x)\partial_tu(t,x)$. Show

$$\frac{\partial p_{j,u}}{\partial t}=\frac{1}{2}\frac{\partial}{\partial x_j}\left((\partial_tu)^2-|\nabla u|^2\right)+\nabla\cdot V,$$

where V is a certain C^1 vector field to be specified.

(2) Assume that (u_0,u_1) has finite energy. Justify that

$$P_j(\vec{u}(t))=\int_{\mathbb{R}^N}p_{j,u}(t,x)dx$$

is defined for all times. Show that this quantity is independent of time. You can start by considering a local version of the momentum

$$\int_{[-R,R]^N}p_{j,u}(t,x)dx\text{ or }\int_{\mathbb{R}^N}p_{j,u}(t,x)\varphi\left(\frac{x}{R}\right)dx$$

then let R tend to $+\infty$. Here φ denotes a C^2 function with compact support equal to 1 in a neighborhood of the origin.

EXERCICE I.8. Suppose $N = 1$ or $N = 2$. Let u be the solution of (LW), (ID), with $(u_0, u_1) \in C^3 \times C^2$ (if $N = 2$) or $C^2 \times C^1$ (if $N = 1$).

Show that if $u_1 \geq 0$ and $u_0 = 0$ then $u(t, x)$ has the sign of t for all x and $t \neq 0$.

When $N = 1$, give a weaker sufficient condition on (u_0, u_1) such that:

$$\forall t \geq 0,\ \forall x \in \mathbb{R}, \quad u(t, x) \geq 0.$$

EXERCICE I.9. Assume $N = 1$ or $N = 2$. Let u be a solution of (I.1.2), with $u_0 = u_1 = 0$, and f of class C^1 (if $N = 1$) or C^2 (if $N = 2$). Express u in terms of f.

EXERCICE I.10.
(1) Prove Theorem I.7.2. You can use compositions of transformations defined in (I.7.1) and (I.7.2).
(2) Prove that Π is timelike if and only if the restriction of the quadratic form g to Π is negatively defined.
(3) Under what condition on A does there exist $B = (b_0, b_1, \ldots, b_N) \in \mathbb{R}^{N+1}$ such that the function
$$e^{A \cdot X + iB \cdot X}$$
is a solution of (LW)?
(4) Now assume that the hyperplane Π is not of timelike type. Let $Y \notin \Pi$. Construct a sequence of solutions $(u_n)_n$ of (LW) such that $u_n(X) = 0$ on Π, such that for any differential operator $D = \prod_{j=1}^{N} \partial_{x_1}^{\alpha_1} \ldots \partial_{x_N}^{\alpha_N}$ (of arbitrarily large order), there exists $C > 0$ such that $|Du_n(X)| \leq Ce^{-n}$ on Π, but $|u_n(Y)| \to +\infty$ as $n \to \infty$.

CHAPTER II

The Linear Equation in Sobolev Spaces

II.1. Reminders on the Fourier Transform

Here, we recall the definition of the Fourier transform on $\mathbb{R}^N$, in the most general framework possible, that of tempered distributions. We omit the proofs. For more details, one can consult, for example, the foundational writings of Laurent Schwartz [**25**], the course of Jean-Michel Bony [**4**], as well as [**2**, Section 1.2] for a quick introduction, and [**20**] for a more in-depth exposition (the first two references are in French).

We begin by introducing a notation: a *multi-index* is an element $\alpha = (\alpha_1, \dots, \alpha_N)$ of $\mathbb{N}^N$. The order of α is $|\alpha| = \sum_{j=1}^N \alpha_j$. The derivative with respect to α of a function f of class $C^{|\alpha|}$ on $\mathbb{R}^N$ is then defined by:

$$\partial_x^\alpha \varphi = \prod_{j=1}^N \partial_{x_j}^{\alpha_j} f.$$

1.a. Fourier Transform on $\mathcal{S}$.

Definition II.1.1. The Schwartz space $\mathcal{S}(\mathbb{R}^N)$ is the space of functions f of class C^∞ on $\mathbb{R}^N$ such that for every $p \in \mathbb{N}$,

$$N_p(f) := \sup_{x\in\mathbb{R}^N \ |\alpha|\leq p} (1+|x|)^p |\partial_x^\alpha f(x)| < \infty.$$

It can be observed that each N_p is a norm on $\mathcal{S}(\mathbb{R}^N)$, but N_p is not complete for any of these norms.

We equip $\mathcal{S}(\mathbb{R}^N)$ with the distance function

$$d(\varphi,\psi) = \sup_{p\geq 0} \frac{1}{2^p} \min\Big(N_p(\varphi-\psi), 1\Big). \tag{II.1.1}$$

It can be seen that $d(\varphi_n, \varphi)$ tends to 0 as n tends to infinity if and only if $N_p(\varphi_n - \varphi)$ tends to 0 for every p.

One can check that the metric space $(\mathcal{S}, d)$ is complete.[1]

The Fourier transform of an element φ of $\mathcal{S}$ is defined by the formula

$$\widehat{\varphi}(\xi) = \mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi}\varphi(x)dx. \tag{II.1.2}$$

It is verified that $\mathcal{F}$ is a continuous application from $\mathcal{S}$ into $\mathcal{S}$.

Fubini's theorem immediately implies the duality formula:

$$\int_{\mathbb{R}^N} \widehat{\varphi}(\xi)\psi(\xi)d\xi = \int_{\mathbb{R}^N} \varphi(x)\widehat{\psi}(x)dx, \tag{II.1.3}$$

for $\varphi, \psi \in \mathcal{S}(\mathbb{R}^N)$.

The Fourier transformation is a bijection of $\mathcal{S}$: by defining

$$\overline{\mathcal{F}}(\psi)(x) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{ix\cdot\xi}\psi(\xi)d\xi = \frac{1}{(2\pi)^N}\widehat{\psi}(-x), \tag{II.1.4}$$

we have the *Fourier inversion formula*: for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$,

$$\mathcal{F}\overline{\mathcal{F}}\varphi = \overline{\mathcal{F}}\mathcal{F}\varphi = \varphi. \tag{II.1.5}$$

By combining the Fourier inversion formula (II.1.5) and the duality formula (II.1.3), we obtain the Plancherel theorem:

$$\int_{\mathbb{R}^N} \varphi(x)\overline{\psi}(x)dx = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} \widehat{\varphi}(\xi)\overline{\widehat{\psi}(\xi)}d\xi. \tag{II.1.6}$$

[1] Such a vector space, equipped with a countable family of semi-norms, and which is complete as a metric space (where the distance function is defined as in (II.1.1)), is called a *Fréchet space*. It is a natural generalization of a Banach space when a unique norm is not sufficient to ensure completeness.

The Fourier transform exchanges multiplication by powers of x and differentiation. For all $\varphi \in \mathcal{S}(\mathbb{R}^N)$

$$\forall \alpha \in \mathbb{N}^N, \quad \frac{1}{i^{|\alpha|}}\mathcal{F}\partial_x^\alpha \varphi = \xi^\alpha \widehat{\varphi}(\xi), \quad i^{|\alpha|}\mathcal{F}(x^\alpha \varphi) = \partial_\xi^\alpha \widehat{\varphi}(\xi). \tag{II.1.7}$$

1.b. Fourier Transform of Tempered Distributions.

Definition II.1.2. The space $\mathcal{S}'(\mathbb{R}^N)$ of *tempered distributions* is the topological dual of $\mathcal{S}(\mathbb{R}^N)$, i.e., the vector space of continuous linear forms on $\mathcal{S}$.

In the definition, continuity must be interpreted in the sense of the topology induced by the distance d defined by (II.1.1). It is easily verified that a linear form f on $\mathcal{S}$ is an element of $\mathcal{S}'$ if and only if:

$$\exists p \in \mathbb{N}, \quad \forall \varphi \in \mathcal{S}, \quad |\langle f, \varphi\rangle| \leq C N_p(\varphi).$$

We equip $\mathcal{S}'$ with the topology of pointwise convergence: a sequence $(f_n)_n$ of elements of $\mathcal{S}'$ converges to f in $\mathcal{S}'$ if and only if

$$\forall \varphi \in \mathcal{S}, \quad \lim_{n\to\infty} \langle f_n, \varphi\rangle = \langle f, \varphi\rangle .$$

Several function spaces continuously embed into $\mathcal{S}'(\mathbb{R}^N)$ in the following manner. If f is a measurable, locally integrable function on f such that

$$\forall R > 0, \quad \int_{|x|\leq R} |f(x)| dx \leq C(1+R)^C$$

for some constant $C > 0$, we define an element L_f of $\mathcal{S}'(\mathbb{R}^N)$ by

$$\langle L_f, \varphi\rangle = \int_{\mathbb{R}^N} f(x)\varphi(x) dx.$$

The preceding application is injective, i.e., L_f is null if and only if f is null almost everywhere on $\mathbb{R}^N$. We then identify f with the linear form L_f, also denoted f. The preceding identification allows us to consider $\mathcal{S}$, Lebesgue spaces $L^p(\mathbb{R}^N)$ $(1 \leq p \leq \infty)$, C_b^k (the space of C^k functions on $\mathbb{R}^N$ that are bounded along with all their derivatives up to order k) as subspaces of $\mathcal{S}'$.

Examples of tempered distributions that are not functions are given by the Dirac delta function at a, denoted δ_a and defined by $\langle \delta_a, \varphi\rangle = \varphi(a)$, as well as the surface measure σ on the sphere S^{N-1}, defined by:

$$\langle \sigma, \varphi\rangle = \int_{S^{N-1}} \varphi(y) d\sigma(y).$$

By duality, several actions can be defined on the elements of $\mathcal{S}'$.

Differentiation. Let $\alpha \in \mathbb{N}^N$ and $f \in \mathcal{S}'$. The *derivative* of f of order α is the element ∂_x^α of $\mathcal{S}'$ defined by:

$$\forall \varphi \in \mathcal{S}, \quad \langle \partial_x^\alpha f, \varphi\rangle = (-1)^{|\alpha|} \langle f, \partial_x^\alpha \varphi\rangle .$$

The integration by parts formula shows that if $f \in C_b^{|\alpha|}$, its derivative of order α in the sense of distributions coincides with its derivative in the classical sense.

Multiplication by a Function. We denote by $\mathcal{P} = \mathcal{P}(\mathbb{R}^N)$ the space of C^∞ functions with *slow growth*, i.e., such that

$$\forall \alpha, \quad \exists M, C > 0 \quad \forall x \in \mathbb{R}^N, \quad |\partial_x^\alpha g(x)| \leq C(1+|x|)^M. \tag{II.1.8}$$

It is easy to check that the multiplication by an element of $\mathcal{P}$ defines a continuous mapping from $\mathcal{S}$ into $\mathcal{S}$. We then define, for $f \in \mathcal{S}'$ and $g \in \mathcal{P}$, the product fg by:

$$\langle fg, \varphi\rangle = \langle f, g\varphi\rangle .$$

The product fg is an element of $\mathcal{S}'$. Fixing $g \in \mathcal{P}$, $f \mapsto fg$ is a continuous mapping from $\mathcal{S}'$ into $\mathcal{S}'$.

Fourier Transform. We define the Fourier transform of an element f of $\mathcal{S}'$ by

$$\forall \varphi \in \mathcal{S}, \quad \left\langle \widehat{f}, \varphi \right\rangle = \langle f, \widehat{\varphi}\rangle$$

The duality formula (II.1.3) shows that if $f \in \mathcal{S}$, its Fourier transform according to formula (II.1.2) and its Fourier transform in the sense of $\mathcal{S}'$ coincide.

We recall that $L^1(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N)$ are identified with subspaces of $\mathcal{S}'(\mathbb{R}^N)$. The Fourier transform on $\mathcal{S}'$ thus applies to elements of these two spaces. On $L^1(\mathbb{R}^N)$, we recover the Fourier transform in the classical sense.

PROPOSITION II.1.3 (Fourier Transform in L^1). *Let $f \in L^1(\mathbb{R}^N)$, and $\widehat{f}$ be its Fourier transform in $\mathcal{S}'$. Then $\widehat{f}$ can be identified with the continuous function given by the formula:*

$$\widehat{f}(\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x)dx.$$

The second proposition immediately follows from the Plancherel theorem:

PROPOSITION II.1.4 (Fourier Transform in L^2). *Let $f \in L^2(\mathbb{R}^N)$ then $\widehat{f} \in L^2(\mathbb{R}^N)$ and*

$$\|f\|_{L^2} = \frac{1}{(2\pi)^{N/2}} \|\widehat{f}\|_{L^2}.$$

Indeed, the Fourier inversion formula in $\mathcal{S}'$ (see below) implies that $f \mapsto \frac{1}{(2\pi)^{N/2}}\hat{f}$ is an isometry of $L^2(\mathbb{R}^N)$.

The properties of the Fourier transform on $\mathcal{S}$ are transmitted by duality to the Fourier transform:

- We define the inverse Fourier transform $\overline{F}$ of an element f of $\mathcal{S}'$ by

$$\left\langle \overline{F} f, \varphi \right\rangle = \left\langle f, \overline{F}\varphi \right\rangle.$$

  Then we have the Fourier inversion formula:

$$\forall f \in \mathcal{S}', \quad \overline{\mathcal{F}}\mathcal{F} f = \mathcal{F}\overline{\mathcal{F}} f = f.$$

- Property (II.1.7) remains valid for $\varphi \in \mathcal{S}'$.

II.2. Sobolev Spaces

2.a. Definition. (cf [**2**, Section 1.3]) Here, we will mainly focus on homogeneous Sobolev spaces based on L^2. We refer to the exercise sheet for classical Sobolev spaces H^σ.

Sobolev spaces on $\mathbb{R}^N$ are easily defined using the Fourier transform:

DEFINITION II.2.1. Let $\sigma \in \mathbb{R}$. The Sobolev space $\dot{H}^\sigma(\mathbb{R}^N)$ is the set of $f \in \mathcal{S}'(\mathbb{R}^N)$ such that $\widehat{f} \in L^1(K)$ for every compact set K, and such that the following quantity is finite:

$$\|f\|^2_{\dot{H}^\sigma} = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} |\xi|^{2\sigma} |\widehat{f}(\xi)|^2 d\xi.$$

The space $\dot{H}^\sigma$, equipped with the inner product:

$$(f, g)_{\dot{H}^\sigma} = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} |\xi|^{2\sigma} \widehat{f}(\xi)\overline{\widehat{g}(\xi)} d\xi$$

is a pre-Hilbert space.

THEOREM II.2.2. *The space $\dot{H}^\sigma(\mathbb{R}^N)$ is complete if and only if $\sigma < N/2$. In this case, the vector space $\mathcal{S}_0$ of functions in $\mathcal{S}$ whose Fourier transform vanishes in a neighborhood of 0 is dense in $\dot{H}^\sigma(\mathbb{R}^N)$.*

Note that $\dot{H}^0$ is exactly the space L^2.

2.b. Sobolev Inequalities. We have the following Sobolev embedding on $\mathbb{R}^N$.

THEOREM II.2.3. *Let $\sigma \in]0, N/2[$, and $p \in (2, \infty)$ such that $\frac{1}{p} = \frac{1}{2} - \frac{\sigma}{N}$. Then $\dot{H}^\sigma(\mathbb{R}^N)$ is contained in L^p with continuous embedding.*

The result is well-known. We give a proof based on the Fourier transform, which yields a slightly stronger result that we will in Chapter VI.

By the density result in Theorem II.2.2, it suffices to show that there exists a constant $C > 0$ such that

$$\forall f \in \mathcal{S}(\mathbb{R}^N), \quad \|f\|_{L^p(\mathbb{R}^N)} \leq C\|f\|_{\dot{H}^\sigma(\mathbb{R}^N)}. \tag{II.2.1}$$

Let $f \in \mathcal{S}$. We denote[2]

$$\|f\|^2_{\dot{B}^\sigma} = \sup_{k\in\mathbb{Z}} \frac{1}{(2\pi)^N} \int_{2^k \leq |x| \leq 2^{k+1}} |\xi|^{2\sigma} |\widehat{f}(\xi)|^2 d\xi,$$

and observe that $\|f\|_{\dot{B}^\sigma} \leq \|f\|_{\dot{H}^\sigma}$. We will prove the following result, which implies (II.2.1):

THEOREM II.2.4 (Improved Sobolev Inequality). *Let σ and p be as in the previous theorem. Then there exists a constant $C > 0$ such that*

$$\forall f \in \mathcal{S}(\mathbb{R}^N), \quad \|f\|^p_{L^p} \leq \|f\|^{p-2}_{\dot{B}^\sigma} \|f\|^2_{\dot{H}^\sigma}.$$

[2]This norm defines the Besov space $\dot{B}^\sigma_{2,\infty}$. See [**2**, Section 2.3] for the defintion of general Besov spaces.

NOTATION II.2.5. Let φ be a function on $\mathbb{R}^N$. For $u \in \mathcal{S}'(\mathbb{R}^N)$, we denote

$$\varphi(D)u = \overline{\mathcal{F}}\left(\varphi(\xi)\widehat{u}(\xi)\right).$$

The operator $\varphi(D)$ is called Fourier multiplier (with symbol φ).

The tempered distribution $\varphi(D)u$ is not well-defined for all functions φ and $u \in \mathcal{S}'$: we need $\varphi\,\widehat{u}$ to define a tempered distribution. This is for example the case if $\varphi \in L^\infty$ and $u \in \dot{H}^\sigma$ (in this case $\varphi(D)u \in \dot{H}^\sigma$), or if $\varphi \in \mathcal{P}(\mathbb{R}^N)$ (the space of C^∞ functions with slow growth i.e. that satisfy (II.1.8)).

PROOF. We use a method introduced by Chemin and Xu in [**6**]. We fix a parameter $A > 0$ and decompose f into a *high-frequency* part $f_{>A}$ and a *low-frequency* part $f_{<A}$:

$$f_{>A} = \overline{\mathcal{F}}\left(\mathbb{1}_{|\xi|>A}\widehat{f}(\xi)\right) = \mathbb{1}_{|D|>A}f, \quad f_{<A} = \mathbb{1}_{|D|<A}f = 1 - f.$$

Let $k(A)$ be the largest integer such that $2^{k(A)} \leq A$. By using the Cauchy-Schwarz inequality, then the fact that $\sigma < N/2$, we obtain:

$$\begin{aligned}|f_{<A}(x)| = \frac{1}{(2\pi)^N}\left|\int_{|\xi|<A} e^{ix\cdot\xi}\widehat{f}(\xi)d\xi\right| &\leq \frac{1}{(2\pi)^N}\sum_{k\leq k(A)}\int_{2^k\leq|\xi|\leq 2^{k+1}}|\widehat{f}(\xi)|d\xi \\ &\leq \frac{1}{(2\pi)^N}\sum_{k\leq k(A)} 2^{k(N/2-\sigma)}\left(\int_{2^k\leq|\xi|\leq 2^{k+1}}|\xi|^{2\sigma}|\widehat{f}(\xi)|^2 d\xi\right)^{1/2} \leq C_N A^{N/2-\sigma}\|f\|_{\dot{B}^\sigma},\end{aligned}$$

where C_N depends only on the dimension N. Then we write (using Fubini's equality):

$$\begin{aligned}\|f\|_{L^p}^p = \int |f(x)|^p dx = \int_{\mathbb{R}^N} p\int_0^{|f(x)|}\lambda^{p-1}d\lambda dx \\ = p\int_0^{+\infty}\lambda^{p-1}\left|\left\{x \in \mathbb{R}^N \;:\; |f(x)| \geq \lambda\right\}\right| d\lambda,\end{aligned}$$

where $|S|$ denotes the Lebesgue measure of the measurable subset S of $\mathbb{R}^N$. Let $A(\lambda)$ be such that

$$C_N A(\lambda)^{\frac{N}{2}-\sigma}\|f\|_{\dot{B}^\sigma} = \lambda/2.$$

For any x in $\mathbb{R}^N$,

$$|f_{<A(\lambda)}(x)| \leq \frac{\lambda}{2}.$$

Thus $|f(x)| > \lambda \Longrightarrow |f_{>A(\lambda)}(x)| > \lambda/2$. Hence:

$$\|f\|_{L^p}^p \leq p\int_0^\infty \lambda^{p-1}\left|\left\{x \in \mathbb{R}^N \;:\; |f_{>A(\lambda)}(x)| > \lambda/2\right\}\right| d\lambda$$

By integrating $|f_{>A(\lambda)}|^2$ over the set $\left\{x \in \mathbb{R}^N \;:\; |f_{>A(\lambda)}(x)| > \lambda/2\right\}$, we get

$$\left|\left\{x \in \mathbb{R}^N \;:\; |f_{>A(\lambda)}(x)| > \lambda/2\right\}\right| \leq \frac{4}{\lambda^2}\|f_{>A(\lambda)}\|_{L^2}^2.$$

Combining with the Plancherel theorem, then Fubini's theorem, we obtain

$$\begin{aligned}\|f\|_{L^p}^p \leq \frac{4p}{(2\pi)^N}\int_0^\infty \lambda^{p-1}\int_{|\xi|>A(\lambda)}|\widehat{f}(\xi)|^2 d\xi d\lambda \\ = \frac{4p}{(2\pi)^N}\int_{\mathbb{R}^N}\left|\widehat{f}(\xi)\right|^2\int_0^{c(f,\xi)}\lambda^{p-3}d\lambda d\xi = C_{p,N}\int_{\mathbb{R}^N}\left|\widehat{f}(\xi)\right|^2 c(f,\xi)^{p-2}d\xi,\end{aligned}$$

where $c(f,\xi) = 2C_N\|f\|_{\dot{B}^\sigma}|\xi|^{\frac{N}{2}-s}$, and $C_{p,N}$ depends only on N and p. It can be easily verified that $(\frac{N}{2}-s)(p-2) = 2s$, which proves the announced inequality. $\square$

We will focus more particularly on the case $s = 1$. According to the above, the Sobolev space $\dot{H}^1(\mathbb{R}^N)$, $N \geq 3$, is a Hilbert space, contained in $L^{\frac{2N}{N-2}}$, which can be defined as the closure of the space $\mathcal{S}(\mathbb{R}^N)$ (or $C_0^\infty(\mathbb{R}^N)$) for the $\dot{H}^1(\mathbb{R}^N)$-norm. We can characterize this norm with the first-order partial derivatives of f. Indeed,

$$\|f\|_{\dot{H}^1}^2 = \frac{1}{(2\pi)^N}\int |\xi|^2\left|\widehat{f}(\xi)\right|^2 d\xi = \sum_{j=1}^N\int\left|\xi_j\widehat{f}(\xi)\right|^2 d\xi,$$

which shows by Plancherel's theorem and formula (II.1.7)

$$\|f\|_{\dot{H}^1}^2 = \int |\nabla f(x)|^2 dx.$$

The attentive reader will have noticed that the space $\dot{H}^1(\mathbb{R}^N)$ is not the set of $\varphi \in \mathcal{S}'(\mathbb{R}^N)$ such that for all j, $\partial_{x_j}\varphi \in L^2(\mathbb{R}^N)$: indeed, constant functions are in this space, but not in $\dot{H}^1(\mathbb{R}^N)$.

II.3. The Wave Equation in the Schwartz Space

Let $(u_0, u_1) \in \mathcal{S}(\mathbb{R}^N)$. We will write the solution u of (LW), (ID) using the Fourier transformation. We start with a formal calculation, assuming that $u(t) \in \mathcal{S}$ for all t (which we will prove later). We denote $\widehat{u}(t)$ as the Fourier transform of u with respect to the spatial variable, i.e.,

$$\widehat{u}(t,\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} u(t,x)dx.$$

Thus, we have

$$\widehat{\Delta u}(t,\xi) = -|\xi|^2 \widehat{u}(t,\xi),$$

and the wave equation (LW) is formally equivalent to the linear differential equation

$$\partial_t^2 \widehat{u}(t,\xi) + |\xi|^2 \widehat{u}(t,\xi) = 0,$$

where the variable ξ is considered as a parameter. The solution to this equation, with initial conditions $(\widehat{u}(0), \partial_t \widehat{u}(0)) = (u_0, u_1)$, yields

$$\widehat{u}(t,\xi) = \cos(t|\xi|)\widehat{u}_0(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\widehat{u}_1(\xi),$$

or, with the previously introduced notation,

$$u(t) = \cos(t|D|)u_0 + \frac{\sin(t|D|)}{|D|}u_1. \tag{II.3.1}$$

THEOREM II.3.1. *Let $(u_0, u_1) \in \mathcal{S}(\mathbb{R}^N)^2$. Then u defined by* (II.3.1) *is an element of $C^\infty(\mathbb{R}\times\mathbb{R}^N)$. It is the unique C^2 solution of* (LW), (ID).

