DYNAMICS OF SEMI-LINEAR WAVE EQUATIONS. 2023-2024. HOMEWORK, CHAPTER 6

Exercise 1. Let $(\varphi)_{n\in\mathbb{N}}$ be a bounded sequence in $\dot{H}^1(\mathbb{R}^3)$.

(1) Prove that there exists $I \sqsubset \mathbb{N}$, and for all integer J, an element ψ_j of $\dot{H}^1(\mathbb{R})$, sequences $(\lambda_{j,n})_{n\in I} \in]0,\infty[^I$, $(x_{j,n})_{n\in I} \in (\mathbb{R}^3)^I$ such that, denoting
$$w_{J,n}(x) = \varphi_n(x) - \sum_{j=1}^{J} \frac{1}{\lambda_{j,n}^{1/2}} \psi_j\left(\frac{x - x_{j,n}}{\lambda_{j,n}}\right),$$
one has
$$\lim_{J\to\infty} \limsup_{n\in I} \|w_{J,n}\|_{L^6} = 0.$$
and
$$j \neq k \Longrightarrow \lim_{n\to\infty} \frac{|x_{j,n} - x_{k,n}|}{\lambda_{j,n}} + \left|\log\left(\frac{\lambda_{j,n}}{\lambda_{k,n}}\right)\right| = +\infty.$$

(2) Let $M = \max_{\substack{\varphi\in\dot{H}^1\\ \|\varphi\|_{\dot{H}^1}=1}} \|\varphi\|_{L^6}$. Prove that there exists $\Phi \in \dot{H}^1$ such that $\|\Phi\|_{L^6} = M$ and $\|\Phi\|_{\dot{H}^1} = 1$.

(3) **Subsidiary questions:** Prove that one can assume that φ is radial (*Hint:* use Schwarz rearrangement). Prove that there exists $\mu \in \mathbb{R}$ such that $-\mu\Delta\Phi = \Phi^5$ in the sense of distributions. Prove that Φ is the ground state W introduced in Chapter V of the course, up to scaling and sign change.

Exercise 2. Let $N \geq 1$, $\sigma \in]0, N/2[$ and $\frac{1}{p} = \frac{1}{2} - \frac{\sigma}{N}$. Recall from Chapter II of the course the Sobolev inequality:

$$\forall f \in \mathcal{S}(\mathbb{R}^N), \quad \|f\|_{L^p(\mathbb{R}^N)} \leq C\|f\|_{\dot{H}^\sigma(\mathbb{R}^N)}. \tag{1}$$

and its improved version (see Theorem II.2.4)
$$\forall f \in \mathcal{S}(\mathbb{R}^N), \quad \|f\|^p_{L^p(\mathbb{R}^N)} \leq C\|f\|^{p-2}_{\dot{B}^\sigma(\mathbb{R}^N)} \|f\|^2_{\dot{H}^\sigma(\mathbb{R}^N)}.$$

Prove the analog of Proposition V.3.2, and of Questions 1 and 2 of Exercise 1 for the Sobolev inequality (1).

Exercise 3. We consider a solution u of the equation:

$$(\partial_t^2 - \Delta)u = u^5, \quad \vec{u}_{\restriction t=0} = (u_0, u_1) \in \dot{\mathcal{H}}^1(\mathbb{R}^3). \tag{W5}$$

in space dimension 3, such that $T_+(u) = +\infty$ and
$$\lim_{t\to\infty} \|\vec{u}(t)\|_{\dot{\mathcal{H}}^1} = +\infty$$

In the first questions we assume $(u_0, u_1) \in \left(C_0^\infty(\mathbb{R}^3)\right)^2$.

(1) Let $y(t) = \int_{\mathbb{R}^3} u^2(t,x)dx$. Compute $y'(t)$ and $y''(t)$.

(2) Using the conservation of the energy, prove that $\lim_{T\to\infty} y''(T) = +\infty$. Deduce that $\lim_{T\to\infty} y'(T) = +\infty$ and $\lim_{T\to\infty} y(T) = +\infty$.

(3) Prove that for large t, $(y'(t))^2 \leq 0.9y(t)y''(t)$.

(4) Prove that there exists $C > 0$ such that for large t, $y(t) \leq Cy'(t)^{0.9}$. Deduce a contradiction.

(5) Adapt the preceding proof to the general case $(u_0, u_1) \in \dot{\mathcal{H}}^1$. One can consider
$$y(t) = \int \varphi\left(\frac{x}{t+1}\right) u^2(t,x)dx,$$
where $\varphi \in C_0^\infty(\mathbb{R}^3)$, $\varphi(x) = 1$ for $|x| \leq 2$, $\varphi(x) = 3$ for $|x| \geq 3$, together with the property
$$\lim_{t\to\infty} \int_{|x|\geq 2|t|} \left(|\nabla u(t,x)|^2 + (\partial_t u(t,x))^2\right) dx = 0$$
(to be proved also).