CHAPITRE 1

Linear wave equation: classical theory

1. Presentation of the equation

The linear wave equation is the equation:

$$\partial_t^2 u - \Delta u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{1.1}$$

where $N \geq 1$ is the spatial dimension (in this course, we will often assume $N = 3$), and

$$\Delta = \sum_{k=1}^{N} \frac{\partial^2}{\partial x_k^2}.$$

(We will use either the notations ∂_y or $\frac{\partial}{\partial y}$ for the derivative with respect to the variable $y \in \{t, x_1, \dots, x_N\}$).

This is an evolution equation: we fix initial data at a certain time $t = t_0$, and we are interested in the evolution of the equation over time t. Since the equation is of order 2, we actually fix initial data for $(u, \partial_t u)$, which we denote $\vec{u}$:

$$\vec{u}_{\restriction t=t_0} = (u_0, u_1) \tag{1.2}$$

where (u_0, u_1) is to be taken in a certain functional space.

We will consider in this course initial data *with real values.* The passage to complex or vector values is immediate for most properties of the equation (1.1) (by working coordinate by coordinate), but can induce drastic changes in the nonlinear case, provided that the nonlinearity mixes the coordinates.

Equation (1.1) is invariant under several obvious space-time transformations. If u is a solution, it is also the case of

$$\mu u(t - t_0, \lambda(Rx - x_0)),$$

where $\mu \in \mathbb{R}$, $t_0 \in \mathbb{R}$, $\lambda > 0$, $R \in \mathcal{O}_N(R)$, $x_0 \in \mathbb{R}^N$[1]

For example, we can limit ourselves, without loss of generality, to the case of an initial time $t_0 = 0$, i.e.

$$\vec{u}_{\restriction t=0} = (u_0, u_1) \tag{1.3}$$

Furthermore, the equation is invariant under time inversion: if u is solution, it is also the case of $t \mapsto u(-t, x)$. In particular, it is a reversible equation.

We will also consider the equation with a force:

$$\partial_t^2 u - \Delta u = f, \tag{1.4}$$

(still with an initial condition of type (1.3)), whose understanding will be crucial for the study of the nonlinear wave equation.

The Cauchy problem (1.1), (1.3) can be approached in at least 3 different ways:

[1] Equation (1.1) is in fact invariant under a larger group of linear transformations, the Lorentz group (cf §7 below)

- The classical approach which consists in finding an explicit formula to express the solution. It works when the initial data are sufficiently regular ($C^3 \times C^2$ in dimension 3 of space) and gives classical solutions (that is to say C^2 in (t,x) and satisfying (1.1) in the sense of the classical derivative).
- The use of the Fourier transformation in space, which is very simple (once the Fourier transformation is known) and particularly effective in Sobolev spaces based on L^2 (which are natural spaces for the study of the equation by virtue of the conservation of energy). This method allows to obtain weak solutions with degrees of regularity lower than the previous one, and to use tools based on the Fourier transformation, which can be useful, for example, to demonstrate certain dispersive properties of the equation.
- The "functional analysis" approach, by the theory of semi-groups, which gives the same type of solutions as the previous method.

In this chapter, we will detail the classical method, first by writing the explicit formula for solutions in dimension 1 of space, then in higher dimensions. We will study in the following chapter the equation in the energy space by the Fourier transformation. This chapter is partly based on Chapter 5 of the beautiful book by Folland on partial differential equations [**1**].

2. Explicit Formula in Dimension 1

In dimension 1, the equation (1.1) can be written as:

$$(\partial_t^2 - \partial_x^2)u = 0, \tag{2.1}$$

which means $(\partial_t - \partial_x)(\partial_t + \partial_x)u = 0$. Thus, we make the change of variables $\eta = x + t$, $\xi = x - t$. Therefore, by setting $v(\eta, \xi) = u\left(\frac{\eta-\xi}{2}, \frac{\eta+\xi}{2}\right)$, or $u(t,x) = v(t+x, t-x)$, we have:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 v}{\partial \eta^2} + \frac{\partial^2 v}{\partial \xi^2} + 2\frac{\partial^2 v}{\partial \xi \partial \eta},$$

and

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial \eta^2} + \frac{\partial^2 v}{\partial \xi^2} - 2\frac{\partial^2 v}{\partial \xi \partial \eta},$$

which gives:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -4\frac{\partial^2 v}{\partial \eta \partial \xi}.$$

Thus, we obtain:

$$(1.1) \iff \frac{\partial^2 v}{\partial \eta \partial \xi} = 0.$$

Let u be a C^2 solution of (2.1), (1.3). Thus, $u_1 \in C^1(\mathbb{R})$ and $u_0 \in C^2(\mathbb{R})$.

The equality $\frac{\partial^2 v}{\partial \eta \partial \xi} = 0$ shows that $\frac{\partial v}{\partial \xi}$ is a (class C^1) function $w(\xi)$ independent of η. Integrating with respect to ξ for η fixed, we deduce:

$$v(\eta, \xi) = \underbrace{\int_0^\xi w(\sigma) d\sigma}\, \varphi(\xi) + \psi(\eta),$$

for a certain function ψ, necessarily C^2 since v is of class C^2 and w of class C^1. Thus, we necessarily have:

$$v(\eta, \xi) = \varphi(\xi) + \psi(\eta), \quad \varphi, \psi \in C^2(\mathbb{R}^2),$$

or equivalently:

$$u(t,x) = \varphi(x-t) + \psi(x+t). \tag{2.2}$$

Using the initial condition (1.3), a direct calculation gives:

$$\psi(\eta) = \frac{1}{2}\int_0^\eta u_1(\sigma)d\sigma + \frac{1}{2}u_0(\eta) + c, \;\; \varphi(\xi) \quad = -\frac{1}{2}\int_0^\xi u_1(y)dy + \frac{1}{2}u_0(\xi) - c,$$

where $c \in \mathbb{R}$ (the choice of this constant is irrelevant). Hence, we deduce:

$$u(t,x) = \frac{1}{2}(u_0(x+t) + u_0(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} u_1(y)dy. \tag{2.3}$$

Conversely, it is easy to verify that formula (2.3) gives a C^2 solution of (2.1), (1.3). Therefore, we have shown:

PROPOSITION 2.1. *Let* $(u_0, u_1) \in C^2(\mathbb{R}) \times C^1(\mathbb{R})$. *Then, there exists a unique solution* $u \in C^2(\mathbb{R}\times\mathbb{R})$ *of* (1.1) *satisfying the initial condition* (1.3). *This solution satisfies formula* (2.3).