PROOF. Uniqueness follows from Theorem I.4.1. Hence, it suffices to prove that u, defined by (II.3.1), is C^∞ and satisfies (LW), (ID). We have

$$u(t,x) = \frac{1}{(2\pi)^N} \int \mathbb{R}^N e^{ix\cdot\xi} \left(\cos(t|\xi|)\widehat{u}_0(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\widehat{u_1}(\xi)\right) d\xi.$$

By writing

$$\frac{\sin(t|\xi|)}{|\xi|} = t\sum_{k\geq 0} \frac{(-1)^k (t|\xi|)^{2k}}{(2k+1)!},$$

we see that it is a C^∞ function of (t,ξ). Moreover, $\frac{\left|\partial_t^j \sin(t|\xi|)\right|}{|\xi|} \leq |t||\xi|^j$. Similarly, $(t,\xi) \mapsto \cos(t|\xi|)$ is C^∞ and $\left|\partial_t^j \cos(t|\xi|)\right| \leq |\xi|^j$. Using the fact that $\widehat{u}_0$ and $\widehat{u}_1$ are elements of $\mathcal{S}(\mathbb{R}^N)$, by the theorem of differentiation under the integral sign, we obtain that u is C^∞ and satisfies (LW). The Fourier inversion formula shows that u also satisfies the initial conditions (ID). ☐

II.4. The wave equation in Sobolev spaces

4.a. The equation in general homogeneous Sobolev spaces. Let $(u_0, u_1) \in \dot{H}^\sigma \times \dot{H}^{\sigma-1}$, $\sigma < N/2$. We define as before u by (II.3.1). We also define the formal derivative of u with respect to time:

$$u'(t,x) = \cos(t|D|)u_1 - |D|\sin(t|D|)u_0.$$

Then u and u' satisfy the following properties:

CLAIM II.4.1. *$u \in C^0(\mathbb{R}, \dot{H}^\sigma)$, $u' \in C^0(\mathbb{R}, \dot{H}^{\sigma-1})$, $u(0) = u_0$, $u'(0) = u_1$.*

PROOF. Using that $\widehat{u}_0 \in L^2(|\xi|^{2\sigma}d\xi)$ and $\widehat{u}_1 \in L^2(|\xi|^{2\sigma-2}d\xi)$, it is easy to see that

$$\widehat{u} \in C^0(\mathbb{R}, L^2(|\xi|^{2\sigma}d\xi)), \quad \widehat{u'} \in C^0(\mathbb{R}, L^2(|\xi|^{2\sigma-2}d\xi)), \tag{II.4.1}$$

which yields the announced continuity property. The facts that $u(0) = u_0$ and $u'(0) = u_1$ follow immediately from the definition. ☐

CLAIM II.4.2. $\forall t, \quad \|(u(t),u'(t))\|_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}} = \|(u_0,u_1)\|_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}}$.

PROOF.

$$\int_{\mathbb{R}^N}|\widehat{u}(t,\xi)|^2|\xi|^{2\sigma}d\xi+\int_{\mathbb{R}^N}\widehat{u'}(t,\xi)|\xi|^{2\sigma-2}d\xi$$
$$=\int_{\mathbb{R}^N}\left|\cos(t|\xi|)\widehat{u}_0(\xi)+\frac{\sin(t|\xi|)}{|\xi|}\widehat{u}_1(\xi)\right|^2|\xi|^{2\sigma}d\xi$$
$$+\int_{\mathbb{R}^N}\Big|-|\xi|\sin(t|\xi|)\widehat{u}_0(\xi)+\cos(t|\xi|)\widehat{u}_1(\xi)\Big|^2|\xi|^{2\sigma-2}d\xi$$
$$=\int_{\mathbb{R}^N}\left(|\widehat{u}_0(\xi)|^2+|\widehat{u}_1(\xi)|^2|\xi|^{-2}\right)|\xi|^{2\sigma}d\xi,$$

which gives the desired property. □

CLAIM II.4.3. *Let $(u_{0,n},u_{1,n})\in(\mathcal{S}_0(\mathbb{R}^N))^2$ such that $(u_{0,n},u_{1,n})$ converges to (u_0,u_1) in $\dot{H}^\sigma\times\dot{H}^{\sigma-1}$. Let u_n be the solution of* (LW) *with data $(u_{0,n},u_{1,n})$. Then*

$$\lim_{n\to\infty}\sup_{t\in\mathbb{R}}\|u_n(t)-u(t)\|_{\dot{H}^\sigma}+\|\partial_t u_n(t)-u'(t)\|_{\dot{H}^{\sigma-1}}=0.$$

PROOF. It follows immediately from the preceding claim, applied to $(u-u_n,u'-\partial_t u_n)$. □

CLAIM II.4.4. *One can identify u with a distribution on $\mathbb{R}\times\mathbb{R}^N$, and it satisfies the wave equation* (LW) *in the distributional sense. Furthermore $u'=\partial_t u$ in the sense of distribution.*

PROOF. We first give a "concrete" proof of these facts for the reader which is not familiar with the theory of distributions, assuming that σ is large enough so that the object considered are all functions on $\mathbb{R}\times\mathbb{R}^N$.

Let $\sigma\geq 0$. We let u_n be as in Claim II.4.3. Using that u_n is a C^∞ solution of (LW) and integrating by parts, we obtain

$$\iint u_n(t,x)(\partial_t^2-\Delta)\varphi dxdt=0.$$

Using the Sobolev embedding $\dot{H}^\sigma\subset L^p$, $\frac{1}{p}=\frac{1}{2}-\frac{\sigma}{N}$, and the point (II.4.3), we see that

$$\lim_{n\to\infty}\|u-u_n\|_{L^p(K)}=0,$$

for all compact K of $\mathbb{R}^N$. This implies

$$0=\lim_{n\to\infty}\iint u_n(t,x)(\partial_t^2-\Delta)\varphi dxdt=\lim_{n\to\infty}\iint u(t,x)(\partial_t^2-\Delta)\varphi dxdt,$$

and thus

$$\forall\varphi\in C_0^\infty(\mathbb{R}\times\mathbb{R}^N),\quad\iint u(\partial_t^2-\Delta)\varphi dtdx=0,$$

which is precisely the meaning of $\partial_t^2u-\Delta u=0$ in the distributional sense.

Let $\sigma\geq 1$. The equality

$$\partial_t u_n=-|D|\sin(t|D|)u_{0,n}+\cos(t|D|)u_{1,n}.$$

holds by differentiation below the integral sign. By integration by parts,

$$\forall\varphi\in C_0^\infty(\mathbb{R}\times\mathbb{R}^N),\quad\iint\partial_t u_n\varphi dtdx=-\iint u_n\partial_t\varphi dtdx,$$

Letting $n\to\infty$, we obtain

$$\forall\varphi\in C_0^\infty(\mathbb{R}\times\mathbb{R}^N),\quad\iint u'\varphi dtdx=-\iint u\partial_t\varphi dtdx,$$

which means that $u'=\partial_t u$ in the distributional sense.

The proof for general σ is essentially the same, and can be skipped by the reader who is not familiar with distributions.

If $\varphi\in C_0^\infty(\mathbb{R}\times\mathbb{R}^N)$ (the space of smooth functions with compact support on $\mathbb{R}\times\mathbb{R}^N$), one defines the action of u on $\mathcal{S}$ by

$$\langle u,\varphi\rangle=\int_{-\infty}^{+\infty}\langle u(t),\varphi(t)\rangle_{\mathcal{S}',\mathcal{S}}\,dt,$$

where $\varphi(t)$ is the function $t\mapsto\varphi(t,\cdot)$. It is a straightforward exercise to prove that u is well-defined and that is is a distribution on $\mathbb{R}\times\mathbb{R}^N$. The facts that u satisfies the wave equation in the distributional sense and that

$u'(t) = \partial_t u(t)$ follow immediately from Claim II.4.3, that implies that $\lim u_n = u$ in the distributional sense, where u_n is a in Claim II.4.3. This last fact is an immediate consequence of Claim II.4.3. $\square$

From now on, we will use the formula (II.1.2) as the definition of the solution u of (LW), (ID) with $(u_0, u_1) \in \left(\mathcal{S}(\mathbb{R}^N)\right)^2$. The preceding claims show that such a u is a limit of smooth, classical solutions of (LW), (ID), and that it satisfies (LW) in a weak sense. Also, we have

$$\partial_t u = -|D|\sin(t|D|)u_0 + \cos(t|D|)u_1$$

in the sense of distribution. In the sequel, we will always use the notation $\partial_t u$ to denote this quantity.

4.b. The wave equation in the energy space. Of particular interest for us is the case $s = 1$. We will call "finite energy solutions" the weak solutions with initial data $\dot{H}^1 \times L^2$ given by the preceding subsection in the case $s = 1$, $N \geq 3$. We will focus on the case $N = 3$. We note that if $(u_0, u_1) \in (C^3 \times C^2)(\mathbb{R}^3) \cap \left(\dot{H}^1 \times L^2\right)(\mathbb{R}^3)$, we have two ways of defining the solution u: by integrals on spheres, as in Theorem I.5.2, and using the Fourier transform, i.e. by formula (II.3.1). Let us prove that these two definitions coincide:

PROPOSITION II.4.5. *Let $u \in C^2(\mathbb{R}\times\mathbb{R}^3)$ be a solution of* (LW), (ID). *Assume furthermore $u_0 = u(0) \in \dot{H}^1$, $u_1 = \partial_t u(0) \in L^2$. Then*

$$u(t) = \cos(t|D|)u_0 + \frac{\sin(t|D|)}{|D|}u_1, \quad \partial_t u(t) = -|D|\sin(t|D|)u_0 + \cos(t|D|)u_1.$$

PROOF. Let $(u_{0,n}, u_{1,n}) \in \left(\mathcal{S}(\mathbb{R}^N)\right)^2$ with

$$\lim_{n\to\infty} \|u_{0,n} - u_0\|_{\dot{H}^1} + \|u_{1,n} - u_1\|_{L^2} = 0.$$

Let u_n be the corresponding solution of (LW) given by (II.3.1) (note that by uniqueness it is also the solution given by Theorem I.5.2). Since $u - u_n$ is a C^2, finite energy solution of (LW), Theorem I.6.1 yields

$$\forall t, \quad \|u(t) - u_n(t)\|^2_{\dot{H}^1} + \|\partial_t u(t) - \partial_t u_n(t)\|^2_{L^2} = \|u_0 - u_{0,n}\|^2_{\dot{H}^1} + \|u_1 - u_{1,n}\|^2_{L^2},$$

which tends to 0 as n goes to infinity. This proves the result, since $u_n(t)$ converges to $\cos(t|D|)u_0 + \frac{\sin(t|D|)}{|D|}u_1$ in $\dot{H}^1(\mathbb{R}^3)$ and $\partial_t u_n(t)$ converges to $-|D|\sin(t|D|)u_0 + \cos(t|D|)u_1$ in L^2 by Claim II.4.3. $\square$

Using the approximation of finite energy solutions by solutions with initial data in $\mathcal{S}$, we can transfer several results of Chapter I to general finite energy solutions. This is the case of the decay of energy on past wave cones, which imply finite speed of propagation. If u is a finite energy solution (in any dimension $N \geq 3$) and $R > 0$, $x_0 \in \mathbb{R}^N$, $t_0 \in \mathbb{R}$, we denote by

$$E_{\text{loc}}(t) = \int_{|x-x_0|<R-|t-t_0|} e_u(t,x)dx.$$

Then

THEOREM II.4.6. *$E_{\text{loc}}(t)$ is nonincreasing for $t \geq t_0$.*

PROOF. From Theorem I.4.1, this quantity is nonincreasing when $(u_0, u_1) \in \mathcal{S}$. Considering the approximation given by Claim II.4.3, we obviously have, as a consequence of this claim,

$$\forall t, \quad \lim_{n\to\infty} \int_{|x-x_0|<R-|t-t_0|} e_{u_n}(t,x)dx = \int_{|x-x_0|<R-|t-t_0|} e_u(t,x)dx.$$

This gives the desired monotonicity property. $\square$

We note that for general finite energy solution the integration by parts used in the proof of Theorem I.4.1 is no longer valid (since the boundary terms are not always well-defined).

4.c. Equation with a source term. We next consider the wave equation with a source term (I.1.2). By linearity, it is sufficient to study the equation with zero initial data:

$$\partial_t^2 u - \Delta u = f, \quad \vec{u}_{\restriction t=0} = (0,0). \tag{II.4.2}$$

PROPOSITION II.4.7. *Assume $f \in C^0\left(\mathbb{R}, \mathcal{S}(\mathbb{R}^N)\right)$. Then u defined by*

$$u(t) = \int_0^s \frac{\sin\left((t-s)|D|\right)}{|D|} f(s)ds \tag{II.4.3}$$

is the unique solution of (II.4.2).

PROOF. The uniqueness follows as usual by Theorem I.4.1. It is thus sufficient to check that u defined by (II.4.3) is of class C^2, and is a solution of (II.4.2). We consider F the function defined on $\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N$ by

$$F(t,s,x)=\left(\frac{\sin\big((t-s)|D|\big)}{|D|}f(s)\right)(x).$$

Thus

$$F(t,s,x)=\frac{1}{(2\pi)^N}\int_{\mathbb{R}^N}e^{ix\cdot\xi}\frac{\sin\big((t-s)|\xi|\big)}{|\xi|}\widehat{f}(s,\xi)d\xi$$

Using that $\widehat{f}\in C^0\Big(\mathbb{R},\mathcal{S}(\mathbb{R}^N)\Big)$, it is easy to check that F is continuous and C^∞ with respect to the variable (t,x), and that one can differentiate below the integral sign. The result follows since by integration by parts in the ξ variable,

$$\Delta F(t,s,x)=-\frac{1}{(2\pi)^N}\int|\xi|^2e^{ix\cdot\xi}\frac{\sin\big((t-s)|\xi|\big)}{|\xi|}\widehat{f}(s,\xi)d\xi$$

□

We note that Duhamel formula (II.4.3) is still valid when $f\in L^1([-T,+T],\dot{H}^{\sigma-1})$ for all T, where σ is a fixed real number (assumed to be $< N/2$ for simplicity), and that it yields a function $u\in C^0(\mathbb{R},\dot{H}^\sigma)$ with $\partial_t u\in C^0(\mathbb{R},\dot{H}^{\sigma-1})$,

$$\partial_t u=\int_0^t\cos\big((t-s)|D|\big)f(s)ds, \tag{II.4.4}$$

in the sense of distribution, and such thatœ

$$\|\vec{u}(t)\|_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}}\leq\int_0^t\|f(s)\|_{\dot{H}^{\sigma-1}}ds. \tag{II.4.5}$$

Note that (II.4.5) is exactly the energy inequality proved in Chapter I when $\sigma=1$.

We can approximate such an f by a sequence of functions (f_n) with

$$f_n\in C^0(\mathbb{R},\mathcal{S}),\quad\forall t,\quad\lim_{n\to\infty}\int_{-T}^{+T}\|f(s)-f_n(s)\|_{\dot{H}^{\sigma-1}}ds=0.$$

The corresponding solutions u_n defined by

$$u_n(t)=\int_0^t\frac{\sin\big((t-s)|D|\big)}{|D|}f_n(s)ds$$

are C^2 solutions of (II.4.2) and satisfy

$$\sup_{-T\leq t\leq T}\|\vec{u}_n(t)-\vec{u}(t)\|_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}}\underset{n\to\infty}{\longrightarrow}0. \tag{II.4.6}$$

As in the case of the free wave equation, this proves that u satisfies (LW) in the sense of distribution. In this situation, we will take the formula (II.4.3) as a definition of the solution u of (LW).

EXERCICE II.1. Assume that $\sigma=1$. Let f defined on $\mathbb{R}\times\mathbb{R}^N$, such that $f\in L^1([-T,+T,L^2(\mathbb{R}^N))$. Prove that there exists a sequence of functions $f_n\in C_0^\infty(\mathbb{R}\times\mathbb{R}^N)$ such that

$$\forall T>0,\quad\lim_{n\to\infty}\|f_n-f\|_{L^1([-T,+T],L^2(\mathbb{R}^N))}=0.$$

EXERCICE II.2. Let u be a C^2 solution of (LW) for some $f\in C^0(\mathbb{R}\times\mathbb{R}^N)$. Assume that $f\in L^1([-T,+T],L^2(\mathbb{R}^N))$ for all $T>0$. Show that u satisfies (II.4.3).

CHAPTER III

Strichartz inequalities

III.1. Introduction

In view of Plancherel theorem and the Fourier representation formulas for the wave equation, it is natural to study the wave equation in $L^2(\mathbb{R}^N)$ or in L^2 based spaces such as the Sobolev spaces $\dot{H}^s$ considered in the preceding chapter. However, this is not sufficient for the study of nonlinear wave equations, since $\||f|^p\|_{L^2(\mathbb{R}^N)} = \|f\|_{L^{2p}}^{2p}$, the appearance of Lebesgue spaces L^q with $q \neq 2$ is unavoidable. A first way to deal with this issue is to use Sobolev inequalities. For example, if one wants to consider solutions in the energy spaces for the equation

$$\partial_t^2 u - \Delta u = u^3, \quad x \in \mathbb{R}^3, \tag{III.1.1}$$

the energy inequality will yields terms of the form[1] $\|u^3\|_{L^1([0,T],L^2)} = \|u\|^3_{L^3([0,T],L^6} \lesssim T\|u\|_{L^\infty([0,T],\dot{H}^1)}$, which is sufficient to prove the existence and uniqueness of finite energy solutions for (III.1.1). However this strategy will not work for higher order nonlinearities, and in particular the quintic one which we will focus on in several chapters of this course. In this chapter I will introduce the celebrated *Strichartz inequalities*, that use the dispersive properties of the wave equation to improve over Sobolev type inequalities. This type of inequalities was introduced by Robert Strichartz in an article published in 1977 [**27**], and generalized later by several authors. See e.g. [**19**] or the book [**26**].

The original inequalities of Strichartz were formulated in terms of Lebesgue spaces $L^q(\mathbb{R}\times\mathbb{R}^N)$ on the whole space time $\mathbb{R}\times\mathbb{R}^N$. Having in minds applications to nonlinear wave equations, it is useful to consider more general spaces where the Lebesgue exponents in space and times are distinct. If I is an interval, we will define $L^p(I, L^q(\mathbb{R}^N))$ as the set of integrable function $f : I \mapsto L^q(\mathbb{R}^N)$ such that

$$\|u\|_{L^p(\mathbb{R},L^q(\mathbb{R}^N))} = \Big\| \|u(\cdot)\|_{L^q(\mathbb{R}^N)} \Big\|_{L^p(\mathbb{R})} = \left(\int_{\mathbb{R}} \|u(t)\|^p_{L^q} dt \right)^{1/p}. \tag{III.1.2}$$

if finite (with the usual modification if $p = \infty$). The notion of integrable functions with values in a Banach space can be rigorously defined by the theory of Bochner's integration, see e.g. section 1.2 in the book [**5**]. An element of $L^p(I, L^q(\mathbb{R}^N))$ can be identified with a (class) of measurable function on $I \times \mathbb{R}^N$. With the identification, we can use the density of $C_0^\infty(\mathbb{R}^N)$ in $L^q(\mathbb{R}^N)$, $q < \infty$, to prove that $C_0^\infty(I\times\mathbb{R}^N)$ is dense in $L^p(I, L^q)$ if q and p are finite. Using this fact, we will mainly work on L^pL^q norms of smooth functions, for which the definition of (III.1.2) is clear.

We will often write $L^p(I, L^q)$ instead of $L^p(I, L^q(\mathbb{R}^N))$ to lighten notations. When $I = \mathbb{R}$, we will also use the notation L^pL^q.

We will use the generalized Hölder inequality in these spaces:

Proposition III.1.1. *Let p, q, p_1, q_1, p_2, q_2 in $[1,\infty]$ with*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Let $f \in L^{p_1}L^{q_1}$ and $g \in L^{p_2}L^{q_2}$. Then $fg \in L^pL^q$ and

$$\|fg\|_{L^pL^q} \leq \|f\|_{L^{p_1}L^{q_1}} \|g\|_{L^{p_2}L^{q_2}}.$$

The proof of Proposition III.1.1, using the standard Hölder inequality, is left as an exercise to the reader. We will also use the following consequence of Hölder inequality:

Exercice III.1. Let $\theta \in [0,1]$, p, q, p_1, q_1, p_2, q_2 in $[1,\infty]$ with

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

Let $f \in L^{p_1}L^{q_1} \cap L^{p_2}L^{q_2}$. Prove that $f \in L^pL^q$ and

$$\|f\|_{L^pL^q} \leq \|f\|^\theta_{L^{p_1}L^{q_1}} \|f\|^{1-\theta}_{L^{p_2}L^{q_2}}.$$

[1] See below for the notations $L^p(I, L^q)$

III.2. Statement of the estimate

The Strichartz inequalities in space dimension 3 with initial data in the energy space read as follows:

THEOREM III.2.1. *Let* $(u_0, u_1) \in (\dot{H}^1 \times L^2)(\mathbb{R}^3)$ *and* $f \in L^1(\mathbb{R} \times L^2(\mathbb{R}^3))$. *Let*

$$u(t) = \cos(t|D|)u_0 + \frac{\sin(t|D|)u_1}{|D|} + \int_0^t \frac{\sin\big((t-s)|D|\big)}{|D|} f(s)ds. \tag{III.2.1}$$

Then for any (p, q) *with* $p > 2$,

$$\frac{1}{p} + \frac{3}{q} = \frac{1}{2}, \tag{III.2.2}$$

one has $u \in L^p(\mathbb{R}, L^q(\mathbb{R}^3))$ *and*

$$\|u\|_{L^p(\mathbb{R},L^q)} \leq C\left(\|(u_0,u_1)\|_{\dot{H}^1\times L^2} + \|f\|_{L^1(\mathbb{R},L^2)}\right).$$

for a constant $C > 0$ *depending only on* p.

REMARK III.2.2. If I is an interval with $0 \in I$, $f \in L^1(I, L^2(\mathbb{R}^N))$, and u satisfies (III.2.1) for $t \in I$, then $u \in L^p(I, L^q(\mathbb{R}^3))$ and

$$\|u\|_{L^p(I,L^q)} \leq C\left(\|(u_0,u_1)\|_{\dot{H}^1\times L^2} + \|f\|_{L^1(I,L^2)}\right). \tag{III.2.3}$$

This follows immediately from the Theorem, extending f by $f(t) = 0$ if $t \notin I$.

REMARK III.2.3. We recall that in the setting of Theorem III.2.1, we also have $\vec{u} \in C^0\left(\mathbb{R}, \dot{H}^1 \times L^2\right)$, and the energy inequality

$$\|\vec{u}(T)\|_{\dot{H}^1\times L^2} \leq \|\vec{u}(0)\|_{\dot{H}^1\times L^2} + \|f\|_{L^1([0,T],L^2)},$$

for any $T > 0$, which can be easily checked using the space Fourier transform of formula (III.2.1)

We have focused on solutions with initial data $\dot{H}^1 \times L^2$ in space dimension 3, in view of application to the quintic wave equation in space dimension 3. Analogs of Theorem III.2.1 exist in all space dimensions $N \geq 2$, with more general assumptions on the initial data (u_0, u_1) and the right hand-side f. The condition (III.2.2) is necessary by the scaling of the equation. For solutions in space dimension N with initial data in $\dot{H}^\sigma \times \dot{H}^{\sigma-1}$, it becomes

$$\frac{1}{p} + \frac{N}{q} = \frac{N}{2} - \sigma.$$

Let us mention that there is in general another condition on p and q. This condition does not appear in Theorem III.2.1 as it is implied by the scaling condition (III.2.2).

Of particular interest is the case $\sigma = 1/2$ in space dimension 3, which was considered by R. Strichartz in his article [**27**], and which is useful to solve the cubic wave equation. We state this inequality and will leave some of the details of the proof to the reader:

THEOREM III.2.4. *Let* u *be defined by* (III.2.1) *with*

$$(u_0, u_1) \in \dot{H}^{1/2}(\mathbb{R}^3) \times \dot{H}^{-1/2}(\mathbb{R}^3), \quad f \in L^{4/3}(\mathbb{R} \times \mathbb{R}^3).$$

Then $u \in L^4(\mathbb{R} \times \mathbb{R}^3)$, $\vec{u} \in C^0\left(\mathbb{R}, \dot{H}^{1/2} \times \dot{H}^{-1/2}(\mathbb{R}^3)\right)$ *and*

$$\sup_{t\in\mathbb{R}} \|\vec{u}(t)\|_{\dot{H}^{\frac{1}{2}}\times\dot{H}^{-\frac{1}{2}}} + \|u\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \leq C\left(\|f\|_{L^{4/3}(\mathbb{R}\times\mathbb{R}^3)} + \|(u_0,u_1)\|_{\dot{H}^{\frac{1}{2}}\times\dot{H}^{-\frac{1}{2}}}\right).$$

In the sequel of this chapter we will prove Theorem III.2.1 for $p \geq 4$, which will be sufficient for our applications to the nonlinear equations below.

We will use the following notations. If A and B are positive quantities, we will write $A \lesssim B$ when there exists a constant C, independent of the parameters, such that $A \leq CB$, and $A \equiv B$ when $A \lesssim B$ and $B \lesssim A$.

By the energy inequality and Sobolev embedding, we have for all t.

$$\|u(t)\|_{L^6} \lesssim \|u(t)\|_{\dot{H}^1} \lesssim \|(u_0,u_1)\|_{\dot{H}^1\times L^2} + \|f\|_{L^1(\mathbb{R},L^2)},$$

which solves the case $p = \infty$, $q = 6$. Next, we notice that by Hölder inequality, if p and q satisfy (III.2.2) with $p \in (4, \infty)$, we have

$$\|u\|_{L^pL^q} \lesssim \|u\|_{L^\infty L^6}^{1-\theta}\|u\|_{L^4L^{12}}^{\theta} \tag{III.2.4}$$

where $\theta = \frac{4}{p}$. Thus the inequality (III.2.3) for this pair (p,q) will follows from the same equality for $p=4$, $q=12$. We are just reduced to prove the estimate (III.2.3) for $p=4$, $q=12$. By density, we can assume $(u_0,u_1)\in \left(C_0^\infty(\mathbb{R}^3)\right)^2$, $f\in C_0^\infty(\mathbb{R}\times\mathbb{R}^3)$.

The inequality will follow from a dispersion inequality which is a quantitative version of the inequality $|u(t)|\lesssim 1/|t|$ obtained for compactly supported, smooth functions in Chapter I. This inequality is proved in Section III.3. To deduce the Strichartz inequality from the dispersion inequality a few tools from harmonic analysis are needed. These tools, that include Hardy-Littlewood-Sobolev inequality, dyadic decomposition, Littlewood-Paley theory and interpolation of Lebesgue spaces, are recalled in Section III.4. Section III.6 is devoted to the end of the proof of Theorem III.2.1.

