DYNAMICS OF SEMI-LINEAR WAVE EQUATIONS. HOMEWORK, CHAPTER 3

Exercise 1. Let $u_0 \in (\dot{H}^{1/2} \times \dot{H}^{-1/2})(\mathbb{R}^3)$, $f \in L^{4/3}(\mathbb{R} \times \mathbb{R}^3)$. Let

$$u(t) = \cos(t|D|)u_0 + \frac{\sin(t|D|)}{|D|}u_1 + \int_0^t \frac{\sin((t-s)|D|)}{|D|} f(s)ds.$$

Using the methods developed in Chapter 3 of the course, prove:

(1) $u \in L^4(\mathbb{R} \times \mathbb{R}^3)$ and

$$\|u\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|f\|_{L^{4/3}(\mathbb{R}\times\mathbb{R}^3)} + \|(u_0, u_1)\|_{\dot{H}^{1/2}\times\dot{H}^{-1/2}}.$$

(2) $(u, \partial_t u) \in C^0(\mathbb{R}, \dot{H}^{1/2} \times \dot{H}^{-1/2})$ and

$$\sup_t \|(u, \partial_t u)\|_{\dot{H}^{1/2}\times\dot{H}^{-1/2}} \lesssim \|f\|_{L^{4/3}(\mathbb{R}\times\mathbb{R}^3)} + \|(u_0, u_1)\|_{\dot{H}^{1/2}\times\dot{H}^{-1/2}}.$$

Exercise 2. For $u_0 \in \mathcal{S}'(\mathbb{R}^N)$, we denote $u(t) = e^{it\Delta}u_0$ the element of $C^0(\mathbb{R}, \mathcal{S}'(\mathbb{R}))$ defined by

$$u(t) = \overline{\mathcal{F}}\left(e^{-it|\xi|^2}\widehat{u_0}(\xi)\right),$$

which is (formally at least) the solution of the linear Schrödinger equation

$$i\partial_t u + \Delta u = 0.$$

One can show, by explicit calculation, the dispersion inequality:

$$\left\|e^{it\Delta}u_0\right\|_{L^\infty(\mathbb{R}^N)} \lesssim \frac{1}{|t|^{N/2}}\|u_0\|_{L^1(\mathbb{R}^N)}.$$

Let $(p, q) \in [2, \infty]^2$, with $p > 2$ and $\frac{2}{p} + \frac{N}{q} = \frac{N}{2}$. Show

$$\left\|e^{it\Delta}u_0\right\|_{L^p(\mathbb{R}, L^q(\mathbb{R}^N))} \lesssim \|u_0\|_{L^2(\mathbb{R}^N)}. \tag{1}$$

Hint: Use the methods from Chapter 3 of the course, but without the dyadic decomposition, which is unnecessary in this case.