DYNAMICS OF SEMI-LINEAR WAVE EQUATIONS. HOMEWORK, CHAPTER II

Exercise 1. Let $N \geq 1$, $\varphi \in \mathcal{S}(\mathbb{R}^N)$, which is not identically 0. Let

$$\varphi_\varepsilon(x) = e^{ix_1/\varepsilon}\varphi(x).$$

Show that for $\sigma > 0$, $\lim_{\varepsilon\to 0} \|\varphi_\varepsilon\|_{\dot{H}^\sigma} = +\infty$ Show that the sequences $\left(\|u_n\|_{L^p}\right)_n$ and $\left(\|u_n\|_{\dot{B}^\sigma}\right)_n$ are bounded for all $\sigma > 0$ and $p \in [1, \infty]$.

Exercise 2. Let $\sigma \in \mathbb{R}$. We define the (inhomogeneous) Sobolev space $H^\sigma(\mathbb{R}^N)$ by the space of $u \in \mathcal{S}'(\mathbb{R}^N)$ such that $\hat{u}$ is in $L^1(K)$ for all compact K and that the following quantity is finite

$$\|f\|^2_{H^\sigma} = \int_{\mathbb{R}^N} (1 + |\xi|^2)^\sigma |\widehat{f}(\xi)|^2 d\xi.$$

(1) Prove that H^σ is an Hilbert space.
(2) Assume that σ is a positive integer. Prove that f is in H^σ is and only if for all $\alpha \in \mathbb{N}^N$ with $|\alpha| \leq \sigma$, $\partial_x^\alpha f \in L^2(\mathbb{R}^N)$. Prove that the norm on H^σ is equivalent to the norm defined by

$$\|f\|^2 = \sum_{|\alpha|\leq\sigma} \|\partial_x^\alpha f\|^2_{L^2}.$$

In the case where σ is even, prove that this norm is also equivalent to the norm defined by

$$\|f\|^2 = \|f\|^2_{L^2} + \|\Delta^{\sigma/2} f\|^2_{L^2}.$$

(3) Prove that for all $(u_0, u_1) \in H^\sigma \times H^{\sigma-1}$ the formula

$$u(t) = \cos(t|D|)u_0 + \frac{\sin(t|D|)}{|D|}u_1 \tag{1}$$

defines $u \in C^0(\mathbb{R}, H^\sigma)$ such that $\partial_t u \in C^0(\mathbb{R}, H^{\sigma-1})$ that satisfies the linear wave equation in the sense of distribution.

Exercise 3. Let $u_1 \in L^2(\mathbb{R}^N)$ and consider

$$u(t) = \frac{\sin(t|D|)}{|D|}u_1,$$

the solution of the wave equation with initial data $(0, u_1)$.

(1) Prove that for all t, $u(t) \in L^2(\mathbb{R}^N)$, and $\|u(t)\|_{L^2} \leq t\|u_1\|_{L^2}$.
(2) Give a sufficient condition on u_1 so that

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{L^2} < \infty \tag{*}$$

(3) Give example of functions $u_1 \in L^2$ such that (*) does not hold.

Exercise 4 (Conservation laws). Let $(u_0, u_1) \in \mathcal{S}'(\mathbb{R}^N) \times \mathcal{S}'(\mathbb{R}^N)$. Assume that $\widehat{u_0}$ and $\widehat{u_1}$ are locally integrable functions. Let

$$u(t,x) = \cos(t|D|)u_0 + \frac{\sin(t|D|)}{|D|}u_1.$$

Let $\varphi \in C^\infty(\mathbb{R}^N)$ such that $\left(|\widehat{u_0}(\xi)|^2|\xi|^2 + |\widehat{u_1}(\xi)|^2\right)\varphi(\xi)$ is integrable on $\mathbb{R}^N$. Prove that

$$\int \left[|\xi|^2 \left|\widehat{u}(t,\xi)\right|^2 + \left|\widehat{\partial_t u}(\xi)\right|^2\right] \varphi(\xi) d\xi$$

is well-defined and independent of t.

Exercise 5. Consider the fonction defined on $\mathbb{R}^3$ by

$$S(x) = \frac{\sin(|x|)}{|x|}. \tag{2}$$

Compute the Fourier transform of S. Hint: compare the two formulas for the wave equation in space dimension 3 obtained in Chapter I and Chapter II.