III.3. Dispersion inequality

For any function in $\mathcal{S}(\mathbb{R}^N)$, and $s\in\mathbb{N}$, we will denote

$$\|\varphi\|_{\dot W^{s,p}} = \sup_{|\alpha|=s} \|\partial_x^\alpha \varphi\|_{L^p(\mathbb{R}^N)}\,. \tag{III.3.1}$$

In this section we prove

THEOREM III.3.1. *Let $(u_0,u_1)\in\left(\mathcal{S}(\mathbb{R}^3)\right)^2$ and u the solution of* (LW), (ID). *Then for all $t>0$,*

$$\|u(t)\|_{L^\infty(\mathbb{R}^3)} \lesssim \frac{1}{t}\left(\|u_0\|_{\dot W^{2,1}} + \|u_1\|_{\dot W^{1,1}}\right).$$

PROOF. By space translation invariance it is sufficient to bound $|u(t,0)|$. We have

$$u(t,0) = t\int_{S^2} u_1(ty)d\sigma(y) + \int_{S^2} u_0(ty)d\sigma(y) + t\int_{S^2} y\cdot\nabla u_0(ty)d\sigma(y).$$

By the divergence theorem (denoting by B^3 the unit ball of $\mathbb{R}^3$),

$$\begin{aligned} t\int_{S^2} u_1(ty)d\sigma(y) \\ &= t\int_{B^3}\nabla\cdot(yu_1(ty))\,dy = 3t\int_{B^3}u_1(ty)dy + t^2\int_{B^3} y\cdot\nabla u_1(ty)dy. \end{aligned}\tag{III.3.2}$$

We have

$$\left|\int_{B^3} y\cdot\nabla u_1(ty)dy\right| \leq \frac{1}{t^3}\int_{tB^3}|\nabla u_1(y)|dy \leq \frac{3}{t^3}\|u_1\|_{\dot W^{1,1}}, \tag{III.3.3}$$

and

$$\int_{B^3}|u_1(ty)|dy \leq t\int_{\mathbb{R}^3}|\partial_{x_1}u_1(ty)|dy \leq \frac{1}{t^2}\|u_1\|_{\dot W^{1,1}}, \tag{III.3.4}$$

where we have used the Sobolev type inequality $\int_{B^3}|\varphi|dx \lesssim \int_{\mathbb{R}^3}|\partial_{x_1}\varphi|$, that follows immediately from the formula $\varphi(x_1,x_2,x_3) = \int_{-\infty}^{x_1}\partial_{x_1}\varphi(s,x_2,x_3)ds$. Combining (III.3.2), (III.3.3) and (III.3.4), we obtain

$$\left|t\int_{S^2}u_1(ty)d\sigma(y)\right| \lesssim \frac{1}{t}\|u_1\|_{\dot W^{1,1}}. \tag{III.3.5}$$

By the same proof, using also the inequality $\int_{B^3}|\varphi|\lesssim\int_{\mathbb{R}^3}|\partial_{x_1}\partial_{x_2}\varphi|$, we have

$$\left|\int_{S^2}u_0(ty)d\sigma(y)\right| + \left|\int_{S^2} y\cdot\nabla u_0(ty)d\sigma(y)\right| \lesssim \frac{1}{t}\|u_0\|_{\dot W^{2,1}}. \tag{III.3.6}$$

This concludes the proof of the dispersion inequality. □

III.4. Some tools from harmonic analysis

We first recall an interpolation Theorem for a linear operator between L^p space.

THEOREM III.4.1 (Riesz–Thorin interpolation Theorem). *Let (X,μ), (Y,ν) be measure spaces. Let*

$$\theta\in]0,1[,\quad (p_0,p_1,q_0,q_1,p,q)\in[1,\infty]^6$$

with

$$\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1},\quad \frac{1}{q} = \frac{\theta}{q_0} + \frac{1-\theta}{q_1}. \tag{III.4.1}$$

Let A be a linear operator defined on $L^{p_0}(X)+L^{p_1}(X)$ which is bounded from $L^{p_0}(X)$ to $L^{q_0}(Y)$ and from $L^{p_1}(X)$ to $L^{q_1}(Y)$. Then A is a bounded linear operator from $L^p(X)$ to $L^q(Y)$, and

$$\|A\|_{L^p(X)\to L^q(Y)} \leq \|A\|^{\theta}_{L^{p_0}(X)\to L^{q_0}(Y)}\|A\|^{1-\theta}_{L^{p_1}(X)\to L^{q_1}(Y)}.$$

In the theorem, $\|A\|_{E\to F}$ denotes the operator norm of the bounded operator $A : E \to F$, where E and F are Banach spaces.

We next recall Young's inequality for the convolution

THEOREM III.4.2. *Let $f \in L^q(\mathbb{R}^N)$, $g \in L^r(\mathbb{R}^N)$ with $1/q+1/r \geq 1$, and p defined by $\frac{1}{p}+1=\frac{1}{q}+\frac{1}{r}$. Then*

$$f * g(x) = \int f(x-y)g(y)dy$$

is defined for almost every $x \in \mathbb{R}^N$ and

$$\|f * g\|_{L^p} \leq \|f\|_{L^q}\|g\|_{L^r}, \tag{III.4.2}$$

EXERCICE III.2. Prove Young's inequality. Hint: start with the cases $(p,q)=(1,1)$, $(p,q)=(\infty,1)$, $(p,q)=(\infty,\infty)$ and use the interpolation theorem III.4.1.

When $N=1$ and $\theta \in]0,1[$, the function $t \mapsto 1/t^{\theta}$, is not in $L^{1/\theta}$ due to a logarithmic divergence at 0 and ∞. The Hardy-Littlewood-Sobolev inequality says that this function behaves as a $L^{1/\theta}$ function from the point of view of convolution. We will use this inequality in the particular case $\theta=1/2$, $p=4/3$, $q=4$. We refer e.g. to [**2**, Theorem 1.7] for the proof.

THEOREM III.4.3 (Hardy Littlewood Sobolev). *Let $\theta \in]0,1[$, $(p,q) \in]1,\infty[^2$ satisfy*

$$\frac{1}{p}+\theta = 1+\frac{1}{q}.$$

Let $f \in L^p(\mathbb{R}^N)$. Let, for $t \in \mathbb{R}$,

$$g(t) = \int_{\mathbb{R}} f(s)\frac{1}{|t-s|^{\theta}}ds. \tag{III.4.3}$$

Then the integral defining g converges for almost every t, and

$$\|g\|_{L^q(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}.$$

We next give a few elements of Littlewood-Paley theory, which is a useful tool to study L^p spaces with $p \neq 2$ by Fourier transformation. What follows is by no mean a complete account on Littlewood-Paley theory: we will just state the needed results, and will give only some of the proofs. We refer to [**2**, Chapter 2] for a complete introduction to the subject.

We start with some inequalities on frequency localized function.

THEOREM III.4.4 (Berstein-type estimates). *Let $\psi \in C_0^{\infty}(\mathbb{R}^N)$. Then if $1 \leq q \leq p \leq \infty$*

$$\forall f \in \mathcal{S}(\mathbb{R}^N), \quad \forall \lambda > 0, \quad \|\psi(\lambda D)f\|_{L^p} \lesssim \lambda^{\left(\frac{N}{p}-\frac{N}{q}\right)}\Big\|f\Big\|_{L^q} \tag{III.4.4}$$

Assume furthermore $\psi(\xi)=0$ for ξ close to 0. Then, if $s \in \mathbb{R}$ and $p \in [1,\infty]$,

$$\forall f \in \mathcal{S}(\mathbb{R}^N), \quad \forall \lambda > 0, \quad \Big\||D|^s\psi(\lambda D)f\Big\|_{L^p} \approx \lambda^{-s}\Big\|\psi(\lambda D)f\Big\|_{L^p}. \tag{III.4.5}$$

Moreover, if $s \in \mathbb{N}$,

$$\forall f \in \mathcal{S}(\mathbb{R}^N), \quad \forall \lambda > 0, \quad \sup_{|\alpha|=s}\Big\|\partial_x^{\alpha}\left(\psi(\lambda D)f\right)\Big\|_{L^p} \approx \lambda^{-s}\Big\|\psi(\lambda D)f\Big\|_{L^p}. \tag{III.4.6}$$

In the theorem, the implicit constants might depend on ψ, but of course not on f and $\lambda > 0$.

PROOF. *Step 1.*

We first prove (III.4.4) for $\lambda = 1$. We have

$$\psi(D)u = (\overline{\mathcal{F}}\psi) * u, \tag{III.4.7}$$

where $f * g$ is the convolution of f and g. This is a classical property of the Fourier transform, which can be checked by an explicit computation of $\mathcal{F}(\psi(D)u)$. Note that $\overline{\mathcal{F}}\psi \in \mathcal{S} \subset \bigcap_{1\leq p\leq\infty} L^p$. Using Young's inequality we obtain that (III.4.4) holds for $\lambda = 1$, i.e. that there exists $C > 0$ such that

$$\forall f \in \mathcal{S}(\mathbb{R}^N), \quad \|\psi(D)f\|_{L^p} \leq \|f\|_{L^q}.$$

Step 2: rescaling. Denote by $T_\lambda u(x) = u(\lambda x)$. By a simple change of variable, one can prove

$$\Psi(D)(T_\lambda u) = T_\lambda\left(\psi(\lambda D)u\right)$$

Thus by Step 1,

$$\|T_\lambda\left(\psi(\lambda D)u\right)\|_{L^p} \lesssim \|T_\lambda u\|_{L^q}.$$

Since $\|T_\lambda f\|_{L^p} = \frac{1}{\lambda^{N/p}}\|f\|_{L^p}$, we obtain (III.4.4) for any $\lambda > 0$.

Step 3: proof of (III.4.5).

Let $\chi \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$, such that $\chi(\xi) = 1$ if $\xi \in \operatorname{supp}(\psi)$. Then

$$|D|^s\psi(\lambda D)u = |D|^s\chi(\lambda D)\psi(\lambda D)u = \frac{1}{\lambda^s}\Xi(\lambda D)\psi(\lambda D)u,$$

where $\Xi(\xi) = |\xi|^s\chi(\xi)$. Using (III.4.4) with $p = q$, we obtain

$$\big\||D|^s\psi(\lambda D)u\big\|_{L^p} \lesssim \frac{1}{\lambda^s}\big\|\psi(\lambda D)u\big\|_{L^p}. \tag{III.4.8}$$

Using (III.4.8), with s replaced by $-s$ and u replaced by $|D|^s\chi(\lambda D)u$, we obtain

$$\big\|\psi(\lambda D)u\big\|_{L^p} = \big\||D|^{-s}\psi(\lambda D)|D|^s u\big\|_{L^p} \lesssim \lambda^s\big\|\psi(\lambda D)|D|^s u\big\|_{L^p}.$$

This concludes the proof of (III.4.5).

Step 4: proof of (III.4.6). First, we have

$$\Big\|\psi(\lambda D)\partial_x^\alpha f\Big\|_{L^p} = \Big\|\partial_x^\alpha\chi(\lambda D)\psi(\lambda D)f\Big\|_{L^p} = \frac{1}{|\lambda|^{|\alpha|}}\Big\|\Xi_\alpha(\lambda D)\psi(\lambda D)f\Big\|_{L^p}, \tag{III.4.9}$$

where χ is as above and $\Xi_\alpha(\xi) = (i\xi)^\alpha\chi(\xi)$. The estimate $\lesssim$ in (III.4.6) then follows from (III.4.4) with $q = p$.

Next, if s is even, we have $|D|^s = (-\Delta)^{s/2}$, which shows that (III.4.5) implies the other estimate in (III.4.6).

If s is odd, we write

$$\Big\|\psi(\lambda D)|D|^s f\Big\| = \|\psi(\lambda D)|D|^{s+1}\frac{1}{|D|}f\|_{L^p} \lesssim \sup_{|\alpha|=s+1}\big\|\partial_x^\alpha|D|^{-1}\psi(\lambda D)f\big\|_{L^p}$$
$$\approx \frac{1}{\lambda}\sup_{|\alpha|=s+1}\|\partial_x^\alpha\psi(\lambda D)f\|_{L^p},$$

and we conclude with (III.4.9) that the inequality $\gtrsim$ in (III.4.6) holds in this case also. $\square$

The Littlewood-Paley theory is based on a dyadic decomposition of a distribution $f \in \mathcal{S}'(\mathbb{R}^N)$. We fix once and for all a radial function $\varphi \in C_0^\infty(\mathbb{R}^N)$ with $\varphi(\xi) = 1$ if $|\xi| \leq 1/2$, and $\varphi(x) = 0$ if $|x| \geq 1$. We let

$$\Theta_j(\xi) = \varphi\left(\frac{\xi}{2^{j+1}}\right) - \varphi\left(\frac{\xi}{2^j}\right) = \Theta\left(\frac{\xi}{2^j}\right), \quad \Theta(\xi) = \varphi(\xi/2) - \varphi(\xi).$$

We have

$$\operatorname{supp}\Theta_j \subset \left\{2^{j-1} \leq |\xi| \leq 2^{j+1}\right\}, \quad \sum_{j=-\infty}^{+\infty}\Theta_j(\xi) = 1,\ (\xi \neq 0),$$

where the sum is, for any fixed ξ, a finite sum. We denote

$$\Delta_j f = \Theta_j(D),$$

so that (at least formarly) $f = \sum_{j\in\mathbb{Z}}\Theta_j(D)f$ (*Dyadic decomposition of f in frequencies*). If $f \in \mathcal{S}_0$, it is easy to prove that this sum converges in $\mathcal{S}$.

We have the inequality

$$\frac{1}{2} \leq \sum_{j\in\mathbb{Z}}\Theta_j^2(\xi) \leq 1. \tag{III.4.10}$$

Exercice III.3. Prove (III.4.10).

Combining with Plancherel identity, it follows that if $f \in \mathcal{S}(\mathbb{R}^N)$,

$$\|f\|_{L^2(\mathbb{R}^N)}^2 \approx \sum_{j\in\mathbb{Z}}\|\Delta_j f\|_{L^2(\mathbb{R}^N)}^2, \tag{III.4.11}$$

and more generally,

$$\|f\|_{\dot{H}^s}^2 \approx \sum_{j\in\mathbb{Z}}\|\Delta_j|D|^s f\|_{L^2}^2 \approx \sum_{j\in\mathbb{Z}}(2^{2j})^s\|\Delta_j f\|_{L^2}^2. \tag{III.4.12}$$

The situation is more complicated for $p \neq 2$. Nevertheless, we have the following estimates:

THEOREM III.4.5. *For all $p \in (1,2]$, for any $f \in \mathcal{S}$*

$$\sum_{j\in\mathbb{Z}} \|\Delta_j f\|_{L^p}^2 \lesssim \|f\|_{L^p}^2 \tag{III.4.13}$$

For all $p \in [2,\infty)$, for any $f \in L^p$,

$$\|f\|_{L^p}^2 \lesssim \sum_{j\in\mathbb{Z}} \|\Delta_j f\|_{L^p}^2 . \tag{III.4.14}$$

We omit the proof refering the interested reader to [**2**, Theorem 2.40].

EXERCICE III.4. Prove:

- For all $p \in [1,2]$, for any $f \in \mathcal{S}$

$$\|f\|_{L^p}^p \lesssim \sum_{j\in\mathbb{Z}} \|\Delta_j f\|_{L^p}^p \tag{III.4.15}$$

- For all $p \in [2,\infty]$, for any $f \in L^p$,

$$\sum_{j\ \in\mathcal{Z}} \|\Delta_j f\|_{L^p}^p \lesssim \|f\|_{L^p}^p \tag{III.4.16}$$

(where the sum has to be interpreted as $\sup_j \|\Delta_j f\|_{L^\infty}$ when $p=\infty$).

Hint: Start with the cases $p=1$ and $p=2$ for (III.4.15) and $p=\infty$ and $p=2$ for (III.4.16), then use an interpolation argument.

The two estimates of Exercise III.4 complete the estimates of Theorem III.4.5. The proofs are simpler than the proof of Theorem III.4.5, but are not detailed here since we will not need these estimates below.

Note that there is no perfect equivalence between the norm $\|f\|_{L^p}$ and a norm defined as a ℓ^q norm of the sequence $\left(\|\Delta_j f\|_{L^p}\right)_j$ if $p \neq 2$.

Let us mention that the quantities

$$\|f\|_{\dot{B}^0_{p,q}}^q = \sum_{j\in\mathbb{Z}} \|\Delta_j f\|_{L^p}^q \tag{III.4.17}$$

appearing in (III.4.13), (III.4.14), (III.4.15) and (III.4.16) defines the norm of the so-called Besov space $\dot{B}^0_{p,q}$. See Sections 2.3, 2.4 and 2.5 of [**2**] for more details on Besov spaces.

III.5. A Strichartz inequality for the half wave equation

It is sometimes useful to decompose the wave equation in two first-order equations in the time-variable. This is particularly the case when dealing with Fourier analysis tools. We thus introduce the half-wave equations

$$\partial_t u + i|D|u = 0, \quad \partial_t u - i|D|u = 0,$$

and their solutions (given in term of Fourier representations) $e^{-it|D|}\varphi$ and $e^{it|D|}\varphi$. Note that the solution to the usual wave equation (LW), (ID) is given by

$$2u(t) = e^{it|D|}u_0 + e^{-it|D|}u_0 + \frac{e^{it|D|}}{i|D|}u_1 - \frac{e^{-it|D|}}{i|D|}u_1$$

Note also that if $v(t) = e^{it|D|}\varphi$, then $e^{-it|D|}u_0 = v(-t)$, thus it is sufficient to consider only the solution $e^{it|D|}\varphi$. The function $e^{it|\xi|}$ is not smooth at $\xi = 0$, so that $e^{it|D|}$ does not map $\mathcal{S}(\mathbb{R}^N)$ to $\mathcal{S}(\mathbb{R}^N)$. However it maps $\mathcal{S}_0(\mathbb{R}^N)$ to $\mathcal{S}_0(\mathbb{R}^N)$ (where as before $\mathcal{S}_0(\mathbb{R}^N)$ is the space of functions φ in $\mathcal{S}(\mathbb{R}^N)$ such that $\hat{\varphi}$ is identically 0 in a neighborhood of the origin).

In this Section, we will prove

PROPOSITION III.5.1. *There exists $C>0$ such that*

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^N), \quad \left\| \frac{e^{i\cdot|D|}}{|D|}\varphi \right\|_{L^4(\mathbb{R},L^{12})} \lesssim \|\varphi\|_{L^2}, \tag{III.5.1}$$

where as usual $e^{i\cdot|D|}\varphi$ denotes $(t,x) \mapsto \left(e^{it|D|}\varphi\right)(x)$.

Proof. *Step 1: frequency-localized dispersion estimate.*

We will use the Littlewood-Paley decomposition of φ, $\varphi = \sum_{j\in\mathbb{Z}} \Delta_j\varphi$. In this step we prove the following frequency localized version of the dispersion inequality for the wave equation

$$\forall j, \quad \left\| \frac{e^{it|D|}}{|D|}\Delta_j\varphi \right\|_{L^\infty} \lesssim \frac{2^j}{t}\|\Delta_j\varphi\|_{L^1}. \tag{III.5.2}$$

We let $\varphi_j = \Delta_j\varphi$. By the dispersion inequality for the full wave equation and Theorem III.4.4, we have

$$\left\| \frac{\sin(t|D|)}{|D|}\varphi_j \right\|_{L^\infty} \lesssim \frac{1}{|t|}\|\varphi_j\|_{\dot{W}^{1,1}} \approx \frac{2^j}{|t|}\|\varphi_j\|_{L^1}$$

and

$$\left\| \frac{\cos(t|D|)}{|D|}\varphi_j \right\|_{L^\infty} \approx \frac{1}{2^j}\|\cos(t|D|)\varphi_j\|_{L^\infty} \lesssim \frac{1}{2^j|t|}\|\varphi_j\|_{\dot{W}^{2,1}} \approx \frac{2^j}{|t|}\|\varphi_j\|_{L^1}.$$

Step 2. A $L^4/L^{4/3}$ dispersion inequality

We next introduce $\widetilde{\Delta}_j f = \Delta_{j-1}f + \Delta_j f + \Delta_{j+1}f$. Noting that $\Theta_{j-1}+\Theta_j+\Theta_{j+1} = 1$ on the support of Θ_j, we see that $\widetilde{\Delta}_j\Delta_j f = \Delta_j f$. For fixed $t>0$ and j, consider the operator $e^{it|D|}|D|^{-1}\widetilde{\Delta}_j$. By Step 1, it is a bounded operator from L^1 to L^∞, with operator norm $\lesssim 2^j/t$. By Plancherel and Theorem III.4.4, it is bounded from L^2 to L^2 with operator norm $\lesssim 2^{-j}$. Using the interpolation Theorem III.4.1, we obtain that $e^{it|D|}|D|^{-1}\widetilde{\Delta}_j$ is a bounded operator from $L^{4/3}$ to L^4 with operator norm $\lesssim t^{-1/2}$. Using that $\widetilde{\Delta}_j\Delta_j = \Delta_j$, we deduce

$$\left\| e^{it|D|}\frac{1}{|D|}\Delta_j\varphi \right\|_{L^4} \lesssim \frac{1}{|t|^{1/2}}\|\Delta_j\varphi\|_{L^{4/3}} \tag{III.5.3}$$

Step 3. A frequency localized Strichartz inequality.

Next, we consider the operator T_j defined by

$$(T_j\varphi)(t,x) = \left(e^{it|D|}|D|^{-1/2}\Delta_j\varphi\right)(x)$$

In this step we prove that T_j extends to a bounded operator from $L^2(\mathbb{R}^3)$ to $L^4(\mathbb{R}\times\mathbb{R}^3)$, with an operator norm that is independent of j, i.e.

$$\forall\varphi\in\mathcal{S}(\mathbb{R}^3), \quad \left\| e^{it|D|}|D|^{-1/2}\Delta_j\varphi \right\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|\varphi\|_{L^2}. \tag{III.5.4}$$

We will use a so-called TT^* argument to reduce the proof of (III.5.4) to the proof of the boundedness of an operator acting on functions on $\mathbb{R}\times\mathbb{R}^3$.

The inequality (III.5.4) is equivalent to the following statement:

$$\forall\varphi\in\mathcal{S}(\mathbb{R}^3), \quad \forall g\in C_0^\infty(\mathbb{R}\times\mathbb{R}^3), \quad \left|\iint (T_j\varphi)\overline{g}\,dxdt\right| \lesssim \|\varphi\|_{L^2(\mathbb{R}^3)}\|g\|_{L^{4/3}(\mathbb{R}\times\mathbb{R}^3)}.$$

Using Plancherel equality in the space variable for every $t\in\mathbb{R}$, we obtain

$$\iint (T_j\varphi)\overline{g}\,dxdt = \int \varphi(x)(T_j^*g)(x)dx,$$

where the (formal) adjoint T_j^* of T_j is defined by

$$T_j^*g(x) = \int_{\mathbb{R}} e^{-it|D|}|D|^{-1/2}\Delta_j g(t)dt.$$

We are thus reduced to prove

$$\forall g\in C_0^\infty(\mathbb{R}\times\mathbb{R}^3), \quad \left\|T_j^*g\right\|_{L^2(\mathbb{R}^3)} \lesssim \|g\|_{L^{4/3}(\mathbb{R}\times\mathbb{R}^3)}. \tag{III.5.5}$$

We have

$$\left\|T_j^*g\right\|_{L^2}^2 = \int_{\mathbb{R}^3} T_j^*g\overline{T_j^*g}\,dx = \iint_{\mathbb{R}\times\mathbb{R}^3} T_jT_j^*g\overline{g}\,dxdt, \tag{III.5.6}$$

and (III.5.5) would follow from the inequality

$$\left\|T_jT_j^*g\right\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|g\|_{L^{4/3}(\mathbb{R}\times\mathbb{R}^3)}. \tag{III.5.7}$$

We have

$$T_jT_j^*g(t,x) = \int_{\mathbb{R}} e^{i(t-s)|D|}|D|^{-1}\Delta_j g(s)ds.$$

Using the $L^4/L^{4/3}$ dispersion inequality of Step 2, we obtain at fixed t,

$$\left\|(T_jT_j^*g)(t)\right\|_{L^4(\mathbb{R}^3)} \lesssim \int_{\mathbb{R}} \frac{1}{|t-s|^{1/2}} \left\|\Delta_j g(s)\right\|_{L^{4/3}(\mathbb{R}^3)} ds$$

By Hardy Littlewood Sobolev inequality, we deduce

$$\|TT^*g\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|\Delta_j g\|_{L^{4/3}(\mathbb{R}\times\mathbb{R}^3)},$$

which yields (III.5.7) and thus concludes the proof of (III.5.4). Note that we can also localize the right-hand side in (III.5.4). Indeed, using $\widetilde{\Delta}_j = \Delta_{j+1} + \Delta_j + \Delta_{j-1}$, we see that (III.5.4) remains valid when Δ_j is replaced by $\widetilde{\Delta}_j$. Applying this inequality to $\Delta_j\varphi$, and using that $\widetilde{\Delta}_j\Delta_j = \Delta_j$, we obtain

$$\text{(III.5.8)} \qquad \forall \varphi \in \mathcal{S}(\mathbb{R}^3), \quad \left\|e^{it|D|}|D|^{-1/2}\Delta_j\varphi\right\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|\Delta_j\varphi\|_{L^2}.$$

Step 5. The L^4L^{12} localized in frequency Strichartz inequality. We next prove

$$\left\|e^{it|D|}\Delta_j\varphi\right\|_{L^4(\mathbb{R},L^{12}(\mathbb{R}^3))} \lesssim \|\Delta_j\varphi\|_{\dot{H}^1}$$

Indeed by the Bernstein type's inequalities of Theorem III.4.4,

$$\begin{aligned}\left\|e^{it|D|}|D|^{-1}\Delta_j\varphi\right\|_{L^4(\mathbb{R},L^{12}(\mathbb{R}^3))} &\approx \frac{1}{2^{j/2}}\left\|e^{it|D|}|D|^{-1/2}\Delta_j\varphi\right\|_{L^4(\mathbb{R},L^{12}(\mathbb{R}^3))}\\ &\lesssim \left\|e^{it|D|}|D|^{-1/2}\Delta_j\varphi\right\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|\Delta_j\varphi\|_{L^2}.\end{aligned}$$

Step 6. Summing up the frequencies.