On formula (2.2), we observe that a solution of the wave equation in dimension 1 is the sum of two waves: one, $\varphi(x-t)$, moving at speed 1 to the right (called a *progressive wave*), and the other $\psi(x+t)$, moving at the same speed to the left.[2]

It is also possible to obtain a formula for the equation with the right-hand side (1.4). We leave this as an exercise to the reader. Further on, we will provide a general method giving the solution of the equation with the right-hand side in terms of the equation without the right-hand side.

We can see from formula (2.3) that $u(t,x)$ depends only on the values of (u_0, u_1) over $\Big[x-|t|, x+|t|\Big]$. This is a prime example of "finite speed of propagation" which holds in all spatial dimensions.

3. Integral on the Sphere and Divergence Theorem

We denote $S^{N-1} = \{x \in \mathbb{R}^N,\ |x| = 1\}$, where $|\cdot|$ represents the Euclidean norm on $\mathbb{R}^N$:

$$|x|^2 = \sum_{j=1}^N x_j^2.$$

More generally, S_R^{N-1} will denote the sphere of radius R: $\{x \in \mathbb{R}^N,\ |x| = R\}$.

We denote $d\sigma$ as the volume element on one of these spheres. Thus, the integral of a function $f \in \mathcal{L}^1(S_R^{N-1})$ (i.e., a function integrable on S_R^{N-1}) is written as

$$\int_{S_R^{N-1}} f(y)d\sigma(y).$$

In dimension 3, this integral can, for example, be calculated using spherical coordinates:

$$\int_{S_R^2} f(y)d\sigma(y) = R^2\int_0^{2\pi}\int_0^\pi f(R\sin\theta\cos\varphi, R\sin\theta\sin\varphi, R\sin\varphi)\sin(\theta)d\theta d\varphi.$$

[2]Note that the equations (1.1), (2.1) have been normalized, so that the speed of propagation is exactly 1.

We denote $B_R^N(x_0)$ as the ball centered at x_0 with radius R:

$$B_R^N(x_0) = \left\{x \in \mathbb{R}^N,\ |x - x_0| < R\right\}$$

and simply $B_R^N = B_R^N(0)$.

We will use the following formulas:

Scale change:

$$\int_{S_R^{N-1}} f(y)d\sigma(y) = R^{N-1}\int_{S^{N-1}} f(Ry)d\sigma(y)n \quad f \in \mathcal{L}^1(S_R^{N-1}).$$

Integral in radial coordinates: if $f \in \mathcal{L}^1(\{|x| \leq R\})$,

$$\int_{B_R^N} f(x)dx = \int_0^R \int_{S_r^{N-1}} f(y)d\sigma(y)dr = \int_0^R \int_{S^{N-1}} f(r\omega)d\sigma(\omega)r^{N-1}dr$$

Divergence theorem: if $F \in C^1(\overline{B_R}, \mathbb{R}^N)$,

$$\int_{|x|\leq R} \nabla \cdot F(x)dx = \int_{S_R^{N-1}} \frac{y}{|y|} \cdot F(y)d\sigma(y),$$

where $\nabla \cdot F = \sum_{j=1}^N \partial_{x_j} F_j$ is the divergence of the vector field F.

4. Energy Density. Uniqueness and Finite Propagation Speed

Before giving an explicit formula for the wave equation in dimension 3, we prove a uniqueness result valid in any dimension:

THEOREM 4.1. *Let $(t_0, x_0) \in \mathbb{R}^{1+N}$, $t_1 > t_0$, $R > 0$. We denote $\Gamma = \Big\{(t,x) \in \mathbb{R} \times \mathbb{R}^N\ :\ t_0 \leq t \leq t_1,\ |x - x_0| \leq R - |t - t_0|\Big\}$ Let $u \in C^2(\Gamma)$ be a solution of* (1.1) *on Γ. We suppose $(u, \partial_t u)(t_0, x) = 0$ for all $x \in B_R(x_0)$. Then u is identically zero on Γ.*

The proof of the theorem is based on a monotonicity law that has its own interest.

We denote, for $(t,x) \in \Gamma$,

$$e_u(t,x) = \frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{2}(\partial_t u(t,x))^2,$$

where $|\nabla u|^2 = \sum_{j=1}^N (\partial_{x_j} u)^2$, and we consider, for $t_0 \leq t \leq t_1$, the local energy

$$E_{\mathrm{loc}}(t) = \int_{B_{R-(t-t_0)}(x_0)} e_u(t,x)dx = \int_{|x-x_0|<R-(t-t_0)} e_u(t,x)dx.$$

LEMMA 4.2. *The function E_{loc} is decreasing on $[t_0, t_1]$.*

The lemma immediately implies Theorem 4.1. Indeed, if $\vec{u}(t_0)$ vanishes on $B(x_0, R)$, then $E_{\mathrm{loc}}(t_0) = 0$, and thus $E_{\mathrm{loc}}(t) = 0$ for all $t \in [t_0, t_1]$, showing that u is zero on Γ.

