CHAPTER II

The Linear Equation in Sobolev Spaces

II.1. Reminders on the Fourier Transform

Here, we recall the definition of the Fourier transform on $\mathbb{R}^N$, in the most general framework possible, that of tempered distributions. We omit the proofs. For more details, one can consult, for example, the foundational writings of Laurent Schwartz [**5**], the course of Jean-Michel Bony [**2**], as well as [**1**, Section 1.2] for a quick introduction, and [**4**] for a more in-depth exposition (the first two references are in French).

We begin by introducing a notation: a *multi-index* is an element $\alpha = (\alpha_1, \dots, \alpha_N)$ of $\mathbb{N}^N$. The order of α is $|\alpha| = \sum_{j=1}^N \alpha_j$. The derivative with respect to α of a function f of class $C^{|\alpha|}$ on $\mathbb{R}^N$ is then defined by:

$$\partial_x^\alpha \varphi = \prod_{j=1}^N \partial_{x_j}^{\alpha_j} f.$$

1.a. Fourier Transform on $\mathcal{S}$.

Definition II.1.1. The Schwartz space $\mathcal{S}(\mathbb{R}^N)$ is the space of functions f of class C^∞ on $\mathbb{R}^N$ such that for every $p \in \mathbb{N}$,

$$N_p(f) := \sup_{x \in \mathbb{R}^N \ |\alpha| \leq p} (1 + |x|)^p |\partial_x^\alpha f(x)| < \infty.$$

It can be observed that each N_p is a norm on $\mathcal{S}(\mathbb{R}^N)$, but N_p is not complete for any of these norms.

We equip $\mathcal{S}(\mathbb{R}^N)$ with the distance

$$d(\varphi, \psi) = \sup_{p \geq 0} \frac{1}{2^p} \min\Big(N_p(\varphi - \psi), 1\Big). \tag{II.1.1}$$

It can be seen that $d(\varphi_n, \varphi)$ tends towards 0 as n tends towards infinity if and only if $N_p(\varphi_n - \varphi)$ tends towards 0 for every p.

It is verified that $\mathcal{S}$, equipped with the topology defined by this distance, is complete.[1]

The Fourier transform of an element φ of $\mathcal{S}$ is defined by the formula

$$\widehat{\varphi}(\xi) = \mathcal{H}\varphi(\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} \varphi(x) dx. \tag{II.1.2}$$

It is verified that $\mathcal{H}$ is a continuous application from $\mathcal{S}$ into $\mathcal{S}$.

[1] A complete and metrizable vector space, whose topology is defined by a family of semi-norms, is called a *Fr'echet space.*

Fubini's theorem immediately implies the duality formula:

$$\int_{\mathbb{R}^N} \widehat{\varphi}(\xi)\psi(\xi)d\xi = \int_{\mathbb{R}^N} \varphi(x)\widehat{\psi}(x)dx, \tag{II.1.3}$$

for $\varphi, \psi \in \mathcal{S}(\mathbb{R}^N)$.

The Fourier transformation is a bijection of $\mathcal{S}$: by defining

$$\overline{\mathcal{F}}(\psi)(x) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{ix\cdot\xi}\psi(\xi)d\xi = \frac{1}{(2\pi)^N}\widehat{\psi}(-x), \tag{II.1.4}$$

we have the *Fourier inversion formula*: for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$,

$$\mathcal{H}\overline{\mathcal{H}}\varphi = \overline{\mathcal{H}}\mathcal{H}\varphi = \varphi. \tag{II.1.5}$$

By combining the Fourier inversion formula (II.1.5) and the duality formula (II.1.3), we obtain the Plancherel theorem:

$$\int_{\mathbb{R}^N} \varphi(x)\overline{\psi}(x)dx = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} \widehat{\varphi}(\xi)\overline{\widehat{\psi}(\xi)}d\xi. \tag{II.1.6}$$

The Fourier transform exchanges multiplication by powers of x and differentiation. For all $\varphi \in \mathcal{S}(\mathbb{R}^N)$

$$\forall \alpha \in \mathbb{N}^N, \quad \frac{1}{i^{|\alpha|}}\mathcal{H}\partial_x^\alpha\varphi = \xi^\alpha\widehat{\varphi}(\xi), \quad i^{|\alpha|}\mathcal{H}(x^\alpha\varphi) = \partial_\xi^\alpha\widehat{\varphi}(\xi). \tag{II.1.7}$$

1.b. Fourier Transform of Tempered Distributions.