In this step, we conclude the proof of Proposition III.5.1, by summing up the estimate of Step 5 with respect to j. We fix $\varphi \in \mathcal{S}_0(\mathbb{R}^3)$. We have

$$\text{(III.5.9)} \qquad \sum_{j\in\mathbb{Z}} \left\|e^{it|D|}\Delta_j\varphi\right\|^2_{L^4L^{12}} \lesssim \sum_{j\in\mathbb{Z}} \left\||D|\Delta_j\varphi\right\|^2_{L^2}.$$

The right-hand side is $\approx \|\varphi\|^2_{\dot{H}^1}$ by Plancherel equality (see (III.4.11)). We must prove that the left-hand side dominates $\left\|e^{it|D|}\varphi\right\|_{L^4L^{12}}$. Let $u = e^{it|D|}\varphi$ and $u_j = \Delta_j u$. By Minkowski inequality (i.e. the triangle inequality for the $L^2(\mathbb{R})$ norm), we see that

$$\sum_{j\in\mathbb{Z}} \|u_j\|^2_{L^4L^{12}} = \sum_{j\in\mathbb{Z}} \left\|\|u_j(t)\|^2_{L^{12}(\mathbb{R}^3)}\right\|_{L^2(\mathbb{R})} \geq \left\|\sum_{j\in\mathbb{Z}} \|u_j(t)\|^2_{L^{12}}\right\|_{L^2(\mathbb{R})}$$

By Theorem III.4.5, at fixed t,

$$\|u(t)\|^2_{L^{12}} \lesssim \sum_{j\in\mathbb{Z}} \|u_j(t)\|^2_{L^{12}}.$$

This shows

$$\sum_{j\in\mathbb{Z}} \|u_j\|^2_{L^4L^{12}} \gtrsim \left\|\|u(t)\|^2_{L^{12}(\mathbb{R}^3)}\right\|_{L^2(\mathbb{R})} = \|u\|^{1/2}_{L^4L^{12}},$$

which together with (III.5.9) concludes the proof of Proposition III.5.1. □

Remark III.5.2. An alternative, somehow simpler approach is to sum up over j the frequency localized dispersion inequality of Step 2 of the preceding proof. Using Theorem III.4.5, one obtains a $L^4/L^{4/3}$ dispersion inequality for the half-wave equation:

$$\left\|e^{it|D|}|D|^{-1}\varphi\right\|_{L^4} \lesssim \frac{1}{|t|^{1/2}}\|\varphi\|_{L^{4/3}}.$$

It is then possible to forget about frequency cut-off and run the preceding arguments to obtain Strichartz inequalities for the half-wave equation directly.

III.6. Proof of the Strichartz estimate for the full wave equation

We are now ready to prove Theorem III.2.1. We can treat separately the terms

$$u_L(t) = \cos(t|D|)u_0 + \frac{\sin(t|D|)u_1}{|D|}$$

and

$$(Bf)(t) = \int_0^t \frac{\sin\big((t-s)|D|\big)}{|D|} f(s)ds. \tag{III.6.1}$$

Using that $\cos(t|D|) = \frac{1}{2}\left(e^{it|D|} + e^{-it|D|}\right)$, $\sin(t|D|) = \frac{1}{2i}\left(e^{it|D|} - e^{-it|D|}\right)$, we obtain immediately from Proposition III.5.1

$$\|u_L\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|u_0\|_{\dot{H}^1} + \|u_1\|_{L^2}.$$

The other term is more delicate. We first consider

$$u_a(t) = \int_0^\infty \frac{e^{i(t-s)|D|}}{|D|} f(s)ds = e^{it|D|}F, \quad F = \int_0^\infty \frac{e^{-is|D|}}{|D|} f(s)ds$$

and

$$u_b(t) = \int_0^\infty \frac{e^{-i(t-s)|D|}}{|D|} f(s)ds$$

Using that $e^{-is|D|}/|D|$ is a bounded operator from L^2 to $\dot{H}^1$, we obtain that $F \in \dot{H}^1$ with

$$\|F\|_{\dot{H}^1} \lesssim \|f\|_{L^1(\mathbb{R},L^2(\mathbb{R}^3))}.$$

By the Strichartz estimate for the half-wave equation, Proposition III.5.1, we deduce

$$\|u_a\|_{L^4(\mathbb{R},L^{12}(\mathbb{R}^3))} \lesssim \|f\|_{L^1(\mathbb{R},L^2(\mathbb{R}^3))}.$$

Similarly

$$\|u_b\|_{L^4(\mathbb{R},L^{12}(\mathbb{R}^3))} \lesssim \|f\|_{L^1(\mathbb{R},L^2(\mathbb{R}^3))}.$$

Combining, we obtain

$$\|Af\|_{L^4(\mathbb{R},L^{12}(\mathbb{R}^3))} \lesssim \|f\|_{L^1(\mathbb{R},L^2(\mathbb{R}^3))}, \tag{III.6.2}$$

where A is the operator defined by

$$Af(t) = \int_0^\infty \frac{\sin\big((t-s)|D|\big)}{|D|} f(s)ds.$$

Note that Af is analogous to Bf defined above, the only difference between the two being that the integral defining Af is on $[0,\infty)$, whereas the integral defining Bf is on $[0,t[$. An important functional analysis result, due to Michael Christ and Alexander Kiselev [**7**], shows that the boundedness of A implies the boundedness of B. We state this result in a version that was proposed by Christopher Sogge:

Lemma III.6.1. *Let X and Y be Banach spaces. Let $1 \leq p < q \leq \infty$. Let K a continuous function from $\mathbb{R}^2$ to the space of bounded linear operators from X to Y. Let*

$$(Af)(t) = \int_{-\infty}^\infty K(t,\tau)f(\tau)d\tau,$$

and assume that A is a bounded operator from $L^p(\mathbb{R},X)$ to $L^q(\mathbb{R},Y)$, with operator norm C. Define the operator B by

$$(Bf)(t) = \int_{-\infty}^t K(t,\tau)f(\tau)d\tau.$$

Then B extends to a bounded operator from $L^p(\mathbb{R},X)$ to $L^q(\mathbb{R},Y)$, with operator norm $\leq \frac{2C\theta^2}{1-\theta}$, where $\theta = 2^{\frac{1}{q}-\frac{1}{p}}$.

Applying Christ and Kiselev Lemma to

$$K(t,\tau) = \mathbb{1}_{\tau>0}\frac{\sin\big((t-\tau)|D|\big)}{|D|}\chi(\varepsilon|D|), \tag{III.6.3}$$

where $\chi \in C_0^\infty(\mathbb{R}^3)$ is equal to 1 close to 0, one obtains

$$\forall \varepsilon > 0, \quad \forall f \in L^1(\mathbb{R},L^2), \quad \|\chi(\varepsilon D)Bf\|_{L^4L^{12}} \lesssim \|f\|_{L^1L^2},$$

where Bf is as in (III.6.1). Letting $\varepsilon \to 0$ we obtain the desired result.

Exercice III.5. Justify this last argument.

CHAPTER IV

Cauchy theory for the non-linear equation

In this chapter we will consider the nonlinear wave equation with a power-like nonlinearity

$$\partial_t u^2 - \Delta u = \sigma u^p, \tag{NLW}$$

on $I \times \mathbb{R}^N$, where N is an interval, where the power p is an integer ≥ 2 and σ is nonzero real parameter. Considering the unknown λu instead of u for a suitable choice of $\lambda > 0$, we see that we can assume

$$\sigma \in \{\pm 1\}.$$

We will briefly consider the general case, then restrict to the quintic case $p = 5$ in space dimension 3. We will also comment on the cubic case $p = 3$, in the same space dimension.

IV.1. Scaling invariance. Critical Sobolev space

Let u be a (nonzero) C^2 solution of (NLW) on $(a, b) \times \mathbb{R}^N$, where $a < b$. Let $u_\lambda(t, x) = \lambda^\alpha u(\lambda t, \lambda x)$, where $\lambda > 0$ and $\alpha = \alpha(p, N)$ will be specified later. We have

$$\partial_t^2 u_\lambda - \Delta u_\lambda = \lambda^{\alpha+2-\alpha p} \sigma u_\lambda^p.$$

Thus, if $\alpha = \frac{2}{p-1}$, we see that u_λ is a solution of (NLW) on $\left(\frac{a}{\lambda}, \frac{b}{\lambda}\right) \times \mathbb{R}^N$. We will assume that α has this particular value in the sequel, denoting

$$u_\lambda(t, x) = \lambda^{\frac{2}{p-1}} u(\lambda t, \lambda x).$$

Let

$$\dot{\mathcal{H}}^s = \dot{H}^s(\mathbb{R}^N) \times \dot{H}^{s-1}(\mathbb{R}^N).$$

The critical Sobolev exponent is by definition the unique s such that

$$\|\vec{u}_\lambda(0)\|_{\dot{\mathcal{H}}^{s_c}} = \|\vec{u}(0)\|_{\dot{\mathcal{H}}^{s_c}}.$$

Since by explicit computation

$$\|\vec{u}_\lambda(0)\|_{\dot{\mathcal{H}}^s} = \lambda^{\frac{2}{p-1}+s-N/2} \|\vec{u}(0)\|_{\dot{\mathcal{H}}^s}. \tag{IV.1.1}$$

We see that

$$s_c = \frac{N}{2} - \frac{2}{p-1}.$$

We observe that s_c grows with p, and is always strictly smaller than $N/2$.

Consider a solution u of (NLW) defined on a finite interval $[0, T[$. The corresponding solution u_λ is defined on $[0, T/\lambda[$. Growing λ has the effect of decreasing the time of existence. If $s > s_c$, the $\dot{\mathcal{H}}^s$ norm of $\vec{u}_\lambda(0)$ becomes larger. If $s < s_c$ it becomes smaller. Thus in this case the effect of scaling is to simultaneously decrease the norm of the initial data in $\dot{\mathcal{H}}^s$, $s < s_c$ and shrinking its interval of existence. This is contrary to the intuition that for smaller solutions, the effect of the nonlinearity is weaker, and the solution should behave in a linear way (and in particular has a long time of existence). This leads to an informal conjecture that s_c is a threshold for local well-posedness. It turns out that this conjecture is true for the wave equation: the equation (NLW) is locally well-posed[1] in $\dot{\mathcal{H}}^s$ for $s \geq s_c$, and ill-posed if $\dot{\mathcal{H}}^s$ for $s < s_c$.

We will focus on the quintic case $p = 5$ in space dimension $N = 3$:

$$(\partial_t^2 - \Delta)u = \sigma u^5. \tag{W5}$$

In this case the critical Sobolev case is $\dot{\mathcal{H}}^1$, and the equation is called "energy critical". We will also sometimes consider the cubic equation

$$(\partial_t^2 - \Delta)u = \sigma u^3, \tag{W3}$$

[1] By "well-posed in X", we mean that there is existence and uniqueness of solutions with initial data in X and a reasonable stability theory. We will not give a more rigorous definition of local well-posedness. See e.g. Definition 3.4, Remark 3.5 of T. Tao's book [**29**]

in dimension $1+3$, for which $s_c = 1/2$. As usual, we will take initial data, say at $t = t_0$:

$$(u, \partial_t u)_{t=t_0} = (u_0, u_1). \tag{ID}$$

In all the sequel, we fix $N = 3$.

IV.2. Definition of solutions

As for the linear wave equation, the notion of classical (C^2) solution is too restrictive for the equation (W5), and we will define the following weaker notion of solution, based on Duhamel's formulation of the equation:

Definition IV.2.1. A *finite energy solution* of (W5), (ID) on an interval I with $t_0 \in I$ is a function $u \in L^5_{\mathrm{loc}}(I, L^{10})$ such that $\forall t \in I$,

$$u(t) = \cos\big((t-t_0)|D|\big)u_0 + \frac{\sin\big((t-t_0)|D|\big)}{|D|}u_1 + \int_{t_0}^{t} \frac{\sin\big((t-s)|D|\big)}{|D|}u^5(s)ds, \tag{IV.2.1}$$

where $(u_0, u_1) \in \dot{\mathcal{H}}^1$.

In the definition, by $u \in L^5_{\mathrm{loc}}(I, L^{10}(\mathbb{R}^3))$, we mean that $u \in L^5(J, L^{10})$ for any compact interval $J \subset I$.

Note that if u is a finite-energy solution in the above sense, one has $u^5 \in L^1_{\mathrm{loc}}(I, L^2(\mathbb{R}^3))$, and thus by energy estimates (see Remark III.2.3),

$$\vec{u} \in C^0(I, \dot{\mathcal{H}}^1).$$

Also, by Chapter II, u satisfies the equation (W5) in the sense of distribution on $I \times \mathbb{R}^3$.

The solutions given by the Duhamel formula as in Definition IV.2.1 are called "strong" solutions in the book of Terence Tao [**29**], by opposition to the weaker notion of distributional solutions (that do not impose continuity in time) and the stronger notion of classical solutions (that are C^2 and satisfy the equation in a classical sense). Note however that this terminology is not universal. For example the solutions of Definition IV.2.1 are called ... "weak" solutions in the book [**26**] of Christopher Sogge.

We refer to Section 3.2 of [**29**] "What is a solution?", for a discussion on different types of solutions.

In the sequel, by "solution to (W5)" we will always mean (unless specified otherwise) a solution in the sense of Definition IV.2.1.

Exercice IV.1. Check that the definition of finite energy solutions above does not depend on the choice of the initial time. In other words, if u is a solution of (W5) on I and $t_1 \in I$, then for all $t \in I$,

$$u(t) = \cos\big((t-t_1)|D|\big)u(t_1) + \frac{\sin\big((t-t_1)|D|\big)}{|D|}\partial_t u(t_1) + \int_{t_1}^{t} \frac{\sin\big((t-s)|D|\big)}{|D|}u^5(s)ds.$$

IV.3. Existence and uniqueness

3.a. A local statement. We introduce the following notations:

$$S_L(t)\vec{u}_0 = \cos(t|D|)\,u_0 + \frac{\sin(t|D|)}{|D|}u_1, \quad \vec{S}_L(t)\vec{u}_0 = \left(S_L(t)\vec{u}_0, \partial_t S_L(t)\vec{u}_0\right),$$

where $\vec{u}_0 = (u_0, u_1)$. We start with the following local statement:

Theorem IV.3.1. *There exists $\delta_0 > 0$ with the following property. Let I be an interval with $t_0 \in I$. Let $\vec{u}_0 \in \dot{\mathcal{H}}^1$. Assume*

$$\|S_L(\cdot - t_0)\vec{u}_0\|_{L^5(I,L^{10})} = \delta \leq \delta_0. \tag{IV.3.1}$$

Then there exists a unique solution u of (W5), (ID) *on I. Furthermore*

$$\sup_{t\in I}\left\|\vec{u}(t) - \vec{S}_L(t-t_0)\vec{u}_0\right\|_{\dot{\mathcal{H}}^1} + \|u - S_L(\cdot - t_0)\vec{u}_0\|_{L^5(I,L^{10})} \lesssim \delta^5. \tag{IV.3.2}$$

In the Theorem, $S_L(\cdot - t_0)\vec{u}_0$ denotes the map $t \mapsto S_L(t-t_0)\vec{u}_0$.

Theorem IV.3.1 has two important consequences:

Local well-posedness: Note that $(5, 10)$ is an admissible couple in dimension 3 (it satisfies (III.2.2)). By Theorem III.2.1, if $\vec{u}_0 \in \dot{\mathcal{H}}^1$, then $S_L(\cdot)\vec{u}_0 \in L^5(\mathbb{R}, L^{10}(\mathbb{R}^3))$. Thus if $T > 0$ is small enough, then

$$\|\vec{u}_0\|_{L^5([-T,+T],L^{10})} \leq \delta_0,$$

and Theorem IV.3.1 implies that there exists a solution to (W5), (ID) on $[-T, +T]$.

Small data global well-posedness: If $\vec{u}_0 \in \dot{\mathcal{H}}^1$ and $\|u_0\|_{\dot{H}^1} \leq \delta_0/C_S$, where C_S is the constant in the Strichartz inequality (III.2.3) with $p=5$, $q=10$, then $\|S_L(\cdot)\vec{u}_0\|_{L^5(\mathbb{R},L^{10})} \leq \delta_0$, and one can use Theorem IV.3.1 with $I=\mathbb{R}$. This shows that the corresponding solution u is globally defined, and that $u \in L^5(\mathbb{R},L^{10})$.

Proof of Theorem IV.3.1. Assume without generality that $t_0=0$. We use the Banach fixed point theorem, proving that the operator A, defined by

$$Av\,(t) = S_L(t)\vec{u}_0 + Bv\,(t), \quad Bv\,(t) = \int_0^t \frac{\sin\big((t-s)|D|\big)}{|D|} v^5(s)ds, \tag{IV.3.3}$$

is a contraction on X defined by

$$X = \left\{ v \in L^5(I,L^{10}),\ \|v\|_{L^5(I,L^{10})} \leq 2\delta_0 \right\}.$$

We first prove that A maps X into X. Indeed, If $v \in X$, then by Theorem III.2.1 (see Remark III.2.2),

$$\|Bv(t)\|_{L^5(I,L^{10})} \leq C_S \left\|v^5\right\|_{L^1(I,L^2)} \leq C_S \|v\|^5_{L^5(I,L^{10})} \leq C_S\delta_0^5 \leq \delta_0,$$

assuming $\delta_0 \leq C_S^{-1/4}$. Thus $Av \in X$.

We next prove that A is a contraction on X. Let $v,w \in X$. Using $w^5 - v^5 = (w-v)(w^4+w^3v+w^2v^2+wv^3+v^4)$ and Young's inequality $ab \leq a^p/p + b^q/q$, $1/p+1/q=1$, one obtains

$$|v^5 - w^5| \leq \frac{5}{2}|v-w|\,(v^4+w^4).$$

Combining with Hölder's inequality, we obtain

$$\left\|v^5-w^5\right\|_{L^1(I,L^2)} \leq \frac{5}{2}\|v-w\|_{L^5(I,L^{10})}\left(\|v\|^4_{L^5(I,L^{10})} + \|w\|^4_{L^5(I,L^{10})}\right). \tag{IV.3.4}$$

By Strichartz estimates

$$\begin{aligned}\|Av-Aw\|_{L^5(I,L^{10})} &= \|Bv-Bw\|_{L^5(I,L^{10})} \\ &\leq C_S\left\|v^5-w^5\right\|_{L^1(I,L^2)} \leq 5C_S\|v-w\|_{L^5(I,L^{10})}\delta_0^4.\end{aligned}$$

If δ_0 is small enough ($\delta_0 = (10C_S)^{-1/4}$ works), one has

$$\|Av-Aw\|_{L^5(I,L^{10})} \leq \frac{1}{2}\|v-w\|_{L^5(I,L^{10})}.$$

This shows that A is a contraction on X.

Let u be the only fixed point of A in X. Since $u=Au$ and $u \in L^5(I,L^{10})$ we see that u is a solution of (W5) on I.[2] Using

$$u - S_L(\cdot)\vec{u}_0 = Bu,$$

and $\|Bu\|_{L^5(I,L^{10})} \leq \delta^5$, and Strichartz inequality, we obtain (IV.3.2). It remains to prove the uniqueness statement. From the contraction argument, we see that u is the unique solution of (W5) such that $\|u\|_{L^5(I,L^{10})} \leq \delta_0$. We prove a stronger statement, Lemma IV.3.2 below, that will conclude the proof. ☐

Lemma IV.3.2. *Let u, v be two solutions of* (W5) *on an interval I with $t_0 \in I$. Assume $\vec{u}(t_0) = \vec{v}(t_0)$. Then $u=v$.*

Proof. Assume again $t_0=0$ to simplify notations. Let $\delta_0>0$ be as in Theorem IV.3.1. We let $K=[a,b]$ be a compact subinterval of I such that $t_0 \in K$. We will prove that $u(t)=v(t)$ for $t \in K$. Since K is compact, we have by Definition IV.2.1,

$$u \in L^5(K,L^{10}), \quad v \in L^5(K,L^{10}).$$

We can thus divide K into p subintervals $[\tau_j,\tau_{j+1}]$, $0 \leq j \leq p-1$, with $\tau_0 < \tau_1 < \ldots < \tau_p$, such that

$$\forall j \in \{0,\ldots,J-1\}, \quad \max\left(\|u\|_{L^5([\tau_j,\tau_{j+1}],L^{10})}, \|v\|_{L^5([\tau_j,\tau_{j+1}],L^{10})}\right) \leq \delta_0.$$

Let j_0 be an index such that $0 \in [\tau_{j_0},\tau_{j_0+1}]$. By the proof of Theorem III.2.1, with $I=[\tau_{j_0},\tau_{j_0+1}]$, noting that u and v are in X, we obtain $u(t)=v(t)$ for $t \in [\tau_{j_0},\tau_{j_0+1}]$. This implies

$$\vec{u}(\tau_{j_0}) = \vec{v}(\tau_{j_0}) \text{ and } \vec{u}(\tau_{j_0+1}) = \vec{v}(\tau_{j_0+1}).$$

We can then iterate the preceding arguments on the intervals $[\tau_j,\tau_{j+1}]$, $j=j_0+1$, $j=j_0+2$ until $j=J-1$, and $j=j_0-1$, $j=j_0-2$ until $j=0$ to obtain that $u(t)=v(t)$ for $t \in K$, concluding the proof. ☐

[2] Recall that "solution" is to be taken in the sense of Definition IV.2.1.

3.b. Maximal solution. Using the above local existence theorem, we can now glue the solutions above to construct a maximal solution of (W5).

Corollary IV.3.3. *Let $\vec{u}_0 \in \dot{\mathcal{H}}^1$ and $t_0 \in \mathbb{R}$. Then there is a unique maximal solution of* (W5), (ID). *Denoting by $I_{\max} = (T_-, T_+)$ its interval of existence, we have the following* blow-up criteria*:*

$$T_+ < \infty \Longrightarrow u \notin L^5([t_0, T_+[, L^{10}), \quad T_- > -\infty \Longrightarrow u \notin L^5(]T_-, t_0], L^{10}). \tag{IV.3.5}$$

The phrase "maximal solution" in the theorem means that if v is another solution of (W5), (ID) defined on an interval I with $t_0 \in I$, then $I \subset I_{\max}$ and $u(t) = v(t)$ for all $t \in I$.

Proof. Let $\mathcal{J}$ be the set of all open intervals I such that $t_0 \in I$, and there exists a solution v of (W5), (ID) on I. Let

$$I_{\max} = \bigcup_{I \in \mathcal{H}} I.$$

By Theorem IV.3.1, $\mathcal{J}$ is nonempty. Thus $I_{\max}$ is an open interval containing t_0. If $t \in I_{\max}$, there exists an interval I and a solution v of (W5), (ID) on I. By the uniqueness Lemma IV.3.2, the value $v(t)$ does not depend on the choice of I. We denote by $u(t)$ this common value. Let K be a compact subinterval of $I_{\max}$. We next prove:

$$u \in L^5(K, L^{10}). \tag{IV.3.6}$$

Indeed, for all $t \in K$, there exist an open interval $I \in \mathcal{J}$ such that $t \in I$ and u is a solution of (W5) on I. This implies in particular that $u \in L^5([t-\varepsilon, t+\varepsilon], L^{10})$ if $\varepsilon = \varepsilon(t)$ is small enough. Using the compactness of K, we can cover K by a finite numbers of interval $]t - \varepsilon(t), t + \varepsilon(t)[$, and thus we obtain (IV.3.6).

If $t \in I_{\max}$, by the definition of $I_{\max}$ and the uniqueness Lemma IV.3.2, we have that

$$u(t) = S_L(t)\vec{u}_0 + \int_0^t \frac{\sin\big((t-s)|D|\big)}{|D|} u^5(s)ds,$$

which concludes the proof that u is a solution of (W5), (ID) on $I_{\max}$. The maximality of u is a direct consequence of the definition of $I_{\max}$ and Lemma IV.3.2. □

Let us mention that it is not possible to improve the blow-up criterion to

$$T_+ < \infty \Longrightarrow \limsup_{t\to\infty} \|\vec{u}(t)\|_{\dot{\mathcal{H}}^1} = +\infty.$$

Indeed, it was proved by Krieger, Schlag and Tataru [**24**] that there exist solutions of (W5) with $\sigma = 1$, with finite time of existence T_+ and such that

$$\limsup_{t\to T_+} \|\vec{u}(t)\|_{\dot{\mathcal{H}}^1} < \infty.$$

Exercice IV.2. Consider the cubic nonlinear wave equation (W3), (ID) with initial data (u_0, u_1) in the critical space $\dot{\mathcal{H}}^{1/2}$, in space dimension 3. Define a concept of "solution" for this equation analogous to the one of Definition IV.2.1. Prove the analogs of Theorem IV.3.1 and Corollary IV.3.3. *Hint*: use the $L^4(I \times \mathbb{R}^3)$ norm instead of the $L^5(I, L^{10}(\mathbb{R}^3))$ norm, and the Strichartz inequality of Theorem III.2.4.

IV.4. Finite speed of Propagation

Remark IV.4.1. The proof of Theorem III.2.1 implies that if I is an interval, $t_0 \in I$, and u is a solution of (W5), (ID) on I such that $\|u\|_{L^5(I,L^{10})} \leq \delta_0/2$, then u is the limit, in $L^5(I, L^{10})$, of the sequence u^n defined by $u^0 = 0$, $u^n = Au^n$, where A is the operator defined in the proof. Indeed, by Strichartz estimates,

$$\|S_L(\cdot - t_0)\vec{u}_0\|_{L^5(I,L^{10})} \leq \|u\|_{L^5(I,L^{10})} + C_S\|u\|^5_{L^5(I,L^{10})} \leq \frac{\delta_0}{2} + C_S\delta_0^5/32 \leq \delta_0.$$

Thus $\vec{u}_0$ satisfies the assumption of Theorem III.2.1 and the conclusion follows from the fact that u is a fixed point of the contraction A.

