DYNAMICS OF SEMI-LINEAR WAVE EQUATIONS. HOMEWORK, CHAPTER 4

In this exercise sheet, we consider the equations:

$$(\partial_t^2 - \Delta)u = \sigma u^5. \tag{W5}$$

Exercise 1. Assume $\sigma = 1$.

(1) Prove that there exists a solution of the ODE: $Y'' = Y^5$ defined on $]-1,+1[$ and such that
$$\lim_{t\to\pm 1} |Y(t)| = +\infty$$

(2) Prove that for all a, b with $a < b$, there exists a solution of (W5) with maximal interval of existence (a, b).

Exercise 2. Assume $\sigma = 1$. Let u be a solution of (W5) such that $u_0 = 0$ and $u_1 \in L^2 \cap C^1$ is nonnegative. Prove that $u(t,x) \geq 0$ for all x and for all $t \geq 0$ in the domain of existence of u.

Exercise 3. (1) Let $R > 0$ and u, v be two solutions of (W5) defined on $[0,\infty[\times\mathbb{R}^3$. Assume that $\vec{u}(0,x) = \vec{v}(0,x)$ for $|x| > R$. Prove that $\vec{u}(t,x) = \vec{v}(t,x)$ for $t \geq 0$, $|x| > R + |t|$.

(2) Prove that there exists a small $\delta_1 > 0$ with the following property: if u is a solution of (W5) with initial data $\vec{u}_0 = (u_0, u_1) \in \dot{\mathcal{H}}^1$ at $t = 0$ with maximal time of existence $T_+ = +\infty$ and such that
$$\int_{|x|>R} |\nabla u_0|^2 + (u_1)^2 dx = \delta^2 \leq \delta_1^2,$$
Then
$$\int_{|x|>R+|t|} |\nabla u(t)|^2 + (\partial_t u(t))^2 dx \leq 2\delta^2 \tag{1}$$

Exercise 4. Exercise IV.1, p.45 of the course.

Exercise 5. Exercise IV.6, p.54 of the course.