Definition II.1.2. The space $\mathcal{S}'(\mathbb{R}^N)$ of *tempered distributions* is the topological dual of $\mathcal{S}(\mathbb{R}^N)$, i.e., the vector space of continuous linear forms on $\mathcal{S}$.

In the definition, continuity must be interpreted in the sense of the topology induced by the distance d defined by (II.1.1). It is easily verified that a linear form f on $\mathcal{S}$ is an element of $\mathcal{S}'$ if and only if:

$$\exists p \in \mathbb{N}, \quad \forall \varphi \in \mathcal{S}, \quad |\langle f, \varphi\rangle| \leq CN_p(\varphi).$$

We equip $\mathcal{S}'$ with the topology of pointwise convergence: a sequence $(f_n)n$ of elements of $\mathcal{S}'$ converges to f in $\mathcal{S}'$ if and only if

$$\forall \varphi \in \mathcal{S}, \quad \lim_{n\to\infty} \langle f_n, \varphi\rangle = \langle f, \varphi\rangle\,.$$

Several function spaces continuously embed into $\mathcal{S}'(\mathbb{R}^N)$ in the following manner. If f is a measurable, locally integrable function on f such that

$$\forall R > 0, \quad \int_{|x|\leq R} |f(x)|dx \leq C(1+R)^C$$

for some constant $C > 0$, we define an element L_f of $\mathcal{S}'(\mathbb{R}^N)$ by

$$\langle L_f, \varphi\rangle = \int_{\mathbb{R}^N} f(x)\varphi(x)dx.$$

It is verified that the preceding application is injective, i.e., L_f is null if and only if f is null almost everywhere on $\mathbb{R}^N$. We then identify f with the linear form L_f, also denoted f. The preceding identification allows us to consider $\mathcal{S}$, Lebesgue spaces $L^p(\mathbb{R}^N)$ $(1 \leq p \leq \infty)$, C_b^k (the space of C^k functions on $\mathbb{R}^N$ that are bounded along with all their derivatives up to order k) as subspaces of $\mathcal{S}'$.

Examples of tempered distributions that are not functions are given by the Dirac delta function at a, denoted δ_a and defined by $\langle \delta_a, \varphi \rangle = \varphi(a)$, as well as the surface measure σ on the sphere S^{N-1}, defined by:

$$\langle \sigma, \varphi \rangle = \int_{S^{N-1}} \varphi(y) d\sigma(y).$$

By duality, several actions can be defined on the elements of $\mathcal{S}'$.

Differentiation. Let $\alpha \in \mathbb{N}^N$ and $f \in \mathcal{S}'$. The *derivative* of f of order α is the element ∂_x^α of $\mathcal{S}'$ defined by:

$$\forall \varphi \in \mathcal{S}, \quad \langle \partial_x^\alpha f, \varphi \rangle = (-1)^{|\alpha|} \langle f, \partial_x^\alpha \varphi \rangle .$$

The integration by parts formula shows that if $f \in C_b^{|\alpha|}$, its derivative of order α in the sense of distributions coincides with its derivative in the classical sense.

Multiplication by a Function. We denote by $\mathcal{P}$ the space of C^∞ functions with *slow growth*, i.e., such that

$$\forall \alpha, \quad \exists M, C > 0 \quad \forall x \in \mathbb{R}^N, \quad |\partial_x^\alpha g(x)| \leq C(1 + |x|)^M .$$

It is easily verified that multiplication by an element of $\mathcal{P}$ is a continuous application from $\mathcal{S}$ into $\mathcal{S}$. We then define, for $f \in \mathcal{S}'$ and $g \in \mathcal{P}$, the product fg by:

$$\langle fg, \varphi \rangle = \langle f, g\varphi \rangle .$$

It is shown that fg is an element of $\mathcal{S}'$ and that $f \mapsto fg$ is a continuous application from $\mathcal{S}'$ into $\mathcal{S}'$.

Fourier Transform. We define the Fourier transform of an element f of $\mathcal{S}'$ by

$$\forall \varphi \in \mathcal{S}, \quad \left\langle \widehat{f}, \varphi \right\rangle = \langle f, \widehat{\varphi} \rangle$$

The duality formula (II.1.3) shows that if $f \in \mathcal{S}$, its Fourier transform according to formula (II.1.2) and its Fourier transform in the sense of $\mathcal{S}'$ coincide.

It is recalled that $L^1(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N)$ are subspaces of $\mathcal{S}'(\mathbb{R}^N)$. The Fourier transform on $\mathcal{S}'$ thus applies to elements of these two spaces. On $L^1(\mathbb{R}^N)$, we recover the Fourier transform in the classical sense.