This remark will be used at least twice in this course to obtain properties of the solution u. We will first use it to prove the finite speed of propagation property for the nonlinear equation:

Theorem IV.4.2. *Let $(t_0, x_0) \in \mathbb{R}^{1+3}$, $t_1 > t_0$, $R > 0$. We denote $\Gamma = \Big\{(t,x) \in \mathbb{R}\times\mathbb{R}^N \,:\, t_0 \leq t \leq t_1,\ |x - x_0| \leq R - |t - t_0|\Big\}$ Let u and v be two solutions of* (W5) *on $[t_0, t_1]$. We suppose $(u, \partial_t u)(t_0, x) = (v, \partial_t v)(t_0, x)$ for all $x \in B_R(x_0)$. Then $u(t,x) = v(t,x)$ for almost all $(t,x) \in \Gamma$.*

PROOF. Dividing the interval $[t_0, t_1]$ into subintervals $[\tau_j, \tau_{j+1}]$, $0 \leq j \leq J-1$, $t_0 = \tau_0 < \tau_1 < \ldots < \tau_J$, such that

$$\forall j \in \{0, \ldots, J-1\}, \quad \max\left(\|u\|_{L^5([\tau_j,\tau_{j+1}],L^{10}(\mathbb{R}^3))}, \|v\|_{L^5([\tau_j,\tau_{j+1}],L^{10}(\mathbb{R}^3))}\right) \leq \delta_0/2,$$

we see that it is sufficient to prove the theorem with the additional assumption

$$\max\left(\|u\|_{L^5([t_0,t_1],L^{10}(\mathbb{R}^3))}, \|v\|_{L^5([t_0,t_1],L^{10}(\mathbb{R}^3))}\right) \leq \delta_0/2.$$

Thus $u = \lim_{n\to\infty} u^n$, $v = \lim_{n\to\infty} v^n$ in $L^5(I, L^{10})$, $I = [t_0, t_1]$, where u^n and v^n are defined by

$$u^0 = v^0 = 0, \quad u^{n+1} = Au_n, \quad v^{n+1} = \widetilde{A}v^n,$$

where A is as in the proof of Theorem IV.3.1 (see (IV.3.3)), and $\widetilde{A}$ is the analog of A for the initial data of v:

$$\widetilde{A}w(t) = S_L(t)\vec{v}(0) + \int_0^t \frac{\sin\left((t-s)|D|\right)}{|D|} w^5(s)ds.$$

(As usual, we assume $t_0 = 0$ to simplify notations).

We prove by induction on n that $u^n(t,x) = v^n(t,x)$ for almost every $(t,x) \in \Gamma$. This is true for $n = 0$, since $u^0 = v^0 = 0$.

Next, we assume that $u^n(t,x) = v^n(t,x)$ for almost every $(t,x) \in \Gamma$. We have

$$u^{n+1}(t) - v^{n+1}(t) = S_L(t)(\vec{u}(0) - \vec{v}(0)) + \int_0^t \frac{(\sin(t-s)|D|)}{|D|} \left(u^n(s) - v^n(s)\right) ds.$$

By finite speed of propagation for the linear wave equation and the assumption that $\vec{u}^0(x) = \vec{v}^0(x)$ for $|x - x_0| < R$, we obtain that $S_L(t)(\vec{u}(0) - \vec{v}(0)) = 0$ for almost all $(t,x) \in \Gamma$. On the other hand, if $s \in [0,t]$, the inductive hypothesis implies that $u^n(s,x) = v^n(s,x)$ for $|x - x_0| < R - s$. Combining with finite speed of propagation, we see that

$$\frac{(\sin(t-s)|D|)}{|D|} \left(u^n(s) - v^n(s)\right) = 0$$

for almost every (t,x) with $|x - x_0| < R - s - (t-s) = R - t$, i.e. for almost every $(t,x) \in \Gamma$.

Thus $u^n = v^n$ almost everywhere on Γ. Passing to the limit, we obtain $u^n = v^n$ on Γ. □

IV.5. Stability

We now prove that the flow of the equation (W5) is continuous in $\dot{\mathcal{H}}^1$, i.e. that if the initial data of two solutions u and v are close in this space, then $\vec{u}(t)$ and $\vec{v}(t)$ are close for all times t in their domain of existence. In the statement, we must take into account the fact that the solutions u and v might have different maximal interval of existence.

THEOREM IV.5.1. *Let $t_0 \in \mathbb{R}$, $\vec{u}_0 = (u_0, u_1) \in \dot{\mathcal{H}}^1$. Let u be the solution of* (W5), (ID). *Let I be a compact interval such that $t_0 \in I \subset I_{\max}(\vec{u}_0)$. Let $(\vec{u}_0^k)_k$ be a sequence in $\dot{\mathcal{H}}^1$ such that $\lim_n \vec{u}_0^k = \vec{u}_0$ in $\dot{\mathcal{H}}^1$. Let u^k be the corresponding solutions. Then for large k, $I \subset I_{\max}(\vec{u}_0^k)$, and*

$$\lim_{k\to\infty} \left(\sup_{t\in I} \left\|\vec{u}^k(t) - \vec{v}^k(t)\right\|_{\dot{\mathcal{H}}^1} + \|u^k - v^k\|_{L^5(I,L^{10})}\right) = 0.$$

PROOF. We will consider $T > 0$ such that

$$\|u\|_{L^5([0,T],L^{10})} \leq \delta_0 \tag{IV.5.1}$$

(where δ_0 is a small parameter), and prove that $T^+(u^k) > T$ for large k and

$$\|u - u^k\|_{L^5([0,T],L^{10})} \sup_{0\leq t\leq T} \left\|\vec{u}(t) - \vec{u}^k(t)\right\|_{\dot{\mathcal{H}}^1} \xrightarrow[k\to\infty]{} 0. \tag{IV.5.2}$$

The conclusion of the theorem will then follow by iteration, dividing as above the interval I into subintervals where the $L^5 L^{10}$ norm of u is small.

We have

$$u(t) - u^k(t) = S_L(t)(\vec{u}_0 - \vec{u}_0^k) + \int_0^t \frac{\sin\left((t-s)|D|\right)}{|D|} \left(u^5(s) - (u^k)^5(s)\right) ds. \tag{IV.5.3}$$

As in (IV.3.4), we have

$$\left\|u^5 - (u^k)^5\right\|_{L^1([0,t],L^2)} \leq \frac{5}{2} \left\|u - u^k\right\|_{L^5([0,t],L^{10})} \left(\|u\|^4_{L^5([0,t],L^{10})} + \left\|u^k\right\|^4_{L^5([0,t],L^{10})}\right).$$

Using the triangle inequality, and (IV.5.1), we deduce

$$\left\|u^5-(u^k)^5\right\|_{L^1([0,t],L^2)} \leq \frac{5}{2}\left\|u-u^k\right\|_{L^5([0,t],L^{10})}\left(2\delta_0^4+\left\|u-u^k\right\|^4_{L^5([0,t],L^{10})}\right).$$

Thus, by (IV.5.3) and Strichartz estimate, we have that for all $t\in[0,T]$

$$a_k(t)\leq C\left(\varepsilon_k+\delta_0^4 a_k(t)+a_k(t)^5\right),$$

where $a_k(t)=\left\|u-u^k\right\|_{L^5([0,t],L^{10})}$, $\varepsilon_k=\left\|\vec{u}_0-\vec{u}_0^k\right\|_{\dot{\mathcal{H}}^1}\underset{k\to\infty}{\longrightarrow}0$, and C is a constant. Taking δ_0 small (so that $C\delta_0^4\leq 1/2$), we deduce

$$a_k(t)\leq 2C\varepsilon_k+2Ca_k(t)^5. \tag{IV.5.4}$$

We temporarily fix k, large enough so that $2C(4C\varepsilon_k)^5\leq C\varepsilon_k$, and prove

$$\forall t\in[0,T],\quad a_k(t)\leq 3C\varepsilon_k. \tag{IV.5.5}$$

Indeed, (IV.5.5) is true for small $t>0$, since a is continuous and $a(0)=0$. If (IV.5.5) does not hold, using again the continuity of a, we see that there exists a $t\in[0,T]$ such that $3C\varepsilon_k<a_k(t)\leq 4C\varepsilon_k$. By (IV.5.4), and the smallness of ε_k we see that $a_k(t)\leq 3C\varepsilon_k$. This is a contradiction, concluding the proof of (IV.5.5). This type of reasoning is called a *bootstrap argument.* By (IV.5.5),

$$\lim_{k\to\infty}a_k(T)=0$$

Using (IV.5.3) and Strichartz estimate again, we deduce

$$\sup_{t\in[0,T]}\left\|\vec{u}(t)-u^k(t)\right\|_{\dot{\mathcal{H}}^1}\underset{k\to\infty}{\longrightarrow}0,$$

which concludes the proof. □

IV.6. Persistence of regularity, conservation of the energy

The energy of a solution u of (W5) is defined as

$$E(\vec{u}(t))=\frac{1}{2}\int(\partial_t u(t,x))^2dx+\frac{1}{2}\int|\nabla u(t,x)|^2dx-\frac{\sigma}{6}\int(u(t,x))^6dx, \tag{IV.6.1}$$

where all integrals are taken over $\mathbb{R}^3$. Multiplying the equation (W5) by $\partial_t u(t,x)$, integrating on $\mathbb{R}^3$ and integrating by part, we would obtain that the derivative of the energy is 0, and thus that it is independent of time. However this computation is purely formal. To make it rigorous, we need to work on more regular solutions. The key ingredient for this is the *persistence of regularity* property:

THEOREM IV.6.1. *Let $\vec{u}_0=(u_0,u_1)\in\dot{\mathcal{H}}^1$, u be the solution of* (W5), (ID) *given by Corollary IV.3.3, and $I_{\max}$ its maximal interval of existence. Let $\ell\geq 2$ be an integer. Assume $\vec{u}_0\in\dot{\mathcal{H}}^\ell$. Then*

$$\vec{u}\in C^0\left(I_{\max},\dot{\mathcal{H}}^\ell\right),\quad \partial_t^2 u\in C^0\left(I_{\max},\dot{H}^{\ell-2}\right) \tag{IV.6.2}$$

In particular, if $\ell\geq 4$, $u\in C^2\left(I_{\max}\times\mathbb{R}^3\right)$

PROOF. We prove the result for $\ell=2$. The proof for $\ell\geq 3$ is very close and left to the reader. As usual, we assume $t_0=0$. We note that the property of $\partial_t^2 u$ in (IV.6.2) follows from $u\in C^0(I_{\max},\dot{H}^\ell)$, the equation $\partial_t^2u=\Delta u+\sigma u^5$ and Sobolev embedding (which implies that $\dot{H}^\ell\cap\dot{H}^1(\mathbb{R}^3)$ is an algebra for $\ell\geq 2$. We are thus left to prove $\vec{u}\in C^0\left(I_{\max},\dot{\mathcal{H}}^\ell\right)$.

Step 1.

We first consider a small $T>0$. By the proof of Theorem IV.3.1, the restriction of u to $[-T,+T]$ is the limit, in $L^5([-T,+T),L^{10})$, of the sequence u^n defined as above by $u^0=0$, $u^{n+1}=Au^n$, where A is defined by (IV.3.3). Let $j\in\{1,2,3\}$. We have

$$\partial_{x_j}(u^{n+1})=S_L(t)\partial_{x_j}\vec{u}_0+5\int_0^t\frac{\sin\left((t-s)|D|\right)}{|D|}\left(u^n(s)\right)^4\partial_{x_j}u^n(s)ds, \tag{IV.6.3}$$

where ∂_{x_j} is the distributional derivative with respect to x_j, and we have used the formula $\partial_{x_j}(v^5)=5v^4\partial_{x_j}v$, which is valid for $v\in\dot{H}^2$ (this can be checked easily using that the functions in $\dot{H}^2$ are continuous[3].

[3]The reader which is not familiar with the theory of distribution can assume in all this proof that $\vec{u}^0\in C^3\times C^2$: all functions u^n will then by also of class C^2, and differentiation can be understood in the classical sense.

We prove by induction on n that $(u^n, \partial_t u^n) \in C^0([-T,+T], \dot{H}^2)$ with

$$\sup_{-T\leq t\leq T} \|(u^n, \partial_t u^n)\|_{\dot{\mathcal{H}}^2} \leq 2M, \quad \sup_{|\alpha|\leq 1} \|\partial_x^\alpha u^n\|_{L^5([-T,+T],L^{10})} \leq 2\delta_0, \tag{IV.6.4}$$

where $M = \|\vec{u}_0\|_{\dot{\mathcal{H}}^2}$, and we have chosen T small enough, so that

$$\sup_{|\alpha|\leq 1} \|S_L(\cdot)\partial_x^\alpha \vec{u}_0\|_{L^5([-T,+T],L^{10})} \leq \delta_0. \tag{IV.6.5}$$

The case $n = 0$ is trivial since $u^0 = 0$.

Next we assume that $\overrightarrow{u^n} \in C^0\left([-T,+T], \dot{H}^2\right)$ and satisfies (IV.6.4). Then by Strichartz estimates, the definition of u^{n+1}, the inductive hypothesis, (IV.6.5) and the smallness of δ_0:

$$\sup_{|\alpha|\leq 1} \|\partial_x^\alpha u^n\|_{L^5([-T,+T],L^{10})} \leq \delta_0 + C\delta_0^5 \leq 2\delta_0. \tag{IV.6.6}$$

The same argument, together with the definition of M yields that $(u^n(t), \partial_t u^n(t)) \in C^0([-T,+T], \dot{\mathcal{H}}^2)$ and

$$\sup_{-T\leq t\leq T} \|(u^n(t), \partial_t u^n(t))\|_{\dot{\mathcal{H}}^2} \leq M + C\delta_0^5 \leq 2M. \tag{IV.6.7}$$

This shows that (IV.6.4) holds for all n as announced.

Step 2. Fixing $j \in \{1,2,3\}$ we will prove that $(\partial_{x_j} u^n)_n$ is a Cauchy sequence in $C^0([-T,+T], \dot{\mathcal{H}}^1)$ and $L^5([-T,+T], L^{10})$. Indeed,

$$\begin{aligned}
&\left|(u^n)^4 \partial_{x_j} u^n - (u^{n-1})^4 \partial_{x_j} u^{n-1}\right| \\
&\qquad = \left|\left((u^n)^4 - (u^{n-1})^4\right) \partial_{x_j} u^n + \left(\partial_{x_j} u^{n-1} - \partial_{x_j} u^n\right)(u^n)^4\right| \\
&\qquad \lesssim \left|u^n - u^{n-1}\right| \left|\partial_{x_j} u^n\right| \left(|u^n|^3 + |u^{n-1}|^3\right) + \left|\partial_{x_j} u^{n-1} - \partial_{x_j} u^n\right| \left((u^n)^4 + (u^{n-1})^4\right).
\end{aligned}$$

Which yields, by (IV.6.6)

$$\begin{aligned}
&\left\|(u^n)^4 \partial_{x_j} u^n - (u^{n-1})^4 \partial_{x_j} u^{n-1}\right\|_{L^1([-T,+T],L^2)} \\
&\qquad \lesssim \delta_0^4 \left\|\partial_{x_j}(u^n - u^{n-1})\right\|_{L^5([-T,+T],L^{10})} + \delta_0^4 \left\|u^n - u^{n-1}\right\|_{L^5([-T,+T],L^{10})}
\end{aligned}$$

By Strichartz estimates and the definition of u^n, letting

$$\begin{aligned}
c_n &= \sup_{|\alpha\leq 1} \left\|\partial_x^\alpha (u^n - u^{n-1})\right\|_{L^5([-T,+T],L^{10})} \\
d_n &= \sup_{-T\leq t\leq T} \|(u^n(t) - u^{n-1}(t), \partial_t u^n(t) - \partial_t u^{n-1}(t))\|_{\dot{\mathcal{H}}^2},
\end{aligned}$$

we obtain, for $n \geq 1$, choosing δ_0 small enough,

$$c_{n+1} \leq \frac{1}{99} c_n, \quad d_{n+1} \lesssim c_n.$$

This proves that $(u^n)_n$ is a Cauchy sequence, and thus has a limit, in $L^5([-T,+T], L^{10})$ and in $C^0([-T,+T], \dot{H}^2)$, and similarly that $(\partial_t u^n)_n$ has a limit in $C^0([-T,+T], \dot{H}^1)$. By uniqueness of limits (for example in L^1_{loc}), we obtain

$$\vec{u} \in C^0([-T,+T], \dot{\mathcal{H}}^2), \quad \forall j, \quad \partial_{x_j} u \in L^5(\mathbb{R}, L^{10}).$$

Step 3. Maximal interval of existence. We next consider

$$\tau_+ = \sup \left\{\tau < T_+, \quad \sup_{|\alpha|\leq 1} \|\partial_x^\alpha u\|_{L^5([0,\tau],L^{10})} < \infty\right\},$$

where T_+ is the maximal time of existence of u as a $\dot{\mathcal{H}}^1$ solution, defined in Corollary IV.3.3. In this step we prove that $\tau_+ = T_+$.

Assume that $\tau_+ < T_+$. Thus $u \in L^5([0,\tau_+], L^{10})$. We let $\tau_0 \in [0, \tau_+[$ such that

$$\|u\|_{L^5([\tau_0,\tau_+],L^{10})} = \delta \ll 1.$$

Using Strichartz estimates and the formula

$$\partial_{x_j} u = S_L(t)\partial_{x_j} \vec{u}_0 + 5 \int_0^t \frac{\sin\left((t-s)|D|\right)}{|D|} (u(s))^4 \partial_{x_j} u(s) ds,$$

we see that $\vec{u} \in C^0([0,\tau], \dot{\mathcal{H}}^2)$ for all $\tau < \tau_+$. As a consequence of Strichartz estimates, we also obtain, for all $\tau < \tau_+$,

$$\forall j, \quad \left\|\partial_{x_j} u\right\|_{L^5([\tau_0,\tau],L^{10})} \lesssim \|S_L(\cdot - \tau_0)\vec{u}(\tau_0)\|_{L^5([\tau_0,\tau],L^{10})} + C\delta^4\|\partial_{x_j} u\|_{L^5([\tau_0,\tau],L^{10})}.$$

Hence for all $\tau < \tau_+$,

$$\forall j, \quad \left\|\partial_{x_j} u\right\|_{L^5([\tau_0,\tau],L^{10})} \lesssim \|S_L(\cdot - \tau_0)\vec{u}(\tau_0)\|_{L^5([\tau_0,\tau_+],L^{10})},$$

This shows that $\partial_{x_j} u \in L^5([0,\tau_+], L^{10})$. Using the energy inequality, we see that $\vec{u} \in C^0([0,\tau_+], \dot{\mathcal{H}}^2)$. Thus $\vec{u}(\tau_+) \in \dot{\mathcal{H}}^2$, which is a contradiction with the definition of τ_+, since by Steps 1 and 2, $\partial_x^\alpha \vec{u} \in L^5([\tau_+ - \varepsilon, \tau_+ + \varepsilon])$ if ε is small $\varepsilon > 0$.

This concludes the proof for $\ell = 2$. The proof for $\ell \geq 3$ is mostly identical, considering all $\partial_x^\alpha u$ with $|\alpha| \leq \ell - 1$ instead of $|\alpha| \leq 1$. □

EXERCICE IV.3. Prove that if $T_+ < \infty$, then

$$\lim_{t \to T_+} \|\vec{u}(t)\|_{\dot{\mathcal{H}}^2} = +\infty.$$

COROLLARY IV.6.2. *Let u be a solution with initial data $(u_0, u_1) \in (C_0^\infty(\mathbb{R}^3))^2$. Then the corresponding solution u of* (W5), (ID) *is in $C^\infty(I_{\max} \times \mathbb{R}^3)$, where $I_{\max} = I_{\max}(\vec{u}_0)$ is the maximal interval of existence of u.*

PROOF. The corollary follows immediately from Theorem IV.6.1, using that

$$C_0^\infty(\mathbb{R}^3) \subset \bigcap_{s \geq 1} \dot{\mathcal{H}}^s \subset C^\infty. \tag{IV.6.8}$$

□

EXERCICE IV.4. Prove (IV.6.8). *Hint*: use the Fourier representation of u:

$$u(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{ix\cdot\xi}\hat{u}(\xi)d\xi.$$

We are now in position to prove rigorously the conservation of the energy:

THEOREM IV.6.3. *Let the energy E be defined by* (IV.6.1). *Let $\vec{u}$ be a solution of* (W5). *Then $E(\vec{u}(t))$ is independent of $t \in I_{\max}(u)$.*

PROOF. Let $t_0, t_1 \in I_{\max}(u)$. Let $\vec{u}_0^n = (u_0^n, u_1^n) \in \left(C_0^\infty(\mathbb{R}^3)\right)^2$ such that

$$\lim_{n\to\infty} \|\vec{u}_0^n - \vec{u}(t_0)\|_{\dot{H}^1} = 0. \tag{IV.6.9}$$

Let u^n be the solution of (W5) with initial data $u^n(0) = u_0^n$, $\partial_t u^n(0) = u_1^n$. By the stability theorem IV.5.1, $[t_0, t_1]$ is included in the maximal interval of existence of u^n for large n.

By Corollary IV.6.2, $u^n \in C^\infty([t_0, t_1] \times \mathbb{R}^3)$. Since it satisfies (W5) in the sense of distribution, it must also satisfy this equation in the classical sense. By finite speed of propagation $u^n(t)$ is a compactly supported function (in space) for all $t \in [t_0, t_1]$. We have

$$\int \partial_t^2 u^n \partial_t u^n - \int \Delta u^n \partial_t u^n - \sigma \int (u^n)^5 \partial_t u^n = 0$$

Since $\int \Delta u^n \partial_t u^n = \int \sum_{j=1,2,3} \partial_{x_j} u^n \partial_t \partial_{x_j} u^n$, we deduce

$$\frac{d}{dt} E(\vec{u}^n(t)) = 0, \quad t_0 \leq t \leq t_1.$$

Thus $E(\vec{u}^n(t_0)) = E(\vec{u}^n(t_1))$. Passing to the limit $n \to \infty$ and using Theorem IV.5.1, we deduce

$$E(\vec{u}(t_0)) = E(\vec{u}(t_1)),$$

concluding the proof. We have used that by the Sobolev embedding $\dot{H}^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$, the convergence in $\dot{H}^1$ implies the convergence in L^6. □

In the case $\sigma = -1$, all the terms in the definitions of the energy are positive, and we have

$$\vec{u}(t) \leq 2E(\vec{u}(t)).$$

This implies that the $\dot{H}^1$ norm of any solution u of (W5) is bounded on its maximal interval of existence. This is not sufficient to ensure global existence. We will see however that in this case, all solutions are indeed global.

DEFINITION IV.6.4. The equation (W5) or the corresponding nonlinearity is called *defocusing* (or *repulsive*) when $\sigma = -1$ and *focusing* (or *attractive*) when $\sigma = 1$.

Let us mention that we can also construct classical solutions of (W5) (or of any equation of the form (NLW) with $p \in \mathbb{N}$, $p \geq 2$, in space dimension 3), without Strichartz estimates, using the representation formulas of Chapter 1 and a fixed point argument. These solutions coincide with the finite energy solutions of Definition IV.2.1 when $\vec{u}_0 \in C_0^3(\mathbb{R}^3) \times C_0^2(\mathbb{R}^3)$ for example. This is an alternative approach to obtain Corollary IV.6.2. We refer to [**26**, Section I.5] for the details.

One can also prove persistence of regularity of the cubic wave equation:

THEOREM IV.6.5. *Let u be the solution of* (W3), (ID) *with initial data $\vec{u}_0 \in \dot{\mathcal{H}}^{1/2} \cap \dot{\mathcal{H}}^k$, for $k \geq 1$ such that $2k$ is an integer*[4]*. Then $\vec{u} \in C^0\left(I_{\max}, \dot{\mathcal{H}}^k\right)$, where $I_{\max}$ is the maximal interval of existence of u. Furthermore the energy of u:*

$$\frac{1}{2}\int (\partial_t u(t,x))^2 dx + \frac{1}{2}\int |\nabla u(t,x)|^2 dx - \frac{\sigma}{4}\int |u(t,x)|^4 dx$$

is conserved.

EXERCICE IV.5. Prove Theorem IV.6.5. *Hint*: for the case $k = 1$, one can use the Hölder-type inequality

$$\|u^3\|_{L^1L^2} \leq \|u\|_{L^4L^4}\|u\|^2_{L^{8/3}L^8}$$

and the fact that $(8,8)$ is a $\dot{\mathcal{H}}^1$ Strichartz admissible couple.‘

IV.7. Blow-up in finite time

In the focusing case $\sigma = 1$, there exists solutions blowing-up in finite time:

THEOREM IV.7.1. *Let $T > 0$. There exists a solution u of* (W5)*, with C^∞, compactly supported initial data $\vec{u}_0$ at $t = 0$, such that $T_+(\vec{u}_0) = T$.*

PROOF. By scaling invariance, it is sufficient to construct one solution of (W5) blowing-up in finite time, with compactly supported, smooth initial data.

Let Y be a solution of the ODE $Y'' = Y^5$ defined on $[0,1[$, and blowing-up at $t = 1$. For example $Y(t) = c(1-t)^{-1/2}$, where $\frac{3}{4} = c^4$. Note that Y is a solution of (W5) (in the classical sense).