PROOF OF LEMMA 4.2. We notice that

$$\frac{\partial e}{\partial t} = \sum_{j=1}^N \left(\partial_{x_j} u \partial_t \partial_{x_j} u + \partial_{x_j}^2 u \partial_t u\right) = \sum_{j=1}^N \frac{\partial}{\partial x_j}\left(\partial_{x_j} u \partial_t u\right) = \nabla \cdot (\partial_t u \nabla u)\,. \tag{4.1}$$

Without loss of generality, we can assume for simplification of notations that $x_0 = 0$ and $t_0 = 0$. By the integration formula in radial coordinates,

$$E_{\text{loc}}(t) = \int_0^{R-t} s^{N-1} \int_{S^{N-1}} e_u(t, s\omega) d\sigma(\omega) ds.$$

By the differentiation formula under the sum sign, we get that E_{loc} is differentiable and

$$E'_{\text{loc}}(t) = -(R-t)^{N-1} \int_{S^{N-1}} e_u(t, (R-t)\omega) d\sigma(\omega) + \int_{B^N_{R-t}} \frac{\partial e_u}{\partial t}(t,x)dx.$$

By formula (4.1), then the divergence formula

$$\int_{B^N_{R-t}} \frac{\partial e_u}{\partial t}(t,x)dx = \int_{B^N_{R-t}} \nabla \cdot (\partial_t u \nabla u)(t,x)dx = \int_{S^{N-1}_{R-t}} \frac{y}{|y|} \nabla u \partial_t u(t,y) d\sigma(y).$$

We thus have

$$\begin{aligned} E'_{\text{loc}}(t) = -\int_{S^{N-1}_{R-t}} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}(\partial_t u)^2 + \frac{y}{|y|}\nabla u \partial_t u(t,y) \right) d\sigma(y) \\ \leq -\frac{1}{2} \int_{S^{N-1}_{R-t}} \left(\frac{y}{|y|} \nabla u + \partial_t u(t,y) \right)^2 d\sigma(y). \end{aligned}$$

□

5. Explicit formulas.

We now consider higher space dimensions. In dimension $N = 3$, we will show that for any initial data $(u_0, u_1) \in C^2 \times C^3$, there exists a unique solution $u \in C^2(\mathbb{R}^{1+3})$ of (1.1), (1.3), and provide an explicit formula for this solution. We will also provide a formula in dimension $N = 2$. We refer the reader to [**1**, Chapter 5B] for expressions of solutions when $N \geq 4$.

5.1. The radial case in dimension 3. When the initial conditions depend only on the variable $r = |x|$, the explicit formula is very simple.

We start by showing that if f depends only on the variable r, then the function f is C^2 as a function on $\mathbb{R}^3$ if and only if it is C^2 as a function of the variable r on $[0, \infty[$, and satisfies $\frac{df}{dr}(0) = 0$. Moreover,

$$\Delta f = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}$$

(cf Exercise 1.1). We notice that we can rewrite this formula as

$$r\Delta f = \frac{d^2}{dr^2}(rf).$$

Now let u be a C^2 solution of (1.1), (1.3) with initial conditions (u_0, u_1) that are radial. We assume that for all t, $u(t)$ is radial. We will show a posteriori that this assumption is satisfied. The previous formula gives

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2} \right)(ru) = 0.$$

The function $(t,r) \mapsto ru(t,r)$ is thus a solution of the wave equation in dimension 1, on $\mathbb{R}_t \times (0,\infty)$. To obtain a function on $\mathbb{R}^2$ as a whole, we extend $ru(t,r)$ to an odd function:

$$v(t,y) = yu(t,|y|).$$

One can verify (using Exercise 1.1) that v is of class C^2 on $\mathbb{R}^2$, and that

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}\right) v = 0.$$

Formula (2.3) then gives:

$$v(t,y) = \frac{1}{2}(v_0(y+t) + v_0(y-t)) + \frac{1}{2}\int_{y-t}^{y+t} v_1(\sigma)d\sigma,$$

where $(v_0, v_1) = \vec{v} \restriction t = 0$, thus

$$u(t,r) = \frac{1}{2r}\Big((r+t)u_0(|r+t|) + (r-t)u_0(|r-t|)\Big) + \frac{1}{2r}\int_{r-t}^{r+t} \sigma u_1(|\sigma|)d\sigma. \tag{5.1}$$

Notice that when $t > 0$ (to fix ideas),

$$\int_{r-t}^{r+t} \sigma u_1(|\sigma|)d\sigma = \int_{|r-t|}^{r+t} \sigma u_1(|\sigma|)d\sigma.$$

The finite speed of propagation is well verified: the solution $u(t,r)$ depends only on the initial condition (u_0, u_1) on the ball centered at r with radius $|t|$.

The formula (5.1) defines a function $u(t,r)$ of class C^2 outside the origin $x = 0$, as soon as the initial conditions (u_0, u_1) have the expected regularity $C^2 \times C^1$. However, there is a subtle phenomenon of loss of regularity at the origin of the solution u compared to the initial data: there exist data $(u_0, u_1) \in C^2 \times C^1$ such that u, defined by formula (5.1), cannot be extended by a C^2 function up to $r = 0$. To convince oneself, the reader can verify that (with t fixed),

$$\lim_{r \to 0} u(t,r) = u_0(t) + tu_0'(t) + tu_1(t), \tag{5.2}$$

which shows that if (u_0, u_1) are $C^k \times C^{k-1}$ functions, then $u(t,0)$ is only C^{k-1} (see also Exercise 1.2). We can interpret this phenomenon physically as follows: a singularity on the circle $r = r_0$ at the initial time 0 that travels at speed 1 towards the origin will concentrate at the origin at time $t = r_0$, causing a stronger singularity.

The limit (5.2) suggests a maximal loss of regularity of a derivative with respect to the initial data, which is indeed the case:

PROPOSITION 5.1. *Let $(u_0, u_1) \in (C^3 \times C^2)(\mathbb{R}^3)$ be radial functions. Then formula* (2.3) *extended by $u(t,0) = u_0(t) + tu_0'(t) + tu_1(t)$, defines a C^2 function on $\mathbb{R} \times \mathbb{R}^3$, radial with respect to the variable x, and satisfying* (1.1)*,* (1.3).

