DYNAMICS OF SEMI-LINEAR WAVE EQUATIONS. HOMEWORK, CHAPTER 5

In this exercise sheet, we consider the equations:

$$(\partial_t^2 - \Delta)u = \sigma u^5. \tag{W5}$$

and

$$(\partial_t^2 - \Delta)u_L = 0. \tag{LW}$$

in space dimension 3. A "solution" will always mean a finite-energy solution which satisfies the equation in the usual Duhamel sense.

Exercise 1.

(1) Let u_L be a solution of (LW). Prove that

$$\lim_{t\to\infty} \|u_L(t)\|_{L^6} = 0.$$

Hint: start with the case where $\vec{u}_L(0)$ is smooth and compactly supported.

(2) Let f be a measurable function defined on $I \times \mathbb{R}^3$. Assume that the $L^\infty(I, L^6)$ and $L^4(I, L^{12})$ norms of f are finite. Prove that $f \in L^5(I, L^{10})$ and give a bound of the norm of f in this space in terms of the 2 preceding norms.

(3) Prove that there exists $\varepsilon > 0$ with the following property. For any solution u of (W5) with $T_+(u) = +\infty$ and

$$\limsup_{t\to\infty} \|u(t)\|_{L^6} \leq \varepsilon.$$

the solution u scatters to a linear solution.

Exercise 2.

(1) Let u be a solution of (W5). Prove that the momentum of u,

$$P(\vec{u})(t) = \int \nabla u(t,x)\partial_t u(t,x)dx,$$

is independent of t. What is the momentum of u when the initial data $\vec{u}_0$ of u is radial?

(2) Assume that $\vec{u}_0$ is smooth and compactly supported. Let $c > 0$ and

$$u_c(t,x) = u\left(\frac{t - cx_1}{\sqrt{1-c^2}}, \frac{x_1 - ct}{\sqrt{1-c^2}}, x_2, x_3\right).$$

Prove that u_c is a solution of (W5). Compute the momentum and energy of u_c in terms of c, the momentum and the energy of u.

Exercise 3. Let Q be a solution of $-\Delta Q = Q^5$ with $Q \in \dot{H}^1(\mathbb{R}^3)$. Let $F(x) := \mathcal{K}(Q)(x) = \frac{1}{|x|}Q(\frac{x}{|x|^2})$ (Kelvin transform of Q). Prove that $F \in \dot{H}^1$ and that $-\Delta F = F^5$. Compute $\mathcal{K}W$, where $W = (1 + |x|^2/3)^{-1/2}$.