Let $\varphi \in C_0^\infty(\mathbb{R}^3)$ such that $\varphi(x) = 1$ for $|x| \leq 2$. Let u be the solution of (LW) with initial data $(\varphi Y(0), \varphi Y'(0))$. Let T_+ be the maximal time of existence of u. By finite speed of propagation,

$$u(t,x) = Y(t), \quad |x| \leq 2 - t, \quad t \in [0, T_+[.$$

If $T_+ > 1$, we have

$$\int_0^1 \left(\int_{|x|\leq 1} u^{10}(t,x)dx\right)^{1/2} dt = c^5 \int_0^1 \frac{1}{(1-t)^{5/2}} dt = +\infty,$$

a contradiction with the fact that u must be in $L^5([0,1], L^{10})$. Thus $T_+ \leq 1$, concluding the proof. ☐

The preceding proof is not completely rigorous: we have used finite speed of propagation for the equation (W5) outside of the framework of Theorem IV.4.2, since Y is not a solution of (W5) in the sense of Definition IV.2.1. We thus need the analog of IV.7.2 for classical solutions:

THEOREM IV.7.2. *Let $(t_0, x_0) \in \mathbb{R}^{1+3}$, $t_1 > t_0$, $R > 0$. We denote by $\Gamma = \Big\{(t,x) \in \mathbb{R} \times \mathbb{R}^N \ : \ t_0 \leq t \leq t_1,\ |x - x_0| \leq R - |t - t_0|\Big\}$ Let $u, v \in C^2(\Gamma)$ be two classical solutions of* (W5) *on Γ. We suppose $(u, \partial_t u)(t_0, x) = (v, \partial_t v)(t_0, x)$ for all $x \in B_R(x_0)$. Then $u(t,x) = v(t,x)$ all $(t,x) \in \Gamma$.*

We leave the proof of Theorem IV.7.2 as an exercise to the reader:

EXERCICE IV.6. Let u and v be as in Theorem IV.7.2. Assume $t_0 = 0$, $x_0 = 0$. Let

$$V(t) = \frac{1}{2}\int_{|x|<R-t} (u(t,x) - v(t,x))^2 dx + \frac{1}{2}\int_{|x|<R-t} (\partial_t u(t,x) - \partial_t v(t,x))^2 dx$$
$$+ \frac{1}{2}\sum_{j=1}^{3}\int_{|x|<R-t} (\partial_{x_j} u(t,x) - \partial_{x_j} v(t,x))^2 dx.$$

[4]See Exercise IV.2

(1) Prove that $V'(t) \leq CV(t)$ for $t \in [0, t_1]$.
(2) Prove that $V(t) = 0$ for all $t \in [0, t_1]$.

CHAPTER V

Examples of dynamics

In this chapter, we give examples of dynamics of (W5). Section V.1 concerns global solutions which behave asymptotically as solutions of the linear wave equation. In Section V.2, we will consider stationary solutions and traveling waves. Section V.3 gives a dynamical characterization of the stationary solutions in a radial context.

V.1. Scattering

1.a. Definition and characterization.

Definition V.1.1. The solution u of (W5) is said to *scatter* in the future to a linear solution if $T_+(u) = +\infty$ and there exists $\vec{v}_0 \in \dot{\mathcal{H}}^1(\mathbb{R}^3)$ such that

$$\lim_{t\to\infty} \left\| \vec{S}_L(t)\vec{v}_0 - \vec{u}(t) \right\|_{\dot{\mathcal{H}}^1} = 0. \tag{V.1.1}$$

In the remainder of this section, we will simply say that a solution as in Definition V.1.1 scatters or is a scattering solution. We next give a characterization of scattering solutions:

Proposition V.1.2. *The solution u of* (W5), (ID) *scatters if and only if $u \in L^5([0,T_+), L^{10})$, where T_+ is the maximal time of existence of u.*

Proof. Let u be a solution such that $u \in L^5([0,T_+), L^{10})$. By the blow-up criterion, we already know that $T_+(u) = +\infty$. Let $\vec{v}_0 \in \dot{\mathcal{H}}^1$. Since $\vec{S}_L(t)$ conserves the $\dot{\mathcal{H}}^1$ norm, we have

$$\text{(V.1.1)} \iff \lim_{t\to\infty} \left\| \vec{v}_0 - \vec{S}_L(-t)\vec{u}(t) \right\|_{\dot{\mathcal{H}}^1} = 0.$$

We are thus reduced to prove that $\vec{S}_L(-t)\vec{u}_0(t)$ has a limit in $\dot{\mathcal{H}}^1$. Since u is a solution in the sense of Definition IV.2.1, we have

$$\vec{S}_L(-t)\vec{u}(t) = \vec{u}_0 + \int_0^t \vec{S}_L(-s)(0,u^5(s))ds.$$

Using $u \in L^5([0,+\infty, L^{10})$ and

$$\left\| \vec{S}_L(-s)(0,u^5(s)) \right\|_{\dot{\mathcal{H}}^1} = \left\| u^5(s) \right\|_{L^2} = \|u(s)\|_{L^{10}}^5,$$

we see that

$$\int_0^\infty \left\| \vec{S}_L(-s)(0,u^5(s)) \right\|_{\dot{\mathcal{H}}^s} ds = \|u\|^5_{L^5([0,\infty),L^{10})} < \infty.$$

Thus $\int_0^t \vec{S}_L(-s)(0,u^5(s))ds$ converges in $\dot{\mathcal{H}}^1$ as t goes to ∞, which shows that u scatters to a linear solution.

Next, we consider a solution u of (W5) that scatters to a linear solution. Thus $T_+(u) = \infty$, and there exists $\vec{v}_0 \in \dot{\mathcal{H}}^1$ such that

$$\lim_{t\to\infty} \left\| \vec{u}(t) - \vec{S}_L(t)\vec{v}_0 \right\|_{\dot{\mathcal{H}}^1} = 0.$$

Fix $T \geq 0$ such that

$$\|S_L(\cdot)\vec{v}_0\|_{L^5([T,\infty[,L^{10})} \leq \delta_0/2,$$

where δ_0 is given by the local well-posedness theory (Theorem IV.3.1). Then, by Strichartz estimates

$$\|S_L(\cdot)\vec{u}(T)\|_{L^5([0,\infty[,L^{10})} \leq \|S_L(\cdot)\vec{v}_0\|_{L^5([T,\infty[,L^{10})} + C_S \left\| \vec{u}(T) - \vec{S}_L(T)\vec{v}_0 \right\|_{\dot{\mathcal{H}}^1} \leq \delta_0$$

for large T. By Theorem IV.3.1 and the uniqueness Lemma IV.3.2,

$$u \in L^5([T,+\infty), L^{10})$$

which concludes the proof. □

Combining Theorem IV.3.1, Strichartz estimates and Proposition V.1.2, we obtain:

Corollary V.1.3 (Small data scattering). *There exists a constant $\varepsilon > 0$ such that for all $\vec{u}_0 \in \dot{\mathcal{H}}^1$ with $\|\vec{u}_0\|_{\dot{\mathcal{H}}^1} \leq \varepsilon$, the solution of* (W5), (ID) *scatter in both time directions.*

Two natural questions arise:

Existence of wave operators: Given $\vec{v}_0 \in \dot{\mathcal{H}}^1$, does there exist a solution u of (W5) with $T_+(u) = +\infty$ and

$$\lim_{t\to\infty} \left\| \vec{u}(t) - \vec{S}_L(t)\vec{v}_0 \right\|_{\dot{\mathcal{H}}^1} = 0? \tag{V.1.2}$$

Asymptotic completeness: Do all solutions of (W5) scatter?

It turns that the answer to the first question is always positive, independently of the sign σ in (W5). The asymptotic completeness is a much more delicate issue. We already know that it is not true in the focusing case $\sigma = 1$, since there exist solutions blowing-up in finite time (see Section IV.7). On the other hand, the asymptotic completeness holds in the defocusing case $\sigma = -1$ (see [**3**]). We will prove this fact for radial solutions. The general proof is more complicated but relies on the same type of arguments.

1.b. Existence of wave operators.

Theorem V.1.4. *Let $\vec{v}_0 \in \dot{\mathcal{H}}^1$. Then there exists a solution u of* (W5) *with $T_+(u) = +\infty$ and such that* (V.1.2) *holds.*

Proof. Let $\vec{v}_0 \in \dot{\mathcal{H}}^1$. Let u be a scattering solution of (W5) such that (V.1.2) holds. Letting $t \to \infty$ in the equality

$$\vec{S}_L(-t)\vec{u}(t) = \vec{u}_0 + \sigma \int_0^t \vec{S}_L(-s)\left(0, \sigma u^5(s)\right) ds,$$

we obtain

$$\vec{v}_0 = \vec{u}_0 + \sigma \int_0^\infty \vec{S}_L(-s)\left(0, \sigma u^5(s)\right) ds. \tag{V.1.3}$$

Note that the integral is convergent in $\dot{\mathcal{H}}^1$ by conservation of the energy for the linear wave equation and since $u \in L^5([0,\infty[, L^{10})$. In view of (V.1.3), we can rewrite Duhamel's formula as

$$u(t) = S_L(t)\vec{v}_0 - \sigma \int_t^\infty S_L(t-s)\left(0, u^5(s)\right) ds. \tag{V.1.4}$$

This shows that the problem of existence of wave operator can be interpreted as a Cauchy problem with initial data at time infinity. To solve this problem, we fix t_0 large such that

$$\|S_L(\cdot)\vec{v}_0\|_{L^5([t_0,\infty), L^{10})} \leq \delta_0,$$

for some small $\delta_0 > 0$ and we prove that the operator A defined by

$$Av(t) = S_L(t)\vec{v}_0 - \sigma \int_t^\infty S_L(t-s)\left(0, v^5(s)\right) ds$$

is a contraction of the metric space X defined by

$$X = \left\{ v \in L^5([t_0,\infty), L^{10}),\ \|v\|_{L^5([t_0,\infty),L^{10})} \leq 2\delta_0 \right\}$$

The details are very close to the ones of the proof of Theorem IV.3.1 and are left to the reader. □

1.c. Asymptotic completeness in the radial defocusing case. We next prove:

Theorem V.1.5. *Let u be a solution of* (W5) *with $\sigma = -1$ and radial initial data. Then u scatters.*

This proof is due to J. Ginibre, A. Soffer, G. Velo (see [**18**]). We divide it into a few Lemmas.

Lemma V.1.6 (Morawetz inequality). *There exists $C > 0$ with the following property. Let u be a solution of* (W5). *Assume that $\vec{u}_0$ is radial, compactly supported and smooth. Let E be the energy of u and $I_{\max}$ its maximal interval of existence. Then*

$$\int_{I_{\max}} \int_{\mathbb{R}^3} \frac{1}{|x|} |u(t,x)|^6 dx dt \leq CE.$$

PROOF. By persistence of regularity and finite speed of propagation, the solution u is C^∞ on $I_{\max}\times\mathbb{R}^3$, and there exists $R>0$ such that $|x|\leq R+|t|$ on the support of u.

Let

$$M(t)=\int_0^\infty \partial_t u(t,r)\partial_r u(t,r) r^2 dr+\int_0^\infty \partial_t u(t,r) u(t,r) r dr.$$

Then

$$M'(t)=\int_0^\infty \partial_t^2 u(u+r\partial_r u) r dr+\underbrace{\int_0^\infty \partial_t u \partial_r(\partial_t u) r^2 dr+\int_0^\infty (\partial_t u)^2 r dr}_{=0},$$

where we use a straightforward integration by parts to prove that the two last terms cancel each other. Using the equation, we have

$$\begin{aligned}M'(t)=\int_0^\infty\left(\partial_r^2 u+\frac{2}{r}\partial_r u-u^5\right)(u+r\partial_r u)\, r dr&\\ =\int_0^\infty \frac{1}{2}\frac{\partial}{\partial r}(u+r\partial_r u)^2 dr-\int_0^\infty u^5(u+r\partial_r u) r dr&\\ =-\frac{1}{2}u^2(t,0)-\int_0^\infty u^6 r dr-\int_0^\infty \frac{1}{6}\frac{\partial}{\partial r}u^6\, r^2 dr=-\frac{1}{2}u^2(t,0)-\frac{2}{3}\int_0^{+\infty} u^6 r dr.&\end{aligned}$$

Next, we notice that $M(t)\lesssim E$. Indeed, this follows easily by the Cauchy-Schwarz inequality and Hardy's inequality

$$\int_0^\infty u^2 dr\leq 4\int_0^\infty (\partial_r u)^2 r^2 dr, \tag{V.1.5}$$

which follows from Cauchy-Schwarz and the equality

$$2\int_0^\infty u\partial_r u r dr=\int_0^\infty \partial_r(u^2) r dr=-\int_0^\infty u^2 dr.$$

Integrating the bound $M'(t)\leq -\frac{2}{3}\int_0^\infty u^6 r dr$ between two times a and b, with $T_-<a<b<T_+$, and letting $b\to T_+$ and $a\to T_-$ we obtain the desired conclusion. □

We next prove

LEMMA V.1.7 (Bound of the L^8 norm). *Let u be a radial solution of* (W5) *with $\sigma=-1$. Then*

$$\|u\|_{L^8(I_{\max}\times\mathbb{R}^3)}\lesssim E^{1/4}. \tag{V.1.6}$$

PROOF.

STEP 1. We prove the bound when $\vec{u}_0$ is C^∞, compactly supported. For this we use the Morawetz estimate of Lemma V.1.6 and the radial Sobolev inequality:

$$u^2(t,r)\leq \frac{1}{r}\int_r^\infty (\partial_\rho u(t,\rho))^2\rho^2 d\rho\lesssim \frac{1}{r}E. \tag{V.1.7}$$

This last inequality can be proved with the fundamental theorem of calculus and Cauchy-Schwarz inequality:

$$|u(r)|=\left|\int_r^\infty \partial_\rho u(\rho) d\rho\right|\leq \sqrt{\int_r^\infty (\partial_\rho u(\rho))^2\rho^2 d\rho}\sqrt{\int_r^\infty \rho^{-2} d\rho}.$$

Combining Lemma V.1.6 with (V.1.7), we obtain

$$\int_{I_{\max}} u^8(t,r) r^2 dr dt\lesssim E\int_{I_{\max}}\int_0^\infty u^6(t,r) r dr dt\lesssim E^2,$$

which give (V.1.6) in this case.

STEP 2. To prove the bound for general solutions, we use a density argument. We consider a sequence of initial data $(\vec{u}_0^n)_n$ with $\vec{u}_0^n\in (C_0^\infty)^2$, radial, such that $\lim_n \vec{u}_0^n=\vec{u}_0$ in $\dot{\mathcal{H}}^1$. Let $K\subset I_{\max}(\vec{u}_0)$ compact. By continuity of the flow (Theorem IV.5.1), $K\subset I_{\max}(\vec{u}_0^n)$ for large n and

$$\lim_{n\to\infty}\|u^n-u\|_{L^\infty(K,L^6)}+\|u^n-u\|_{L^5(K,L^{10})}=0.$$

Since Hölder inequality implies $L^5(K,L^{10}(\mathbb{R}^3))\cap L^\infty(K,L^6(\mathbb{R}^3))\subset L^8(K\times\mathbb{R}^3)$ with the bound

$$\|f\|^8_{L^8(K\times\mathbb{R}^3)}\leq \|f\|^3_{L^\infty(K,L^6)}\|f\|^5_{L^5(K,L^{10})},$$

we deduce

$$\lim_{n\to\infty} \left\|u^n - u^0\right\|_{L^8(K\times\mathbb{R}^3)} = 0. \tag{V.1.8}$$

By Step 1,

$$\|u^n\|^8_{L^8(K\times\mathbb{R}^3)} \lesssim (E(\vec{u}_0^n))^2 \underset{n\to\infty}{\longrightarrow} E^2,$$

which concludes the proof.

□

We are now ready to end the proof of Theorem V.1.5

Proof of Theorem V.1.5. We fix $\vec{u}_0 \in \dot{\mathcal{H}}^1$. By Lemma V.1.7, we have $u \in L^8(I_{\max}\times\mathbb{R}^3)$. By Proposition V.1.2, and since $u \in L^5(K, L^{10})$ for all $K \in I_{\max}$ it is sufficient to prove $u \in L^5([\tau, T_+[, L^{10})$ for some $\tau \in I_{\max}$. We fix $\tau \in I_{\max}$ such that

$$\|u\|_{L^8([\tau,T_+[\times\mathbb{R}^3)} \leq \varepsilon, \tag{V.1.9}$$

where the small parameter $\varepsilon > 0$ is will be specified later. For $t \in [\tau, T_+[$, we have by Hölder's inequality

$$\|u\|_{L^5([\tau,t],L^{10}(\mathbb{R}^3))} \leq \|u\|^{2/5}_{L^8([\tau,t]\times\mathbb{R}^3)} \|u\|^{3/5}_{L^4([\tau,t],L^{12}(\mathbb{R}^3))}. \tag{V.1.10}$$

Thus it is sufficent to prove $u \in L^4([\tau, T_+[, L^{12})$. For this we use Strichartz estimate, (V.1.10) and (V.1.9):

$$\|u\|_{L^4([\tau,t,L^{12})} \leq C_S\|\vec{u}(\tau)\|_{\dot{\mathcal{H}}^1} + C_S\|u\|^5_{L^5([t_0,t],L^{10})} \leq 2C_S\sqrt{E} + C_S\varepsilon^2\|u\|^3_{L^4([t_0,t],L^{12})}.$$

We prove by a bootstrap argument:

$$\forall t \in [\tau, T_+[, \quad \|u\|_{L^4([\tau,t],L^{12})} \leq 3C_S\sqrt{E}. \tag{V.1.11}$$

Indeed if (V.1.11) holds for some t, we have

$$\|u\|_{L^4([\tau,t[,L^{12})} \leq 2C_S\sqrt{E} + C_S\varepsilon^2(3C_S\sqrt{E})^3 \leq \frac{5}{2}C_S\sqrt{E},$$

where we have chosen ε so small that $\varepsilon^2(3C_S)^3E \leq \frac{1}{2}$. This proves (V.1.11) by the intermediate value theorem.

By the same proof in a neighborhood of T_-, we obtain, $u \in L^5(I_{\max}, L^{10})$, which concludes the proof that u scatters in both time directions. □

Exercice V.1. In the setting of Theorem V.1.5, prove

$$\|u\|_{L^4(\mathbb{R},L^{12})} \leq C\big(E(u_0,u_1)\big)^2. \tag{V.1.12}$$

V.2. Stationary solutions and travelling waves

2.a. Stationary solutions. We are interested by stationary solutions of the equation (W5), i.e. nonzero, $\dot{H}^1$ solutions of the elliptic equation $-\Delta Q = \sigma Q^5$. In the defocusing case $\sigma = -1$, the equation is

$$-\Delta Q + Q^5 = 0,$$

to be interpreted in the sense of distribution on $\mathbb{R}^3$. This means

$$\forall \varphi \in C_0^\infty(\mathbb{R}^3), \quad \int \nabla Q\cdot\nabla\varphi + \int Q^5\varphi = 0.$$

Approximation Q by smooth, compactly supported functions, we obtain

$$\int |\nabla Q|^2 + \int Q^6 = 0,$$

which implies $Q = 0$ a.e. Thus in the defocusing case, the only nonstationary solution is the constant null solution. This was already known, since in this case, all solutions scatter and a scattering solution cannot be stationary since it is in $L^5(\mathbb{R}, L^{10})$.

We next consider the focusing case $\sigma = 1$. The equation is:

$$-\Delta Q = Q^5, \quad Q \in \dot{H}^1(\mathbb{R}^3). \tag{Ell}$$

Since Q must be a solution of (W5) in the sense of definition IV.2.1, we can also assume $Q \in L^{10}$.

Exercice V.2. Prove that a solution of (Ell) with $Q \in L^{10}$ is C^∞, and that it is bounded as well as all of its derivative. *Hint:* use the equation and Sobolev embeddings.

Let us mention that the assumption $Q \notin L^{10}$ is not necessary, and that it is possible (but not trivial), to prove that Q satisfying (Ell) must be in $C^\infty \cap L^\infty$ (see [**30**]). Note that in this case, a simple elliptic regularity argument based on Sobolev inequalities does not work. Indeed, we have:

$$Q \in \dot{H}^1 \Longrightarrow Q \in L^6 \Longrightarrow \Delta Q = -Q^5 \in L^{6/5} \Longrightarrow \nabla Q \in L^2, \tag{V.2.1}$$

where we used the Sobolev embeddings $\dot{H}^1 \subset L^6$ and $\dot{W}^{2,6/5} \subset \dot{H}^1$. Of course (V.2.1) does not give any improvement on the regularity of Q.

The equation (Ell) has nonzero solutions. In the radial case, the solutions are completely classified.

Theorem V.2.1. *Let*

$$W(x) = \frac{1}{\left(1 + \frac{|x|^2}{3}\right)^{1/2}}, \quad x \in \mathbb{R}^3 \tag{V.2.2}$$

Then W is a solution of (Ell). *Furthermore the set of radial solutions of* (Ell) *is given by*

$$\Sigma = \{0\} \cup \left\{ \frac{\iota}{\lambda^{1/2}} W\left(\frac{\cdot}{\lambda}\right),\ \lambda > 0,\ \iota \in \{\pm 1\} \right\}.$$

It can be checked by explicit computations that W is a solution of (Ell) . Since the equation is invariant by scaling and sign change, we obtain also that $\frac{\iota}{\lambda^{1/2}} W\left(\frac{\cdot}{\lambda}\right)$ is also a solution for any $\lambda > 0$, $\iota = 1$ or -1. The fact that these are the only radial solutions of (Ell) can be proved by ODE arguments. This will be a consequence of a stronger rigidity theorem below (see Theorem V.3.3) and we thus omit the proof.

Let us mention that W is the maximizer for the Sobolev inequality on $\mathbb{R}^3$: $\|f\|_{L^6} \lesssim \|\nabla f\|_{L^2}$. Thas is, if $f \in \dot{H}^1$, one has

$$\int |f|^6 \leq C_s \left(\int |\nabla f|^2 \right)^3, \quad C_s = \int |W|^6 \times \left(\int |\nabla W|^2 \right)^{-3}, \tag{V.2.3}$$

with equality if and only if $f = 0$ or $f = W$, up to scaling, space translation and sign change. This was proved independtly by Aubin [**1**] and Talenti [**28**] in the mid 70's.

Much less is known about the equation (Ell) without symmetry assumption. Multiplying (Ell) by Q and integrating by parts, we obtain

$$\int |\nabla Q|^2 = \int |Q|^6 = 3E(Q, 0).$$

In particular, the energy of a nonzero solution Q of (Ell) (considered as a solution of (W5)) is positive. Combining with (V.2.3), obtain that the energy of any nonzero solution of (Ell) is greater or equal to $E(W, 0)$. The least-energy nonzero solution W of (Ell) is sometimes called the *ground state* of (W5). It was proved by Ding in 1986 (see [**10**]) that one can also construct arbitrarily large solutions of (Ell).

2.b. Travelling waves. Travelling wave solutions of (W5) are by definitions solutions of the form $\varphi(x - \mathbf{c}t)$, where the speed $\mathbf{c} \in \mathbb{R}^3$ is fixed, and $\varphi \in \dot{H}^1$. Using the invariance of (W5) by rotation, we can assume $\mathbf{c} = (c, 0, 0)$, where $c \in \mathbb{R}$. We are thus lead to study solutions of (W5) of the form

$$u(t, x) = \varphi(x_1 - tc, x_2, x_3), \quad c \in \mathbb{R}, \quad \varphi \in \dot{H}^1. \tag{V.2.4}$$

These solutions can be deduced from solutions of the elliptic equation (Ell).

Theorem V.2.2. *Let u be a nonzero solution of* (W5) *of the form* (V.2.4). *Then $\sigma = 1$, $|c| < 1$, and Q defined by*

$$Q(x_1, x_2, x_3) = \varphi\left(x_1 \sqrt{1 - c^2}, x_2, x_3\right) \tag{V.2.5}$$

is a solution of (Ell).

Remark V.2.3. Recall from (I.7.2) the definition of the Lorentz boost of a function $u : \mathbb{R}^4 \to \mathbb{R}$. One can check that the Lorentz boost of a C^2, global solution u of (W5) is also a solution of (W5). The travelling waves are exactly given by applying Lorentz boosts to solutions of (Ell).

Proof of the Theorem. Let u be a nonzero travelling wave solution.

The fact that $|c| < 1$ follows from finite speed of propagation. Indeed, arguing by contradiction, we consider a solution u of (W5) of the form (V.2.4), with $c \geq 1$ (where we have assumed c positive to fix ideas, the case $c \leq -1$ can be deduced by the transformation $x_1 \mapsto -x_1$).

We fix $L > 0$ such that

$$\int_{x_1>L} |\nabla u_0|^2 + u_1^2 = \varepsilon^2, \tag{V.2.6}$$

where $\varepsilon > 0$ is small. Let $(v_0, v_1) \in \dot{\mathcal{H}}^1(\mathbb{R}^3)$ such that

$$(v_0, v_1)(x) = (u_0, u_1)(x), \quad \int |\nabla v_0|^2 + v_1^2 dx \leq 2\varepsilon^2. \tag{V.2.7}$$

(Defining v_j for $j = 0, 1$ by $v_j(x) = u_j(2L - x)$ for $x \leq L$ would work for example). Let v be the solution of (W5) with initial data (v_0, v_1) at $t = 0$. By the small data theory (Theorem IV.3.1), $v \in L^5(\mathbb{R}, L^{10})$. By (V.2.7) and finite speed of propagation,

$$\forall t \geq 0, \quad \forall x \in \mathbb{R}^3, \quad x_1 \geq L + t \Longrightarrow v(t, x) = u(t, x) = \varphi(x_1 - ct, x_2, x_3).$$

Thus

$$\int_{\mathbb{R}^3} |v(t, x)|^{10} dx \geq \int_{x_1 \geq L+t} |\varphi(x_1 - ct, x_2, x_3)|^{10} dx \geq \int_{x_1 \geq L+(1-c)t} |\varphi(x)|^{10} dx \geq a,$$

where $a = \int_{x_1 \geq L} |\varphi|^{10} dx > 0$ by (V.2.6). This concludes the proof.

We thus have $c < 1$. In this case, it is easy to check, using (W5), that Q defined by (V.2.5) satisfies $-\Delta Q = \sigma Q^5$. This implies since Q is not identically 0, that $\sigma = 1$ and that Q is solution to (Ell), which concludes the proof. □

We will now consider exclusively the case $\sigma = 1$, i.e. the equation

$$\partial_t^2 u - \Delta u = u^5. \tag{W5f}$$

We have identified 3 types of solutions to (W5f).