The Proposition 5.1 is left as an exercise to the reader.

The formula (5.1) is remarkably simple. In higher space dimensions, we also have an explicit formula for radial solutions, which becomes more complicated as the dimension increases (see Exercise 1.3). The loss of regularity observed in dimension 3 (and absent in dimension 1) increases with dimension, as the reader can verify.

There is no simple formula in the radial case in even dimensions.

We also have explicit formulas (of course more complicated) without radiality assumptions, in all dimensions. We will explicitly state these formulas when $N = 3$, then $N = 2$.

5.2. General solutions in dimension 3: averaging over spheres. If $f \in C^0(\mathbb{R}^3)$, we define

$$(M_f)(t,x) = \frac{1}{4\pi}\int_{S^2} f(x+ty)d\sigma(y) = \frac{1}{4\pi t^2}\int_{S^2_{|t|}} f(x+z)d\sigma(z). \tag{5.3}$$

the average of f over the sphere of radius $|t|$ and center x. The function M_f inherits the regularity of f (cf exercise 1.5).

THEOREM 5.2. *Let $(u_0, u_1) \in C^3(\mathbb{R}^3) \times C^2(\mathbb{R}^3)$. Then the unique C^2 solution of the wave equation* (1.1) *with initial conditions* (1.3) *is given by*

$$u(t,x) = tM_{u_1}(t,x) + \frac{\partial}{\partial t}(tM_{u_0}(t,x)).$$

PROOF. We start by verifying that $tM_{u_1}(t,x)$ is the solution of the wave equation (1.1), with initial condition $(0,u_1)$. By the theorem of differentiation under the sum sign, if $g \in C^2(\mathbb{R}^3)$,

$$\frac{\partial}{\partial t}\left(M_g(t,x)\right) = \frac{1}{4\pi}\int_{S^2}(y\cdot\nabla g)(x+ty)d\sigma(y).$$

Using the divergence formula,

$$\begin{aligned}\int_{S^2}(y\cdot\nabla g)(x+ty)d\sigma(y) &= t\int_{|y|\leq 1}\left(\nabla\cdot(\nabla g)\right)(x+ty)dy\\ &= t\int_{|y|\leq 1}(\Delta g)(x+ty)dy \ = \frac{1}{t^2}\int_0^t\int_{|y|=1}(\Delta g)(x+sy)s^2ds.\end{aligned}$$

Thus:

$$\frac{\partial}{\partial t}\left(tM_{u_1}(t,x)\right) = M_{u_1}(t,x) + \frac{1}{t}\int_0^t\int_{|y|=1}(\Delta u_1)(x+sy)dys^2ds.$$

and therefore

$$\begin{aligned}&\frac{\partial^2}{\partial t^2}\left(tM_{u_1}(t,x)\right) = \frac{1}{4\pi t^2}\int_0^t\int_{|y|=1}(\Delta g)(x+sy)d\sigma(y)s^2ds\\ &-\frac{1}{4\pi t^2}\int_0^t\int_{|y|=1}(\Delta u_1)(x+sy)d\sigma(y)s^2ds+\frac{t}{4\pi}\int_{|y|=1}(\Delta u_1)(x+ty)d\sigma(y) \ = \Delta\left(tM_{u_1}(t,x)\right).\end{aligned}$$

This shows that tM_{u_1} satisfies the wave equation (1.1). Furthermore, since $M_{u_1}(0,x) = u_1(0,x)$, the initial condition at $t = 0$ is indeed $(0,u_1)$.

Now let $v(t,x) = tM_{u_0}(t,x)$. Then, by the same reasoning, v is a solution of the wave equation (1.1) with initial condition $(0,u_0)$. We deduce that $\partial_t v$ is a solution of the wave equation with initial condition $(u_0,0)$, which concludes the proof. □

Notice that we can rewrite the formula of the theorem as:

$$u(t,x) = tM_{u_1}(t,x) + M_{u_0}(t,x) + tM_{y\cdot\nabla u_0}(t,x). \tag{5.4}$$

We now give three consequences of the previous formula.

Corollary 5.3 (Strong Huygens' principle). *The solution $u(t,x)$ depends only on the values of u_0, ∇u_0, and u_1 on the sphere centered at x and of radius $|t|$.*

Remark 5.4. The strong Huygens' principle is a stronger version of the speed of propagation, which states that $u(t,x)$ depends only on the values of (u_0,u_1) on the *ball* centered at x and of radius $|t|$. This principle remains valid in any odd dimension ≥ 3 (the number of derivatives of u_0 and u_1 in the statement increases with the dimension). In even dimension, solutions only satisfy the finite speed of propagation: see §5.3. In dimension 1, as shown by formula (2.3), only solutions even in time (with initial condition of the form $(u_0,0)$) satisfy the strong Huygens' principle.

Corollary 5.5 (Dispersion). *Let $(u_0,u_1) \in (C^3 \times C^2)(\mathbb{R}^3)$, with compact support included in the ball $\overline{B}(0,R)$. Then $|t| - R \leq |x| \leq t + R$ on the support of x and*

$$|u(t,x)| \lesssim \frac{C}{|t|}.$$

Proof. The assertion on the support follows from the strong Huygens' principle (Corollary 5.3). The second assertion is a consequence of formula (5.4). Indeed, we have: $M_{u_1}(t,x) = \frac{1}{4\pi t^2}\int_{S_t^2} u_1(x+y)dy$. The integrand is zero outside the set

$$\{y \in S_t^2 \ : \ x+y \in \operatorname{supp}(u)\},$$

whose measure is uniformly bounded independently of t and x. Thus we have

$$|tM_{u_1}(t,x)| \leq \frac{C}{t},$$

where the constant C depends only on $\sup_x |u_1(x)|$ and R. The same reasoning allows to bound the other terms. □

Finally, we state a positivity property of the wave equation in space dimension 3. This property also holds if $N = 1,2$, but is false if $N \geq 4$.

Corollary 5.6 (Positivity). *Let $u_1 \in C^2(\mathbb{R}^3)$ such that*

$$\forall t \geq 0,\ \forall x \in \mathbb{R}^3, \quad u_1(x) \geq 0.$$

Then

$$\forall t \geq 0,\ \forall x \in \mathbb{R}^3, \quad u(t,x) \geq 0.$$

Proof. This follows immediately from formula (5.4). □

5.3. Dimension $1+2$. A solution u of equation (1.1) with $N=2$ is also a solution of the same equation with $N=3$, constant with respect to the 3rd spatial coordinate. From Theorem 5.2, one can derive an expression of u from the initial data. This strategy is called "descent method".

Theorem 5.7. *Let $(u_0,u_1) \in (C^3 \times C^2)(\mathbb{R}^2)$. Then equation (1.1) has a unique C^2 solution on $\mathbb{R} \times \mathbb{R}^2$, given by the formula*

$$u(t,x) = \frac{1}{2\pi}\left[\frac{\partial}{\partial_t}\left(t\int_{|y|\leq 1}\frac{u_0(x+ty)}{\sqrt{1-|y|^2}}dy\right) + t\int_{|y|\leq 1}\frac{u_1(x+ty)}{\sqrt{1-|y|^2}}dy\right]. \tag{5.5}$$

Proof. Uniqueness follows from Theorem 4.1. Moreover, as in the proof of Theorem 5.2, the formula for even solutions in time (with initial condition $(u_0, 0)$) can be easily deduced from the formula for odd solutions in time (with initial condition $(0, u_1)$). So we only consider this second case.

Let u be a C^2 solution of (1.1) on $\mathbb{R}\times\mathbb{R}^2$, with initial data $(u, \partial_t u)(0) = (0, u_1)$, where $u_1 \in C^2(\mathbb{R}^2)$. By Theorem 5.2, considering u as a solution on $\mathbb{R} \times \mathbb{R}^3$, we obtain:

$$u(t, x_1, x_2) = \frac{t}{4\pi}\int_{S^2} \tilde{u}_1((x_1, x_2, 0) + ty)d\sigma(y)dy,$$

where by definition $\tilde{u}_1(x_1, x_2, x_3) = u_1(x_1, x_2)$. Passing to spherical coordinates, we get

$$\begin{aligned}\int_{S^2} \tilde{u}_1((x_1, x_2, 0)+ty)d\sigma(y) &= \int_0^{2\pi}\int_0^{\pi} u_1(x_1+t\sin\theta\cos\varphi, x_2+t\sin\theta\sin\varphi)\sin\theta d\theta d\varphi \\ &= 2\int_0^{2\pi}\int_0^{\pi/2} u_1(x_1 + t\sin\theta\cos\varphi, x_2 + t\sin\theta\sin\varphi)\sin\theta d\theta d\varphi.\end{aligned}$$

The announced formula then follows from the change of variable $y_1 = t\sin\theta\cos\varphi$, $y_2 = t\sin\theta\sin\varphi$. □

It can be seen from the formula in Theorem 5.7 that the strong Huygens principle is not verified in dimension $1+2$: the solution $u(t, x)$ depends on the values of the initial condition over the entire ball $B_{|t|}(x)$, not just on the sphere $x\ :\ |x| = |t|$.

6. Conservation Laws

The energy of a solution u on $\mathbb{R} \times \mathbb{R}^N$ is defined as:

$$E(\vec{u}(t)) = \int_{\mathbb{R}^N} e_u(t, x)dx = \frac{1}{2}\int_{\mathbb{R}^N}\Big((\partial_t u(t, x))^2 + |\nabla u(t)|^2\Big)dx.$$

This is the global version of the local energy considered in §4. The energy of a solution is conserved over time.

Theorem 6.1. *Let $u \in C^2(\mathbb{R}^{1+N})$ be a solution of* (1.1), (1.3). *Assume* (u_0, u_1) *has finite energy. Then for any t, $E(\vec{u}(t))$ is finite and $E(\vec{u}(t)) = E(u_0, u_1)$.*

Proof. One might be tempted to write

$$\frac{d}{dt}(E(\vec{u}(t))) = \int \partial_t e_u(t, x)dx = \int \nabla\cdot(\partial_t u\nabla u)dx = 0,$$

but the last equality, obtained by integration by parts ignoring the "boundary" term (i.e., when $|x| \to \infty$) is purely formal. To justify the preceding calculation, we can use the decay of the local energy (Lemma 4.2). For $R > 0$, we define:

$$E_{<R}(\vec{u}(t)) = \int_{|x|<R} e_u(t, x)dx.$$

Notice that this quantity is finite as soon as $u \in C^1(\mathbb{R}^{1+N})$. Let's fix $t > 0$. By Lemma 4.2, for any $R > t$,

$$E_{<R-t}(\vec{u}(t)) \le E_{<R}(\vec{u}(0)) \le E(u_0, u_1).$$

As we let R tend to $+\infty$, we obtain that $E(\vec{u}(t))$ is finite, and

$$E(\vec{u}(t)) \le E(u_0, u_1).$$

Reversing the direction of time, we also obtain the inequality

$$E(u_0, u_1) \leq E(\vec{u}(t)).$$

We have shown that the energy is conserved for $t \geq 0$. By applying this result to the solution $(t, x) \mapsto u(-t, x)$, we obtain energy conservation for $t \leq 0$, which concludes the proof. □

There exists another conserved quantity (vectorial), the momentum, defined as

$$P(\vec{u}(t)) = \int \partial_t u(t,x) \nabla u(t,x) dx \in \mathbb{R}^N.$$

PROPOSITION 6.2. *Let $u \in C^2(\mathbb{R}^{1+N})$ be a solution of* (1.1) *with finite energy. Then*

$$\forall t \in \mathbb{R}, \quad P(\vec{u}(t)) = P(u_0, u_1).$$

The proof of this proposition is left as an exercise (see Exercise 1.7).