1. Solutions of the ordinary differential equation $y'' = y^5$, such as $\left(\frac{3}{4t^2}\right)^{\frac{1}{4}}$, that can be truncated to obtain finite time blow-up solutions with finite energy.
2. Scattering solutions, that are global and asymptotically close to solutions of the linear wave equation, that move with the speed of light (1 in our normalization).
3. Travelling wave solutions, with velocity < 1.

If we believe that the ODE solution will not play any role for the asymptotics of global solutions[1], we are lead to conjecture that this asymptotics will only be influenced by the travelling wave and linear solutions. Moreover, the different speeds of propagation would decouple asymptotically the linear and travelling wave dynamics. We will come back to a more precise form of this *resolution conjecture* in the next chapter.

In the sequel, we will focus on radial solutions, for which more things are known. Note that in this case, there is no travelling wave, and that the only nonzero solutions of (Ell) are given by the transforms of W.

We will identify a nondispersive property of solutions of (W5f), that turns out to characterize the stationary solutions of (W5f).

V.3. Nonradiative solutions

3.a. Definition and classification. In order to study the dynamics of nonlinear dispersive equation, it is common to classify solutions that are "completely nondispersive" in a certain sense. These solutions tend to play an important role on the dynamics, and their classification is crucial for its understanding. We will give here the notion of "nonradiative solutions", that was introduced to prove the resolution into stationary solutions for radial solutions of (W5) (See Theorem VI.2.1 below and [**14**]).

DEFINITION V.3.1. Let u be a global solution of (W5) or of the linear wave equation (LW). Let $R \in \mathbb{R}$ and $t_0 \in \mathbb{R}$. The solution u is (R, t_0)-*nonradiative* when

$$\sum_{\pm} \lim_{t \to \pm\infty} \int_{|x| \geq R + |t - t_0|} e_u(t, x) dx = 0.$$

In the definition, we have used the notation $e_u(t, x) = \frac{1}{2}|\nabla u(t, x)|^2 + \frac{1}{2}(\partial_t u(t, x))^2$. To simplify notations, we will restrict without generality to the case $t_0 = 0$, and call the corresponding solutions R-nonradiative solutions. If $R = 0$, the solution with simply be called "nonradiative".

It is possible, using the explicit formulas of Chapter 1, to prove that the only 0-nonradiative solution of the linear wave equation (LW) is the constant null solution. In the nonlinear case, using that the speed of

[1] This belief is false in general for semi-linear wave equation, but turns out to be true in the energy-critical case. See [**15**] for an example of a global solution of the cubic wave equation which is asymptotically close to a solution of the corresponding ODE.

travelling waves is always < 1, we see that travelling waves are also R-nonradiative solutions for all R. The rigidity conjecture for nonradiative solution says that this should be the only ones:

CONJECTURE V.3.2 (Rigidity conjecture for nonradiative solutions). *Let u be a nonradiative solution of* (W5f). *Then u is a travelling wave.*

We prove this conjecture in the radial case:

THEOREM V.3.3 (Dynamical characterization of W). *Let $R_0 \geq 0$ and u be a radial, R_0-nonradiative solution of* (W5f). *Then one of the following occurs:*

- $u(t,x) = 0$ *for* $|x| > R_0 + |t|$.
- *there exists* $\lambda > 0$, $\iota \in \{\pm 1\}$,

$$\forall |x| > R_0 + |t|, \quad u(t,x) = \iota W_\lambda(x),$$

where

$$W_\lambda(x) = \frac{1}{\lambda^{1/2}} W\left(\frac{x}{\lambda}\right).$$

REMARK V.3.4. In the case where $R_0 = 0$, we see that the theorem implies that $(u_0, u_1) = (\iota W_\lambda, 0)$, and thus that u is the stationary solution W_λ. This implies the uniqueness part in Theorem V.2.1, since any solution of the elliptic equation (Ell) is also a nonradiative solution of (W5f).

3.b. A lower bound of the exterior energy for the linear equation. The proof of Theorem V.3.3 is based on its (quantitative) analog for the linear equation (LW):

PROPOSITION V.3.5. *Let $R \geq 0$. Let $(u_0, u_1) \in \dot{\mathcal{H}}^1$ and $u_L(t) = S_L(t)(u_0, u_1)$. Then*

$$\sum_{\pm} \lim_{t \to \pm\infty} \int_{R+|t|}^{+\infty} e_u(t,r) r^2 \, dr = \frac{1}{2} \int_R^{+\infty} \left(\partial_r(ru_0)\right)^2 + r^2 u_1^2 \, dr. \tag{V.3.1}$$

The right-hand side of (V.3.1) can be compared to the $\dot{\mathcal{H}}^1(\{|x| > R\})$-norm by a simple integration by parts. Indeed, if $R > 0$, we have, for any radial $\dot{H}^1$ function f on $\mathbb{R}^3$,

$$\int_R^\infty \Big(\partial_r(rf(r))\Big)^2 dr = \int_R^\infty \Big(\partial_r f(r)\Big)^2 r^2 dr - Rf(R)^2. \tag{V.3.2}$$

When $R = 0$, the boundary term vanishes and we have

$$\int_0^\infty \Big(\partial_r(rf(r))\Big)^2 dr = \int_0^\infty \Big(\partial_r f(r)\Big)^2 r^2 dr.$$

The formula (V.3.1) reads

$$\sum_{\pm} \lim_{t \to \pm\infty} \int_{|x| \geq |t|}^{+\infty} e_u(t,x) dx = \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla u_0(x)|^2 + u_1^2(x)\right) dx. \tag{V.3.3}$$

Let us mention that (V.3.3) remains valid without the assumption that (u_0, u_1) is radial, and can be proved with the explicit formulas of Theorem I.5.2. It is still valid in any *odd* space dimension, as proved in [**13**], but not in *even* space dimension, even for radial solutions (see [**9**]).

Investigating (V.3.1), we see that the only radial R-nonradiative solutions of (W5) are the solutions that are equal to ℓ/r for $r > R + |t|$, (where $\ell \in \mathbb{R}$). Since ℓ/r is not in $\dot{H}^1(\mathbb{R}^3)$, we also obtain that 0 is the only 0-nonradiative solution.

PROOF OF PROPOSITION V.3.5. This follows from the explicit formula for radial, 3D solutions (see (I.5.1)),

$$u(t,r) = \frac{1}{r}(\varphi(r+t) - \varphi(t-r)), \tag{V.3.4}$$

where

$$\varphi(\eta) = \frac{1}{2}\eta u_0(|\eta|) + \frac{1}{2}\int_0^\eta \sigma u_1(|\sigma|) d\sigma. \tag{V.3.5}$$

Using this formula, we see that

$$\Big(\partial_r(ru)\Big)^2 + \Big(\partial_t(ru)\Big)^2 = 2(\varphi'(r+t))^2 + 2(\varphi'(t-r))^2.$$

This gives

$$\sum_{\pm} \lim_{t\to\pm\infty} \int_{R+|t|}^{+\infty} |\partial_{t,r}(ru_L(t,r))|^2\,dx = 2\int_R^{\infty} (\varphi'(\eta))^2\,d\eta + 2\int_{-\infty}^{-R} (\varphi'(\eta))^2\,d\eta$$

$$= \int_R^{+\infty} (\partial_r(ru_0))^2 + r^2u_1^2\,dr.$$

Using (V.3.2) we obtain (V.3.1). Indeed by by the formula (V.3.4), we have

$$\lim_{t\to\infty} (R+t)u^2(t,R+T) = 0,$$

since $|\varphi(\eta)|/\sqrt{|\eta|}$ goes to 0 as $|\eta|\to\infty$. □

3.c. Proof of the rigidity result. We next prove the rigidity Theorem V.3.3. The proof takes several steps.

Step 1. Let $(u_0,u_1)\in\dot{\mathcal{H}}^1$ be as in Theorem V.3.3. Let $\varepsilon>0$ be a small parameter to be specified. In all the proof we fix $R_\varepsilon\geq R_0$ such that

$$\int_{R_\varepsilon}^{+\infty} \left((\partial_r u_0)^2+u_1^2\right) r^2\,dr \leq \varepsilon^2. \tag{V.3.6}$$

In this step, we prove

$$\forall R\geq R_\varepsilon,\quad \int_R^{+\infty} (\partial_r(ru_0))^2 + r^2u_1^2\,dr \leq CR^5u_0^{10}(R). \tag{V.3.7}$$

Let $R\geq R_\varepsilon$. We define the radial functions $v_0\in\dot{H}^1(\mathbb{R}^3)$, $v_1\in L^2(\mathbb{R}^3)$ as follows:

$$\begin{cases} (v_0,v_1)(r) = (u_0,u_1)(r) & \text{if } r>R\\ (v_0,v_1)(r) = (u_0(R),0) & \text{if } r\in(0,R). \end{cases} \tag{V.3.8}$$

We let $v(t)$ be the solution of (W5f) with initial data (v_0,v_1), and $v_L(t,r)=S_L(t)(v_0,v_1)$ be the corresponding solution to the free wave equation. We note that by final speed of propagation

$$v(t,r)=u(t,r),\quad r>R+|t|.$$

By the small data theory, since ε is small,

$$\sup_{t\in\mathbb{R}} \|\vec{v}(t)-\vec{v}_L(t)\|_{\dot{H}^1\times L^2} \leq C\,\|(v_0,v_1)\|^5_{\dot{H}^1\times L^2}. \tag{V.3.9}$$

By Proposition V.3.5,

$$\sum_{\pm}\lim_{t\to\pm\infty}\int_{R+|t|}^{+\infty} |\partial_{t,r}(v_L(t,r))|^2r^2\,dr \geq \int_R^{+\infty} (\partial_r(ru_0))^2+u_1^2\,dr. \tag{V.3.10}$$

By (V.3.9), and finite speed of propagation

$$\int_{R+|t|}^{+\infty} \left|\partial_{t,r}(v_L(t,r))-\partial_{t,r}(u(t,r))\right|^2 r^2\,dr \leq C\left(\int_R^{+\infty} ((\partial_r u_0)^2+u_1^2)r^2dr\right)^5.$$

Combining with (V.3.10) and using that the solution is R-nonradiative, we obtain

$$\int_R^{+\infty} (\partial_r(ru_0))^2+r^2u_1^2\,dr \leq C\left(\int_R^{+\infty}\left((\partial_r u_0)^2+u_1^2\right)r^2dr\right)^5.$$

With the integration by parts formula (V.3.2) and the smallness of ε, we deduce (V.3.7).

Step 2. In this step we prove that there exists $\ell\in\mathbb{R}$ and $C>0$ such that for large R,

$$\left|u_0(r)-\frac{\ell}{r}\right| \leq \frac{C}{r^3},\quad \int_r^{+\infty}\rho^2u_1(\rho)\,d\rho \leq \frac{C}{r^5}. \tag{V.3.11}$$

First fix R and R' such that $R_\varepsilon\leq R\leq R'\leq 2R$. Letting $\zeta_0(r)=ru_0(r)$, we have, using Cauchy-Schwarz, then Step 1

$$|\zeta_0(R)-\zeta_0(R')| \leq \int_R^{R'} |\partial_r\zeta_0(r)|\,dr \leq \sqrt{R}\sqrt{\int_R^{R'}(\partial_r\zeta_0)^2dr} \leq \frac{1}{R^2}\zeta_0^5(R). \tag{V.3.12}$$

Since by the definition (V.3.6) of R_ε and the integration by parts formula (V.3.2) one has

$$\frac{1}{R}\zeta_0^2(R) \leq \varepsilon^2, \tag{V.3.13}$$

we deduce from (V.3.12):

$$|\zeta_0(R) - \zeta_0(R')| \leq C\varepsilon^4 \zeta_0(R), \qquad R_\varepsilon \leq R \leq R' \leq 2R. \tag{V.3.14}$$

Let $\alpha = \log_2\left(1 + C\varepsilon^4\right)$, so that $2^\alpha = (1 + C\varepsilon^4)$. By (V.3.14), for all k, $|\zeta_0(2^{k+1}R_\varepsilon)| \leq 2^\alpha |\zeta_0(2^k R_\varepsilon)|$. Thus the sequence $\left(\frac{|\zeta_0(2^k R_\varepsilon)|}{(2^k)^\alpha}\right)_{k\geq 0}$ is nonincreasing. This implies that $|\zeta_0(2^k R_\varepsilon)| \lesssim \left(2^k R_\varepsilon\right)^\alpha$, for $k \geq 0$ and thus, using (V.3.14) again,

$$|\zeta_0(R)| \lesssim R^\alpha.$$

We can take ε small enough, so that $\alpha \leq 1/5$. The inequality (V.3.12) yields

$$|\zeta_0(R) - \zeta_0(R')| \lesssim \frac{1}{R}, \quad R_\varepsilon \leq R \leq R' \leq 2R. \tag{V.3.15}$$

This shows that $\sum_{k\geq 0} \left|\zeta_0(2^k R) - \zeta_0(2^{k+1}R_\varepsilon)\right| < \infty$, and thus that $\left(\zeta_0(2^k R_\varepsilon)\right)_k$ has a limit ℓ as $k \to \infty$. By (V.3.15)

$$\lim_{R\to\infty} \zeta_0(R) = \ell.$$

This implies that ζ_0 is bounded. The inequality (V.3.12) then yields

$$\forall k \geq 0, \quad \forall R \geq R_\varepsilon, \quad \left|\zeta_0\left(2^k R\right) - \zeta_0\left(2^{k+1}R\right)\right| \lesssim \frac{1}{2^{2k}R^2}.$$

Summing over $k \geq 0$ and using the triangle inequality, we obtain,

$$|\zeta_0(R) - \ell| \lesssim \frac{1}{R^2},$$

which is the first inequality in (V.3.11). Combining with Step 1, we obtain the second inequality in (V.3.11).

STEP 3. In this step, we assume $\ell = 0$ and prove that $(u_0, u_1) \equiv (0, 0)$. Indeed by (V.3.14), if $R \geq R_\varepsilon$ and $k \in \mathbb{N}$,

$$\left|\zeta_0\left(2^{k+1}R\right)\right| \geq (1 - C\varepsilon^4)\left|\zeta_0\left(2^k R\right)\right|.$$

Hence by induction on k,

$$\left|\zeta_0\left(2^k R\right)\right| \geq (1 - C\varepsilon^4)^k |\zeta_0(R)|.$$

Since by the preceding step and the assumption $\ell = 0$, $|\zeta_0(2^k R)| \lesssim 1/(2^k R)^2$, we deduce, choosing ε small enough and letting $k \to \infty$ that $\zeta_0(R) = 0$. Combining with (V.3.7) we deduce

$$R \geq R_\varepsilon \Longrightarrow \int_R^{+\infty} (\partial_r \zeta_0)^2 + u_1^2(r)\, dr = 0,$$

that is $u_0(r)$ and $u_1(r)$ are 0 for almost every $r \geq R_\varepsilon$. Going back to the definition of R_ε we see that we can choose any $R_\varepsilon > R_0$, which concludes this step.

STEP 4. We next assume $\ell \neq 0$. To fix ideas, we assume that ℓ is positive. By the definition (V.2.2) of W and the definition of W_λ we have, for $\lambda > 0$

$$W_\lambda(r) = \frac{\sqrt{3\lambda}}{r} + \mathcal{O}\left(\frac{1}{r^3}\right), \quad r \to \infty.$$

We choose $\lambda > 0$ such that $\sqrt{3\lambda} = \ell$ so that

$$\left|W_\lambda(r) - \frac{\ell}{r}\right| \lesssim \frac{1}{r^3} \tag{V.3.16}$$

for large r. In this step we prove that $(u_0 - W_\lambda, u_1)$ has compact support. Let $f = u - W_\lambda$. Then

$$\begin{cases} \partial_t^2 f - \Delta f = D_\lambda(f) := \sum_{k=1}^{5} \binom{5}{k} W_\lambda^{5-k} f^k. \\ \vec{f}_{\restriction t=0} = (f_0, f_1) := (u_0 - W_\lambda, u_1), \end{cases} \tag{V.3.17}$$

For $\varepsilon > 0$ small, we fix $R'_\varepsilon \gg 1$ such that

$$\int_{R'_\varepsilon}^{+\infty} \left(|\partial_r f_0(r)|^2 + |f_1(r)|^2\right) r^2 dr \leq \varepsilon^2 \tag{V.3.18}$$

$$\int_{\mathbb{R}} \left(\int_{R'_\varepsilon+|t|}^{+\infty} W_\lambda^{10}(r) r^2\, dr \right)^{\frac{1}{2}} dt \leq \varepsilon^5. \tag{V.3.19}$$

Let f_L be the solution of $\partial_t^2 f_L = \Delta f_L$ with

$$\vec{f}_{L\restriction t=0} = (\tilde{f}_0, \tilde{f}_1),$$

where $(\tilde{f}_0, \tilde{f}_1)$ coincides with (f_0, f_1) for $r > R'_\varepsilon$ and is defined as in (V.3.8). Using (V.3.17) and the assumptions (V.3.18) and (V.3.19) on R'_ε, we obtain

$$\sup_{t\in\mathbb{R}} \left\| 1\!\!1_{\{|x|>|t|+R'_\varepsilon\}} \left|\nabla_{t,x}(\tilde{f}(t) - \tilde{f}_L(t))\right| \right\|_{L^2} \lesssim \varepsilon^4 \left\|(\tilde{f}_0, \tilde{f}_1)\right\|_{\dot{H}^1\times L^2}. \tag{V.3.20}$$

Let $R \geq R'_\varepsilon$. Using that by Proposition V.3.5,

$$\sum_{\pm} \lim_{t\to\pm\infty} \int_R^{+\infty} \left(\partial_{t,r}\big(\tilde{f}_L(t,r)\big)\right)^2 r^2 dr \gtrsim \int_R^{+\infty} \left(\left(\partial_r(r\tilde{f})\right)^2 + r^2 \tilde{f}_1^2 \right) dr,$$

and since

$$\sum_{\pm} \lim_{t\to\pm\infty} \int_R^{+\infty} \left(\partial_{t,r}\big(\tilde{f}(t,r)\big)\right)^2 r^2 dr = 0,$$

we deduce from (V.3.20)

$$\varepsilon^8 \int_R^{+\infty} \left((\partial_r f_0)^2 + f_1^2\right) r^2 dr \gtrsim \int_R^{+\infty} \left(\left(\partial_r(rf_0)\right)^2 + r^2 f_1^2 \right) dr,$$

and thus

$$\varepsilon^8 R f_0^2(R) \gtrsim \int_R^{\infty} \left(\left(\partial_r(rf_0)\right)^2 + r^2 f_1^2 \right) dr. \tag{V.3.21}$$

Letting $g_0 = rf_0$, we deduce by Cauchy-Schwarz that for $R \geq R'_\varepsilon$, $k \in \mathbb{N}$,

$$\left|g_0\left(2^{k+1}R\right) - g_0\left(2^k R\right)\right| \lesssim \int_{2^k R}^{2^{k+1}R} |\partial_r g_0| dr \lesssim \varepsilon^4 \left|g_0(2^k R)\right|.$$

This yields by an easy induction $|g_0(2^k R)| \geq \left(1 - C\varepsilon^4\right)^k |g_0(R)|$, where $C > 0$ is a constant which is independent of ε. Since by Step 2,

$$\frac{C}{\left(2^k R\right)^2} \geq \left|g_0\left(2^k R\right)\right|,$$

we obtain choosing ε small enough that $g_0(R) = 0$ for large R. Combining with (V.3.21), we deduce that $(f_0(r), f_1(r)) = 0$ a.e. for large R, concluding this step.

Step 5. In this step we still assume $\ell \neq 0$ and conclude the proof. We let

$$\rho = \inf\left\{ R > R_0 \;:\; \int_R^{+\infty} \left((\partial_r f_0)^2 + f_1^2\right) r^2 dr = 0 \right\}$$

and prove that $\rho = R_0$ i.e.that $u_0(r) = W_\lambda(r)$ for $r > R_0$.

We argue by contradiction, assuming $\rho > R_0$. By the preceding step and finite speed of propagation, the essential support of f is included in $\{r \leq \rho + |t|\}$. Thus f is solution of

$$\begin{cases} \partial_t^2 f - \Delta f = 1\!\!1_{\{|x|\leq\rho+|t|\}} D_\lambda(f). \\ \qquad \vec{f}_{\restriction t=0} = (f_0, f_1) := (u_0 - W_\lambda, u_1), \end{cases}$$

Fix $R''_\varepsilon \in (1, \rho)$ such that,

$$\int_{R''_\varepsilon}^{+\infty} \left(|\partial_r f_0(r)|^2 + |f_1(r)|^2\right) r^2 dr \leq \varepsilon^2$$

$$\int_{\mathbb{R}} \left(\int_{R''_\varepsilon+|t|}^{\rho+|t|} W_\lambda^{10}(r) r^2\, dr \right)^{\frac{1}{2}} dt \leq \varepsilon^5.$$

The same argument as in the preceding step, replacing R'_ε by R''_ε, yields that $(f_0, f_1) = 0$ for almost every $r > R''_\varepsilon$, which contradicts the definition of ρ. The proof is complete.

CHAPTER VI

Resolution into stationary solutions. Profile decomposition

This chapter concerns the focusing quintic wave equation (W5f). We will state a result on the classification of global radial solutions of (W5f). We will not give the full proof. We will however present the profile decomposition for bounded sequences of solutions of (LW) and (W5), which is a crucial ingredient of this proof.

VI.1. General property of global solutions

We start to give properties of global solutions of (W5f).

PROPOSITION VI.1.1. *Let u be a solution of* (W5f) *such that $T_+(u) = +\infty$. Then there exists a sequence $t_n \to \infty$ such that*

$$\limsup_{n\to\infty} \|\vec{u}(t_n)\|_{\dot{\mathcal{H}}^1(\mathbb{R}^3)} < \infty.$$

The proof relies on the computation of the second derivative of $\int |u(t,x)|^2 dx$ (assuming that this derivative is finite). We will leave it as an exercise (see Exercises sheet of Chapter VI).

We will denote $|\nabla_{t,x} u|^2 = (\partial_t u)^2 + \sum_{j=1}^3 (\partial_{x_j} u)^2$

PROPOSITION VI.1.2. *Let u be a radial solution of* (W5f) *such that $T_+(u) = +\infty$. Then there exists a solution v_L of the linear wave equation* (LW) *such that for all $A \in \mathbb{R}$,*

$$\lim_{t\to\infty} \int_{|x|\geq A+|t|} |\nabla_{t,x}(v_L(t,x) - u(t,x))|^2 \, dx = 0 \tag{VI.1.1}$$

Let us give the first step of the proof, that is that there exists $A > 0$ and v_L solution of (LW) such that (VI.1.1) holds. Let A be such that

$$\int_{|x|\geq A+|t|} \left(|\nabla u_0|^2 + u_1^2\right) dx \leq \varepsilon,$$

where ε is small, to be specified. Using a standard extension theorem, we can find $(v_0, v_1) \in \dot{\mathcal{H}}^1$ such that $(v_0, v_1)(x) = (u_0, u_1)(x)$ for $|x| \geq A$, and $\|(v_0, v_1)\|_{\dot{\mathcal{H}}^1} \leq C\varepsilon$. If ε is small enough, by small data scattering (Corollary V.1.3), there exists a solution v_L of the linear wave equation such that

$$\lim_{t\to\infty} \|\vec{v}_L(t) - \vec{v}(t)\|_{\dot{\mathcal{H}}^1} = 0.$$

Since $u(t) = v(t,x)$ for $|x| > A + |t|$, (VI.1.1) follows immediately. We omit the full proof of Proposition VI.1.2, that uses the profile decomposition of Section VI.4.

VI.2. Resolution into stationary solutions

We next state the main result of this chapter. We denote as before $W_\lambda(x) = \frac{1}{\lambda^{\frac{1}{2}}} W\left(\frac{x}{\lambda}\right)$, where W is the ground state.

THEOREM VI.2.1. *Let u be a solution of* (W5f) *such that $T+(u) = +\infty$. Then there exists $J \geq 0$, and, for $j \in \{1, \dots, J\}$, a sign $\iota_j \in \{\pm 1\}$ and a scaling parameter $\lambda_j(t)$ such that*

$$\lim_{t\to\infty} \frac{\lambda_J(t)}{t} = 0 \quad \textit{and} \quad \forall j \in \{1, \dots, J-1\},\ \lim_{t\to\infty} \frac{\lambda_j(t)}{\lambda_{j+1}(t)} = \infty \tag{VI.2.1}$$

$$\lim_{t\to\infty} \left(u(t) - v_L(t) - \sum_{j=1}^J \iota_j W_{\lambda_j(t)} \right) = 0 \ \textit{in}\ \dot{H}^1 \tag{VI.2.2}$$

$$\lim_{t\to\infty} \left(\partial_t u(t) - \partial_t v_L(t)\right) = 0 \ \textit{in}\ L^2. \tag{VI.2.3}$$

A few remarks are in order:

- Theorem VI.2.1 was proved in [**14**] by the author, Carlos Kenig and Frank Merle. Similar results are available in higher dimension for the energy-critical equation (see [**12**], [**8**], [**21**], but the proof is simpler in space dimension 3, and the result is slightly weaker if $N \geq 5$ (one must assume that the solution is bounded in $\dot{\mathcal{H}}^1$, whereas it is known that is bounded only up to a sequence of times, see Proposition III.5.1).
- As a consequence of (VI.2.1), (VI.2.2) and (VI.2.3), we have
$$E(u_0, u_1) = JE(W, 0) + \frac{1}{2}\|\vec{v}_L(0)\|^2_{\dot{\mathcal{H}}^1}.$$
This proves that the number J of profile of a global solution is at most equal to the integer part of $E(u_0, u_1)/E(W, 0)$. In particular, a global solution with energy $< E(W, 0)$ must scatter to a linear solution. This is coherent with the fact, proved by C. Kenig and F. Merle [**22**], that solutions with energy $< E(W, 0)$ scatter or blow-up in finite times.
- An analogous description exists for solutions such that $T_+(u) < \infty$ and
$$\limsup_{t\to T_+(u)} \|\vec{u}(t)\|_{\dot{\mathcal{H}}^1} < \infty$$
(the so-called *Type II blow-up solutions*). The proof is similar.
- Solutions such that $J = 0$ in the conclusion of the theorem are scattering solution. The ground state solution satisfies (VI.2.2), (VI.2.3) with $J = 1$ (and $v_L = 0$, $\lambda_1(t) = 1$). It is also possible to construct solutions with $J = 1$ and nonzero v_L (see [**23**]) and also nonconstant λ_1 (see [**11**]). These constructions are intricate and out of the scope of this course. The existence or non-existence of solutions as in the conclusion of the Theorem with $J \geq 2$ is still open.