7. Lorentz Transformations. Time-like Hyperplanes

The *Minkowski spacetime* of dimension N is the space $\mathbb{R}^{1+N}$, equipped with the quadratic form of signature $(1, N)$:

$$g(X) = x_0^2 - \sum_{j=1}^{N} x_j^2 = t^2 - |x|^2 = {}^tXJX,$$

where ${}^tX$ is the transpose of X,

$$X = (x_0, x_1, \dots, x_N),\ t = x_0,\ x = (x_1, \dots, x_N),$$

and $J = [J_{\mu,\nu}]0 \leq \mu\ \nu \leq N$ is the matrix such that $J0, 0 = 1$, $J_{\ell,\ell} = -1$ if $\ell \in 1, \dots, N$, and $J_{\mu,\nu} = 0$ if $\mu \neq \nu$.

The Lorentz group $\mathrm{O}(1, N)$ is the group of real square matrices P of size $1 + N$ which leave the quadratic form g invariant, i.e., such that $g(PX) = g(X)$ for all X in $\mathbb{R}^{1+N}$. In other words, if P is a $(1 + N) \times (1 + N)$ matrix,

$$P \in \mathrm{O}(1, N) \iff {}^tPJP = J.$$

LEMMA 7.1. *Let $P \in \mathrm{O}(1, N)$, $v \in C^2(\mathbb{R}^{1+N})$, and $w(X) = v(PX)$. Then*

$$(\partial_t^2 - \Delta)v = 0 \iff (\partial_t^2 - \Delta w) = 0.$$

PROOF. It can be noted that a function v of class C^2 on $\mathbb{R}^{1+N}$ satisfies the wave equation (1.1) if and only if $\mathrm{Tr}(Jv'') = 0$, where v'' is the Hessian matrix $\left[\partial_{x_\mu} \partial_{x_\nu} v\right]_{0 \leq \mu\ \nu \leq N}$.

An explicit calculation yields $w''(X) = {}^tPv''(Px)P$, and thus

$$\mathrm{Tr}(Jw''(X)) = \mathrm{Tr}(J, {}^tPv''(PX)P) = \mathrm{Tr}(PJ, {}^tPv''(PX)) = \mathrm{Tr}(v''(PX)),$$

which proves the claimed result. □

Two important examples of elements in $\mathrm{O}(1, N)$ are given by space rotations:

$$\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & R \end{bmatrix}, \quad R \in \mathrm{O}N \tag{7.1}$$

and Lorentz transformations, such as:

$$\mathcal{R}\sigma = \begin{bmatrix} R_\sigma & \mathbf{0} \\ \mathbf{0} & I_{N-1}. \end{bmatrix}, \quad R_\sigma = \begin{bmatrix} \cosh(\sigma) & \sinh(\sigma) \\ \sinh(\sigma) & \cosh(\sigma) \end{bmatrix}, \tag{7.2}$$

where I_{N-1} denotes the identity matrix $(N-1)\times(N-1)$ and $\sigma \in \mathbb{R}$. In these formulas, $\mathbf{0}$ always denotes a matrix of appropriate size.

In the preceding sections, we considered the Cauchy problem with initial conditions on a hyperplane in $\mathbb{R}^{1+N}$ of the form $\{t = t_0\}$. We now seek to solve the same problem by prescribing an initial condition on other hyperplanes. Therefore, we consider a hyperplane of the form

$$\Pi = \{X \in \mathbb{R}^{1+N} \ : \ {}^tAX = 0\} \tag{7.3}$$

where $A \in \mathbb{R}^{1+N} \setminus \{0\}$, $A = (a_0, a_1, \dots, a_N) = (a_0, a)$.

We have:

THEOREM 7.2. *Suppose* $|a_0| > |a|$. *Then there exists a transformation* $P \in \mathrm{O}(A, N)$ *such that*

$$\Pi = P\left(\{(0,x),\ x \in \mathbb{R}^N\}\right).$$

The proof of this theorem is left as an exercise. See Exercise 1.10.

If the condition of the preceding theorem is satisfied, we can therefore reduce the Cauchy problem with an initial condition

$$u_{\restriction \Pi} = u_0, \quad A \cdot \nabla u_{\restriction \Pi} = u_1,$$

to a Cauchy problem with initial conditions at $t = 0$ as treated above.

DEFINITION 7.3. The hyperplane Π is called *time-like* when $A = (a_0, a)$ with $a_0 \in \mathbb{R}$, $A \in \mathbb{R}^N$, and $|a_0| > A$.

It can be shown that Π is time-like if and only if the restriction of the quadratic form g to Π is negatively defined.

8. Equation with a source term

We now consider the equation with a source term (1.4). We will express this solution in terms of the propagator of the free equation (1.1). For $(u_0, u_1) \in C^3 \times C^2(\mathbb{R}^3)$, let $S_L(t)(u_0, u_1)$ denote the solution of (1.1) with initial data (u_0, u_1) at $t = 0$. We denote $S(t)u_1 = S_L(t)(0, u_1)$, such that

$$S_L(t)(u_0, u_1) = \frac{\partial}{\partial t}\left(S(t)u_0\right) + S(t)u_1.$$

For $u_1 \in C^2$, we recall that

$$(S(t)u_1)(x) = tM_{u_1}(t,x) = t\int_{S^2} u_1(x + ty)d\sigma(y).$$

THEOREM 8.1 (Duhamel's Formula). *Let* $(u_0, u_1) \in (C^2 \times C^3)(\mathbb{R}^3)$ *and* $f \in C^2(\mathbb{R} \times \mathbb{R}^3)$. *Then the equation* (1.4)*,* (1.3) *has a unique* C^2 *solution, given by the formula:*

$$u(t) = S_L(t)(u_0, u_1) + \int_0^t S(t-s)f(s)ds.$$

REMARK 8.2. The term involving Duhamel's formula can be explicitly expressed, see (8.1).

PROOF OF THEOREM 8.1. Uniqueness follows immediately from Theorem 4.1, since the difference of 2 solutions of (1.4) with the same source term f is a solution of (1.1). For existence, taking into account Theorem 5.2, it suffices to verify that the function

$$U : (t,x) \mapsto \int_0^t S(t-s)f(s)ds$$

is C^2 and satisfies equation (1.4) with zero initial conditions.

We have:

$$U(t,x) = \frac{1}{4\pi}\int_0^t (t-s)\int_{S^2} f(s, x+(t-s)y)d\sigma(y)ds, \tag{8.1}$$

and the fact that U is C^2 follows from the theorem on differentiation under the integral sign.

Furthermore, using that $S(0)g = 0$ for any function g,

$$\frac{\partial U}{\partial t} = \int_0^t \frac{\partial}{\partial t}\Big(S(t-s)f(s)\Big)ds.$$

Upon further differentiation, we obtain

$$\begin{aligned}\frac{\partial^2 U}{\partial t^2} &= \frac{\partial}{\partial t}\Big(S(t-s)f(s)\Big)\restriction s = t + \int_0^t \frac{\partial^2}{\partial t^2}\Big(S(t-s)f(s)\Big)ds \\ &= f(t) + \int_0^t \Delta\Big(S(t-s)f(s)\Big)ds = f(t) + \Delta U.\end{aligned}$$

where we used that $\frac{\partial}{\partial t}(S(t)g)_{\restriction t=0} = g$ for any function g of class C^2. □

Remark 8.3. Duhamel's formula is certainly not specific to dimension 3, as shown by the calculation leading to this formula, which is completely independent of dimension. The reader is invited to explicitly rewrite the solution of equation (1.4) when $N = 1$ and $N = 2$.

From Duhamel's formula, we deduce the energy inequality:

Proposition 8.4. *Let u be a C^2 solution of* (1.4) *with $N = 3$ with initial data (u_0, u_1), such that $f \in C^2(\mathbb{R}^{1+3})$. Suppose furthermore that (u_0, u_1) has finite energy, and for all $T > 0$, $\int_{[-T,+T]} \sqrt{\int_{\mathbb{R}^3} |f(t,x)|^2 dx} dt < \infty$. Then for all $t > 0$,*

$$\sqrt{E(\mathbf{u}(t))} \leq \sqrt{E(u_0,u_1)} + \int_0^t \sqrt{\int_{\mathbb{R}^3} |f(s,x)|^2 dx} ds.$$

Proof. From Duhamel's formula and the conservation of energy for the free equation (1.1), it suffices to verify that for all $T > 0$,

$$\sqrt{E\left(\int_0^t S(t-s)f(s)ds, \partial_t \int_0^t S(t-s)f(s)ds\right)} \leq \int_0^t \sqrt{\int_{\mathbb{R}^3} |f(s,x)|^2 dx} ds.$$

By conservation of energy (Theorem 6.1), we have $\sqrt{E}(S(t-s)f(s)ds, \partial_t \int_0^t S(t-s)f(s)ds) = |f(s)|L^2$. This implies (using that $\sqrt{E}$ is a norm)

$$\sqrt{E\left(\int_0^t S(t-s)f(s)ds, \partial_t \int_0^t S(t-s)f(s)ds\right)} \leq \int_0^t |f(s)|_{L^2} ds,$$

completing the proof. □

9. Exercises

EXERCICE 1.1. Let $f : \mathbb{R}^N \to \mathbb{R}$ ($N \geq 1$). Suppose f is radial, meaning it depends only on the variable $r = |x| = \sqrt{x_1^2 + x_2^2 + \ldots + x_N^2}$. Denote $f(x) = g(|x|)$, where $g : [0, \infty[\to \mathbb{R}$.

(1) Show that f is continuous on $\mathbb{R}^N$ if and only if g is continuous on $[0, \infty[$.
(2) Show that f is C^1 on $\mathbb{R}^N$ if and only if g is C^1 on $[0, \infty[$ and $g'(0) = 0$.
(3) Show that for any $k \geq 2$, f is C^k on $\mathbb{R}^N$ if and only if g is C^k on $\mathbb{R}^N$ and $g^{(j)}(0) = 0$ for all odd integers $j \leq k$.
(4) Assuming f is C^1, determine $\frac{\partial f}{\partial x_j}$ in terms of g', $j = 1, \ldots, N$. Determine $g'(r)$ in terms of ∇f.
(5) Assuming f is C^2 on $\mathbb{R}^N$, prove the formula

$$\Delta f(x) = g''(|x|) + \frac{N-1}{|x|} g'(|x|).$$

In practice, we use the same notation (f) for functions f and g, and denote $g' = \frac{df}{dr}$, etc...

EXERCICE 1.2. Let $k \geq 0$ and $f \in C^0(\mathbb{R}^3)$ be a *radial* function. Define a function u on $\mathbb{R} \times (\mathbb{R}^3 \setminus \{0\})$, radial with respect to the space variable, by

$$u(t, r) = \frac{1}{2r} \left((r+t) f(|r+t|) + (r-t) f(|r-t|) \right).$$

It is noted that u defines a function of class C^k on $\mathbb{R} \times \left(\mathbb{R}^3 \setminus \{0\}\right)$.

(1) Suppose f has support in the annulus $\{\frac{1}{2} \leq |x| \leq 2\}$, such that for $|\eta - 1| \leq 1/10$,

$$f(\eta) = \begin{cases} 2 - \eta & \text{if } \eta > 1 \\ \eta & \text{if } \eta < 1 \end{cases}.$$

Calculate $\lim_{r \to 0} u(t, r)$ when $t = 1$, $t > 1$, and $t < 1$ (close to 1). Conclude that u cannot be extended to a continuous function on $\mathbb{R} \times \mathbb{R}^3$.
(2) Similarly, give an example of a C^2 function f such that u cannot be extended to a C^2 function on $\mathbb{R} \times \mathbb{R}^3$.
(3) Assume f is C^3. Show that u defines a C^2 function on $\mathbb{R} \times \mathbb{R}^3$.
(4) Let g be a radial function on $\mathbb{R}^3$, C^2. Show that

$$u(t, r) = \frac{1}{2r} \int_{r-t}^{r+t} \sigma g(|\sigma|) d\sigma,$$

extends to a C^2 function on $\mathbb{R}^3$.