The core of the proof of Theorem VI.2.1 is the rigidity Theorem V.3.3. We will not give the full proof of the theorem in this course. We will only present one important tool, the profile decomposition, due to Hajer Bahouri and Patrick Gérard in this context, and which has its own interest and is also useful to prove Proposition VI.1.2. This decomposition is related to the full understanding of the defect of compactness of the Strichartz estimates of Theorem III.2.1. We will first work on this defect of compactness in Section VI.3, then state and prove the profile decomposition in Section VI.4.

VI.3. Characterization of the defect of compactness of the Strichartz estimates

3.a. Preliminaries. Let A and B be 2 Banach spaces. We recall that a bounded linear operator $L : A \to B$ is compact when for any bounded sequence $(a_n)_{n\in\mathbb{N}}$ of A, one can extract from the sequence $(La_n)_{n\in\mathbb{N}}$ a subsequence that converges in B. If A is reflexive, an equivalent formulation is that for any sequence $(a_n)_n$ in A that converges weakly to $a \in A$, the sequence La_n converges strongly (to La since a bounded operator is continuous for the weak topologies).

Let (p, q) such that $p > 2$ and $\frac{1}{p} + \frac{3}{q} = \frac{1}{2}$. By Strichartz estimates (Theorem III.2.1), the map $\vec{u}_0 \mapsto S_L(\cdot)\vec{u}_0$ is a bounded operator from $\dot{\mathcal{H}}^1$ to $L^p(\mathbb{R}, L^q)$. This map is not compact. Indeed, let $\varphi(t)$ be a fixed nonzero solution of the linear wave equation (LW) and $(\lambda_n)_{n\in\mathbb{N}} \in]0, \infty[^{\mathbb{N}}$ be such that $\lim_n |\log(\lambda_n)| = \infty$. Consider the sequence $(u_{Ln})_n$ of solutions of (LW) defined by
$$u_{Ln}(t, x) = \lambda_n^{1/2}\varphi(\lambda_n t, \lambda_n x)$$
Then the sequence $(\vec{u}_{Ln}(0))_n$ converges weakly to 0 $\dot{\mathcal{H}}^1$. However the sequence $(u_{Ln})_n = (S_L(\cdot)\vec{u}_{0n})_n$ does not converge strongly to 0 in L^pL^q (the norm of u_{Ln} in this space is independent of n).

Similarly, let $(t_n, x_n)_n \in \left(\mathbb{R}^{1+3}\right)^{\mathbb{N}}$ such that
$$\lim_{n\to\infty} |t_n| + |x_n| = +\infty,$$
and
$$u_{L,n}(t, x) = \varphi(t + t_n, x + x_n),$$
then $(\vec{u}_{L,n}(0))_n$ converges to 0 weakly in $\dot{\mathcal{H}}^1$, and $(u_{L,n})_n$ does not converge strongly in $L^p(\mathbb{R}, L^q(\mathbb{R}^3))$.

The next statement says that these scaling and translations are the only causes of the defect of compactness of the map $S_L(\cdot) : \dot{\mathcal{H}}^1 \to L^pL^q$.

We introduce a few notations:

- $\mathcal{LW}$ is the set of solutions of (LW) with initial data in $\dot{\mathcal{H}}^1$.
- We denote by $\mathcal{G} =]0, \infty[\times\mathbb{R}\times\mathbb{R}^3$.

- We will use the following convention for limits, sequences and subsequences. A *sequence of elements of* X or *sequence in* X, is a family $(x_n)_{n\in I}$ of elements of X, indexed by an infinite countable set I. The real sequence $(x_n)_{n\in I}$ converges to ℓ if for all $\varepsilon>0$, there exists a finite subset F of I such that $\forall n\in I\setminus F$, $|\ell-x_n|\leq\varepsilon$. In this case, we will write $\lim_{n\in I}x_n=\ell$. We can similarly define strong and weak convergence in a Banach space. When $I=\mathbb{N}$, this notion is of course identical to the usual notion of convergence for sequences. If I is an infinite countable set. We will write $I'\sqsubset I$ when I' is infinite and $I'\subset I$. A subsequence of $(x_n)_{n\in I}$ is thus any sequence of the form $(x_n)_{n\in I'}$ with $I'\sqsubset I$.

3.b. Defect of compactness for the Strichartz estimate.

Theorem VI.3.1. *Let (p,q) with $p>2$, $1/p+3/q=1/2$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{LW}$ such that $\overrightarrow{u_n}(0)$ is bounded in $\dot{\mathcal{H}}^1$. Assume that for all $(g_n)_{n\in\mathbb{N}}\in\mathcal{G}^{\mathbb{N}}$ one has*

$$\Big(\lambda_n^{1/2}u_n(t_n,\cdot+x_n),\lambda_n^{3/2}\partial_tu_n(t_n,\cdot+x_n)\Big)\xrightarrow[n\to\infty]{}(0,0). \tag{VI.3.1}$$

Then

$$\lim_{n\to\infty}\|u_n\|_{L^p(\mathbb{R},L^q)}=0. \tag{VI.3.2}$$

Proof. We first observe that it is sufficient to prove the result for $p=\infty$, $q=6$. Indeed, assume that (VI.3.2) holds $p=\infty$ and $q=6$, and let $p>2$, q with $3/p+1/q=1/2$. By the assumptions that $\overrightarrow{u_n}(0)$ is bounded and Strichartz estimates, $(u_n)_n$ is bounded in L^aL^b for all $a>2$ and b with $1/a+3/b=1/2$. Using this fact for a pair (a,b) with $a<p<\infty$ (and thus $b<q<Q$), we obtain, by Hölder's inequality, that (VI.3.2) holds for this (p,q) also.

We prove the theorem for $p=\infty$, $q=6$ by contradiction, assuming that (VI.3.1) holds and that

$$\limsup_{n\to\infty}\|u_n\|_{L^\infty(\mathbb{R},L^6)}>0. \tag{VI.3.3}$$

Thus there exists $\varepsilon>0$ and $I\sqsubset\mathbb{N}$ such that

$$\forall n\in I,\quad \|u_n\|_{L^\infty(\mathbb{R},L^6)}\geq\varepsilon. \tag{VI.3.4}$$

As a consequence, for all $n\in I$, we can choose $\tau_n\in\mathbb{R}$ such that

$$\|u_n(\tau_n)\|_{L^6}\geq\varepsilon/2. \tag{VI.3.5}$$

Also, we observe that (VI.3.1) implies that for all sequence $(\lambda_n,x_n)_{n\in I}\in\big(]0,\infty[\times\mathbb{R}^3\big)^I$, we have

$$\lambda_n^{1/2}u_n(\tau_n,\lambda_n\cdot+x_n)\xrightarrow[n\to\infty]{}0\text{ weakly in }\dot{H}^1, \tag{VI.3.6}$$

We are reduced to prove the following proposition (due to Patrick Gérard [**17**]), which is the analog to Theorem VI.3.1 for the defect of compactness of the Sobolev embedding of $\dot{H}^1(\mathbb{R}^3)$ into $L^6(\mathbb{R}^3)$.

Proposition VI.3.2. *Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\dot{H}^1(\mathbb{R}^3)$. Assume that for all sequence $(\lambda_n,x_n)\in\big(]0,\infty[,\mathbb{R}^3\big)_n$,*

$$\lambda_n^{1/2}f_n(\lambda_n\cdot+x_n)\xrightarrow[n\to\infty]{}0. \tag{VI.3.7}$$

Then

$$\lim_{n\to\infty}\|f_n\|_{L^6}=0 \tag{VI.3.8}$$

Note that Proposition VI.3.2, together with (VI.3.5) and (VI.3.6) yield a contradiction. We next prove Proposition VI.3.2. □

3.c. Defect of compactness for the Sobolev inequality. In this subsection we prove Proposition VI.3.2. This is a consequence of the improved Sobolev inequality (Theorem II.2.3). We recall the definition of the norm

$$\|f\|^2_{\dot{B}^1}=\sup_{k\in\mathbb{Z}}\frac{1}{(2\pi)^N}\int_{2^k\leq|x|\leq 2^{k+1}}|\xi|^2|\widehat{f}(\xi)|^2d\xi,$$

and note that

$$\|f\|_{\dot{B}^1}\approx\sup_{k\in\mathbb{Z}}\|\Delta_kf\|_{\dot{H}^1}, \tag{VI.3.9}$$

where the Δ_k are in Section III.4. We also recall the improved Sobolev inequality

$$\|f\|^3_{L^6}\lesssim\|f\|^2_{\dot{B}^1}\|f\|_{\dot{H}^1} \tag{VI.3.10}$$

We first prove the following lemma:

Lemma VI.3.3. *Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\dot{H}^1$ and such that*

$$\limsup_{n\to\infty} \|f_n\|_{L^6} > 0. \tag{VI.3.11}$$

Then there exists a subsequence $(u_n)_{n\in I}$ of $(f_n)_{n\in\mathbb{N}}$ and a sequence $(j_n)_{n\in I}\in\mathbb{Z}^I$ such that

$$\lim_{n\in I} \|\Delta_{j_n} f_n\|_{L^6} > 0.$$

Proof. We argue by contradiction, assuming that for all sequence $(j_n)_{n\in\mathbb{N}}\in\mathbb{Z}^{\mathbb{N}}$,

$$\lim_{n\to\infty} \|\Delta_{j_n} f_n\|_{L^6} = 0. \tag{VI.3.12}$$

Fixing $\varepsilon > 0$, we will decompose $(f_n)_{n\in\mathbb{N}}$ as $f_n = v_n + w_n$, where

$$\lim_{n\to\infty} \|v_n\|_{L^6} = 0, \quad \forall n,\ \|w_n\|_{\dot{B}^1} \leq \varepsilon, \tag{VI.3.13}$$

$$\max\left(\|v_n\|_{\dot{H}^1}, \|w_n\|_{\dot{H}^1}\right) \leq \|f_n\|_{\dot{H}^1}, \tag{VI.3.14}$$

which contradicts (VI.3.11), in view of the Sobolev inequality (VI.3.10).

For any $n\in\mathbb{N}$, we let

$$J_n = \left\{ j\in\mathbb{Z},\ :\ \|\Delta_j u_n\|_{\dot{H}^1} \geq \frac{\varepsilon}{3}\right\}, \quad \widetilde{J}_n = \{j\in\mathbb{Z},\ \exists k\in J_n \text{ s.t. } |j-k|\leq 1\}.$$

Let

$$v_n = \sum_{j\in\widetilde{J}_n} \Delta_j u_n, \quad w_n = \sum_{j\in\mathbb{Z}\setminus\widetilde{J}_n} \Delta_j u_n = u_n - v_n.$$

We have (using the notation Θ_j of Section III.4),

$$(2\pi)^3\|v_n\|^2_{\dot{H}^1} = \int \left|\widehat{f}_n(\xi)\right|^2 \left(\sum_{j\in\widetilde{J}} \Theta_j(\xi)\right)^2 |\xi|^2 d\xi \leq \int \left|\widehat{f}_n(\xi)\right|^2 \left(\sum_{j\in\mathbb{Z}} \Theta_j(\xi)\right)^2 |\xi|^2 d\xi,$$

where we have used that $\Theta_j \geq 0$ for all j. Since $\sum_{j\in\mathbb{Z}}\Theta_j = 1$ a.e. we have proved $\|v_n\|_{\dot{H}^1} \leq \|f_n\|_{\dot{H}^1}$. Similarly, $\|w_n\|_{\dot{H}^1} \leq \|f_n\|_{\dot{H}^1}$, and thus (VI.3.14) holds.

We next prove the limit in (VI.3.13). For this we prove that the cardinal $|J_n|$ of J_n is uniformly bounded. This implies that $|\widetilde{J}_n|$ is uniformly bounded, which in turn implies, in view of (VI.3.12), the desired property. We have (see (III.4.12))

$$\|f_n\|^2_{\dot{H}^1} \geq \frac{1}{C}\sum_{j\in J_n} \|\Delta_j f_n\|^2 \geq \frac{\varepsilon}{3C}|J_n|,$$

and thus $|J_n| \leq 3C\varepsilon^{-1}\limsup_n \|f_n\|^2_{\dot{H}^1}$, which concludes the proof of the fact that $\|v_n\|_{L^6}$ goes to 0.

We next prove the statement about w_n in (VI.3.13). We have

$$\|w_n\|_{\dot{B}^1} \leq \sup_{k\in\mathbb{Z}} \|\Delta_k w_n\|_{\dot{H}^1}. \tag{VI.3.15}$$

We recall that $\Delta_k\Delta_j = 0$ if $|j-k|\geq 2$. If $k\in J_n$, then

$$\Delta_k w_n = \sum_{j\in\mathbb{Z}\setminus\widetilde{J}_n} \Delta_k\Delta_j f_n = 0,$$

by the definition of $\widetilde{J}_n$. If $k\notin J_n$, then

$$\Delta_k w_n = \sum_{j\in T_{k,n}} \Delta_j\Delta_k f_n,$$

where $T_{k,n}$ is a subset of $\{k-1, k, k+1\}$. By the triangle inequality and the definition of J_n,

$$\forall k\in J_n, \quad \|\Delta_k w_n\|_{\dot{H}^1} \leq \frac{3\varepsilon}{3} = \varepsilon.$$

The desired bound follows from (VI.3.15). This concludes the proof. □

Proof of Proposition VI.3.2. Let $(f_n)_{n\in\mathbb{N}}$ be as in the proposition. We argue by contradiction, assuming that (VI.3.8) does not hold, i.e. that there exists $I\sqsubset\mathbb{N}$ such that

$$\lim_{n\in I} \|f_n\|_{L^6} > 0.$$

By Lemma VI.3.3, there exists $I'\sqsubset I$ and a sequence $(j_n)_{n\in I'}$ such that

$$\liminf_{n\in I'} \|\Delta_{j_n} f_n\|_{L^6} > 0.$$

Rescaling f_n, we can assume $j_n = 0$ for all n, i.e.

(VI.3.16) $$\lim_{n\in I'} \|\Delta_0 f_n\|_{L^6} > 0.$$

Since $(f_n)_n$ is bounded in $\dot{H}^1$, we have

(VI.3.17) $$\limsup_{n\in I'} \|\Delta_0 f_n\|_{L^2} < \infty.$$

Next we prove

(VI.3.18) $$\lim_{n\in I'} \|\Delta_0 f_n\|_{L^\infty} = 0.$$

Note that $\Delta_0 f_n$ is a continuous bounded function (convolution of a function in $\mathcal{S}$ with an element of L^6). Let $x_n \in \mathbb{R}^3$ such that $|\Delta_0 f_n(x_n)| \geq \frac{1}{2}\|\Delta_0 f_n\|_{L^\infty}$. Then

$$\|\Delta_0 f_n\|_{L^\infty} \leq \frac{1}{2}\left|\Delta_0 f_n(x_n)\right| = \left|\int_{\mathbb{R}^3} \Theta(-y) f_n(x_n+y)dy\right|.$$

Using that $f_n(x_n+\cdot)$ converges weakly to 0 in $\dot{H}^1$, we obtain (VI.3.18). Combining (VI.3.17) and (VI.3.18), we deduce that $\Delta_0 f_n$ goes to 0 in L^6, which contradicts (VI.3.16) and concludes the proof. □

VI.4. Profile decomposition

We will say that two sequences $(g_n)_{n\in I}$, $(h_n)_{n\in I}$ in $\mathcal{G}$ are *orthogonal* when

$$\lim_{n\in I} \frac{|x_n-y_n|+|t_n-s_n|}{\lambda_n} + \frac{\mu_n}{\lambda_n} + \frac{\lambda_n}{\mu_n} = +\infty,$$

where $g_n = (\lambda_n, t_n, x_n)$, $h_n = (\mu_n, s_n, y_n)$.

Claim VI.4.1. *Let $(\lambda_{j,n}, t_{j,n}, x_{j,n})_{n\in I}$, $j=1,2$ be two sequences in $\mathcal{G}$ that are orthogonal. Let u^1 and u^2 be two solutions of the linear wave equation and*

$$u_n^j(t,x) = \frac{1}{\lambda_{j,n}^{1/2}} u\left(\frac{t-t_{j,n}}{\lambda_{j,n}}, \frac{x-x_{j,n}}{\lambda_{j,n}}\right), \quad j=1,2.$$

Then

$$\lim_{n\to\infty} \left(\vec{u}_n^1(0), \vec{u}_n^2(0)\right)_{\dot{\mathcal{H}}^1} = 0.$$

Proof. Rescaling and translating in space, we can assume $\lambda_n^2 = 1$, $x_n^2 = 0$. By conservation of the energy, we have

$$\left(\vec{u}_n^1(0), \vec{u}_n^2(0)\right)_{\dot{\mathcal{H}}^1} = \left(\vec{u}_n^1(t_n^2), \vec{u}_n^2(t_n^2)\right)_{\dot{\mathcal{H}}^1},$$

so that we can also translate the solutions in time and assume $t_n^2 = 0$. Finally, we are left with proving

(VI.4.1) $$\lim_{n\to\infty} \left(\vec{u}_n^1(0), \vec{u}^2(0)\right)_{\dot{\mathcal{H}}^1} = 0,$$

where

$$\vec{u}_n^1(0,x) = \left(\frac{1}{\lambda_n^{1/2}} u^1\left(\frac{-t_n}{\lambda_n}, \frac{x-x_n}{\lambda_n}\right), \frac{1}{\lambda_n^{3/2}} \partial_t u^1\left(\frac{-t_n}{\lambda_n}, \frac{x-x_n}{\lambda_n}\right)\right).$$

and

$$\lim_{n\to\infty} \frac{|t_n|+|x_n|}{\lambda_n} + |\log(\lambda_n)| = +\infty.$$

(We omit the superscript 1 in the parameters to enlighten notations).

By density and energy conservation, we can assume $(u_0^1, u_1^1) \in \left(C_0^\infty(\mathbb{R}^3)\right)^2$. If $\lim_{n\to\infty} \frac{|t_n|}{\lambda_n} = \infty$, then by the explicit formulas of Theorem I.5.2, the two components of $\vec{u}_n^1(0)$ go to 0 uniformly as n goes to infinity, and (VI.4.1) follows. If $\lambda_n \to \infty$, (VI.4.1) is immediate (since the scalar product is bounded by $C/\lambda_n^{1/2}$ for large n). If $\lambda_n \to 0$, one can perform the change of variable $x \mapsto x/\lambda_n$ and obtain (VI.4.1), using that the scalar product is bounded by $C\lambda_n^{1/2}$ for large n. It remains to treat the case where If $\frac{|t_n|}{\lambda_n}$, λ_n and $1/\lambda_n$ are bounded. In this case, $\lim_{n\to\infty} |x_n| = +\infty$, and (VI.4.1) follows by the strong Huygens principle. □

We next prove, as a consequence of Theorem VI.3.1:

THEOREM VI.4.2 (Profile decomposition). *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{LW}$ such that $(\overrightarrow{u_n}(0))_{n\in\mathbb{N}}$ is bounded in $\dot{\mathcal{H}}^1$. Then there exists $I \sqsubset \mathbb{N}$, and, for all $j \geq 1$, there exists $\varphi_j \in \mathcal{LW}$, and a sequence $(g_{j,n})_{n\in I}=(\lambda_{j,n}, t_{j,n}, x_{j,n})_{n\in I} \in \mathcal{G}^I$ such that the following properties hold:*

$$j \neq k \Rightarrow \lim_{n\in I} \frac{|x_{j,n}-x_{k,n}|+|t_{j,n}-t_{k,n}|}{\lambda_{j,n}} + \frac{\lambda_{j,n}}{\lambda_{k,n}} + \frac{\lambda_{k,n}}{\lambda_{j,n}} = +\infty, \tag{VI.4.2}$$

$$\left(\lambda_{j,n}^{1/2} u_n(t_{j,n}, \lambda_{j,n}\cdot + x_{j,n}), \lambda_{j,n}^{3/2}\partial_t u_n(t_{j,n}, \lambda_{j,n}\cdot + x_{j,n})\right) \xrightarrow[n\in I]{} \vec{\varphi}_j(0), \tag{VI.4.3}$$

(with weak convergence in $\dot{\mathcal{H}}^1$) and, denoting by

$$w_{J,n}(t,x) = u_n(t,x) - \sum_{j=1}^{J} \frac{1}{\lambda_{j,n}^{1/2}} \varphi_j\left(\frac{t-t_{j,n}}{\lambda_{j,n}}, \frac{x-x_{j,n}}{\lambda_{j,n}}\right), \tag{VI.4.4}$$

we have

$$\lim_{J\to\infty} \limsup_{n\in I} \|w_{J,n}\|_{L^\infty(\mathbb{R},L^6)} = 0. \tag{VI.4.5}$$

Furthermore, for all J, the following Pythagorean expansion holds:

$$\lim_{n\to\infty} \|\vec{u}_n(0)\|^2_{\dot{\mathcal{H}}^1} - \sum_{j=1}^{J} \|\overrightarrow{\varphi_j}(0)\|^2_{\dot{\mathcal{H}}^1} - \|\overrightarrow{w_{J,n}}(0)\|^2_{\dot{\mathcal{H}}^1} = 0. \tag{VI.4.6}$$

PROOF. We construct the profile φ_j and parameters $g_{j,n}$ by induction on j.

Let $J \geq 1$ and assume that there exist $I \sqsubset \mathbb{N}$, and, for $1 \leq j \leq J-1$, a profile φ_j and a sequence of parameters $(g_{j,n})_{n\in I}$ such that (VI.4.2) hold for $1 \leq j < k \leq J-1$ and (VI.4.3) hold for $1 \leq j \leq J-1$. (if $J=1$ we do not assume anything and set $w_{0,n} = u_n$).

Let $\mathcal{A}_J$ be the set of $(U_0, U_1) \in \dot{\mathcal{H}}^1$ such that there exist a sequence $(\lambda_n, t_n, x_n)_n$ of parameters and $I' \sqsubset I$ such that

$$\left(\lambda_n^{\frac{1}{2}} w_{J-1,n}(t_n, \lambda_n\cdot + x_n), \lambda_n^{\frac{3}{2}}\partial_t w_{J-1,n}(t_n, \lambda_n\cdot + x_n)\right) \xrightarrow[n\in I]{} (U_0, U_1),$$

weakly in $\dot{\mathcal{H}}^1$, where $w_{J-1,n}$ is defined by (VI.4.4). We distinguish two cases.

Case 1. $\mathcal{A}_J = \left\{(0,0)\right\}$. In this case we stop the process and let $\varphi_j = 0$ for all $j \geq J$.

Case 2. There exists a nonzero element in $\mathcal{A}_J$. In this case, we choose $(\varphi_{0,J}, \varphi_{1,J}) \in \mathcal{A}_J$ such that

$$\|(\varphi_{0,J}, \varphi_{1,J})\|_{\dot{\mathcal{H}}^1} \geq \frac{1}{2} \sup_{(U_0,U_1)\in\mathcal{A}_J} \|(U_0,U_1)\|_{\dot{\mathcal{H}}^1}, \tag{VI.4.7}$$

and we choose sequences $(\lambda_{J,n})_n$ and $(t_{J,n})_n$ such that, (after extraction of subsequences in n),

$$\left(\lambda_{J,n}^{\frac{1}{2}} w_n^{J-1}(t_{J,n}, \lambda_{J,n}\cdot + x_{J,n}), \lambda_{J,n}^{\frac{3}{2}}\partial_t w_n^{J-1}(t_{J,n}, \lambda_{J,n}\cdot + x_{J,n})\right) \xrightarrow[n\to\infty]{} (\varphi_{0,J}, \varphi_{1,J}) \tag{VI.4.8}$$

weakly in $\dot{\mathcal{H}}^1$.

Note that (VI.4.3) holds for $j = J$ thanks to (VI.4.8). Furthermore (VI.4.2) for $j \in \{1, \ldots J-1\}$, $k = J$ follows from (VI.4.3) (for $j \in \{1, \ldots, J-1\}$), (VI.4.8) and the fact that $(\varphi_{0,J}, \varphi_{1,J}) \neq (0,0)$. Finally, (VI.4.6) is a consequence of (VI.4.2), (VI.4.3) and Claim VI.4.1.

If there exists a $J \geq 1$ such that Case 1 above holds, then we are done: indeed, in this case, $w_{J,n}$ does not depend on J for large J, and (VI.4.5) is an immediate consequence of the definition of $\mathcal{A}_J$ and Theorem VI.3.1.

Next assume that Case 2 holds for all $J \geq 1$. Using a diagonal extraction argument, we obtain, for all $j \geq 1$, profiles U_{L}^j, and sequences of parameters $(\lambda_{j,n}, t_{j,n}, x_{j,n})_{n\in I}$ such that (VI.4.2), (VI.4.3) and (VI.4.6) hold for all j, k, J. It remains to prove (VI.4.5). In view of Theorem VI.3.1, it is sufficient to prove:

$$\lim_{J\to\infty} \sup_{(A_0,A_1)\in\mathcal{A}_J} \|(A_0,A_1)\|_{\dot{\mathcal{H}}^1} = 0.$$

This follows from (VI.4.7) and the fact that, by (VI.4.6),

$$\lim_{J\to\infty} \|(\varphi_{0,J}, \varphi_{1,J})\|_{\dot{\mathcal{H}}^1} = 0.$$

The proof is complete.

This general principle of proof is sometimes called "*exhaustion method*". □

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