EXERCICE 1.3 (Solution of the radial wave equation in odd dimension). Let $N \geq 3$ be an odd integer, written as $N = 2k + 1$. Let T_k be the operator defined by

$$T_k \phi = \left(r^{-1} \frac{d}{dr} \right)^{k-1} \left(r^{2k-1} \phi(r) \right).$$

(1) Show that

$$T_k \varphi = \sum_{j=0}^{k-1} c_j r^{j+1} \phi^{(j)} r,$$

for some $c_j \in \mathbb{R}$. Determine c_0 and c_{k-1}.

(2) Show that for any function $\varphi \in C^{k+1}([0,+\infty[)$,

$$\frac{d^2}{dr^2}(T_k\varphi) = \left(r^{-1}\frac{d}{dr}\right)^k (r^{2k}\varphi'(r)).$$

Hint: You can start by verifying that the formula is true when $\varphi(r) = r^m$ for any integer m.

(3) Given a solution $u(t,x)$ of the linear wave equation in space dimension N, assumed to be radial with respect to the space variable. Suppose u is C^{k+1} on $\mathbb{R}^{1+N}$. Show

$$(\partial_t^2 - \partial_r^2)(T_k u) = 0.$$

Deduce an expression of $T_k u$ in terms of u_0 and u_1.

(4) Express $u(t,r)$ in terms of u_0 and u_1 when $N = 5$. What regularity of u_0 and u_1 is required for u to be C^2 on $\mathbb{R}^{1+5}$?

Exercice 1.4. Let u be a solution of the wave equation (1.1) in spatial dimension $N \geq 3$, radial with respect to the space variable. Recall that $\Delta u = \frac{d^2}{dr^2} + \frac{N-1}{r}\frac{d}{dr}$. Suppose $u \in C^2(\mathbb{R}^{1+N})$, with compactly supported initial data. Let

$$v(t,r) = \int_r^\infty \rho\partial_t u(t,\rho)d\rho.$$

Show that v defines a radial solution, of class C^2, to the wave equation in spatial dimension $N-2$.

Exercice 1.5. Let $f \in C^k(\mathbb{R}^3)$. Show that the function M_f, defined by (5.3), is also of class C^k.

Exercice 1.6. Let $u \in C^2(\mathbb{R}\times\mathbb{R}^N)$ be a solution of (1.1) with finite energy. Show

$$\forall \varepsilon > 0,\ \exists R > 0,\ \forall t \in \mathbb{R}, \quad \int_{|x|>R+|t|} e_u(t,x)dx \leq \varepsilon.$$

Exercice 1.7 (Conservation of momentum). (1) Let u be a C^2 solution of (1.1) on $\mathbb{R}\times\mathbb{R}^N$, and $j \in 1,\dots N$. Let $p_{j,u}(t,x) = \partial_{x_j}u(t,x)\partial_t u(t,x)$. Show

$$\frac{\partial p_{j,u}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x_j}\left((\partial_t u)^2 - |\nabla u|^2\right) + \nabla\cdot V,$$

where V is a certain C^1 vector field that you will specify.

(2) Justify that

$$P_j(\vec{u}(t)) = \int_{\mathbb{R}^N} p_{j,u}(t,x)dx$$

is defined for all times. Show that this quantity is independent of time. You can start by considering a local version of the momentum

$$\int_{[-R,R]^N} p_{j,u}(t,x)dx \text{ or } \int_{\mathbb{R}^N} p_{j,u}(t,x)\varphi\left(\frac{x}{R}\right)dx$$

then let R tend to $+\infty$. Here φ denotes a C^2 function with compact support equal to 1 in a neighborhood of the origin.

EXERCICE 1.8. Suppose $N = 1$ or $N = 2$. Let u be the solution of (1.1), (1.3), with $(u_0, u_1) \in C^3 \times C^2$ (if $N = 2$) or $C^2 \times C^1$ (if $N = 1$).

Show that if $u_1 \geq 0$ and $u_0 = 0$ then $u(t, x)$ has the sign of t for all x and $t \neq 0$. When $N = 1$, give a weaker sufficient condition on (u_0, u_1) such that:

$$\forall t \geq 0,\ \forall x \in \mathbb{R}, \quad u(t, x) \geq 0.$$

EXERCICE 1.9. Assume $N = 1$ or $N = 2$. Let u be a solution of (1.4), with $u_0 = u_1 = 0$, and f of class C^1 (if $N = 1$) or C^2 (if $N = 2$). Express u in terms of f.

EXERCICE 1.10. (1) Prove Theorem 7.2. You can use compositions of transformations defined in (7.1) and (7.2).

(2) Prove that Π is of timelike type if and only if the restriction of the quadratic form g to Π is negatively defined.

(3) Under what condition on A does there exist $B = (b_0, b_1, \ldots, b_N) \in \mathbb{R}^{N+1}$ such that the function

$$e^{A \cdot X + iB \cdot X}$$

is a solution of (1.1)?

(4) Now assume that the hyperplane Π is not of timelike type. Let $Y \notin \Pi$. Construct a sequence of solutions $(u_n)n$ of (1.1) such that $u_n(X) = 0$ on Π, such that for any differential operator $D = \prod j = 1^N \partial_{x_1}^{\alpha_1} \ldots \partial_{x_N}^{\alpha_N}$ (of arbitrarily large order), there exists $C > 0$ such that $|Du_n(X)| \leq Ce^{-n}$ on Π, but $|u_n(Y)| \to +\infty$ as $n \to \infty$.

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