Proposition II.1.3 (Fourier Transform in L^1). *Let $f \in L^1(\mathbb{R}^N)$, and $\widehat{f}$ be its Fourier transform in $\mathcal{S}'$. Then $\widehat{f}$ is identified with the continuous function given by the formula:*

$$\widehat{f}(\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x) dx.$$

The second proposition immediately follows from the Plancherel theorem:

Proposition II.1.4 (Fourier Transform in L^2). *Let $f \in L^2(\mathbb{R}^N)$ then $\widehat{f} \in L^2(\mathbb{R}^N)$ and*

$$\|f\|_{L^2} = \frac{1}{(2\pi)^{N/2}} \|\widehat{f}\|_{L^2}.$$

The properties of the Fourier transform on $\mathcal{S}$ are transmitted by duality to the Fourier transform:

- We define the inverse Fourier transform $\overline{F}$ of an element f of $\mathcal{S}'$ by

  $$\left\langle \overline{F} f, \varphi \right\rangle = \left\langle f, \overline{F} \varphi \right\rangle .$$

  Then we have the Fourier inversion formula:

  $$\forall f \in \mathcal{S}', \quad \overline{\mathcal{H}}\mathcal{H} f = \mathcal{H}\overline{\mathcal{H}} f = f.$$

- Property (II.1.7) remains valid for $\varphi \in \mathcal{S}'$.

II.2. Sobolev Spaces

2.a. Definition. (cf [**1**, Section 1.3]) Here, we will mainly focus on homogeneous Sobolev spaces based on L^2. We refer to the exercise sheet for classical Sobolev spaces H^σ.

Sobolev spaces on $\mathbb{R}^N$ are easily defined using the Fourier transform:

Definition II.2.1. Let $\sigma \in \mathbb{R}$. The Sobolev space $\dot{H}^\sigma(\mathbb{R}^N)$ is the set of $f \in \mathcal{S}'(\mathbb{R}^N)$ such that $\widehat{f} \in L^1(K)$ for every compact set K, and such that the following quantity is finite:

$$\|f\|_{\dot{H}^\sigma}^2 = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} |\xi|^{2\sigma} |\widehat{f}(\xi)|^2 d\xi.$$

The space $\dot{H}^\sigma$, equipped with the inner product:

$$(f,g)_{\dot{H}^\sigma} = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} |\xi|^{2\sigma} \widehat{f}(\xi)\overline{\widehat{g}(\xi)} d\xi$$

is a pre-Hilbert space.

Theorem II.2.2. *The space $\dot{H}^\sigma(\mathbb{R}^N)$ is complete if and only if $\sigma < N/2$. In this case, the vector space $\mathcal{S}_0$ of functions in $\mathcal{S}$ whose Fourier transform vanishes in a neighborhood of 0 is dense in $\dot{H}^\sigma(\mathbb{R}^N)$.*

Note that $\dot{H}^0$ is exactly the space L^2.

2.b. Sobolev Inequalities. We have the following Sobolev inclusion on $\mathbb{R}^N$.

Theorem II.2.3. *Let $\sigma \in]0, N/2[$, and $p \in (2,\infty)$ such that $\frac{1}{p} = \frac{1}{2} - \frac{\sigma}{N}$. Then $\dot{H}^\sigma(\mathbb{R}^N)$ is contained in L^p, and this inclusion is continuous.*

The result is well-known. We give a proof based on the Fourier transform, which yields a slightly stronger result that we will need later.

By the density result in Theorem II.2.2, it suffices to show that there exists a constant $C > 0$ such that

$$\forall f \in \mathcal{S}(\mathbb{R}^N), \quad \|f\|_{L^p(\mathbb{R}^N)} \leq C \|f\|_{\dot{H}^\sigma(\mathbb{R}^N)}. \tag{II.2.1}$$

Let $f \in \mathcal{S}$. We denote

$$\|f\|_{\dot{B}^\sigma}^2 = \sup_k \frac{1}{(2\pi)^N} \int_{2^k \leq |x| \leq 2^{k+1}} |\xi|^{2\sigma} |\widehat{f}(\xi)|^2 d\xi,$$

and observe that $\|f\|_{\dot{B}^\sigma} \leq \|f\|_{\dot{H}^\sigma}$. We will prove the following result, which implies (II.2.1):

Theorem II.2.4 (Improved Sobolev Inequality). *Let σ and p be as in the previous theorem. Then there exists a constant $C > 0$ such that*

$$\forall f \in \mathcal{S}(\mathbb{R}^N), \quad \|f\|_{L^p}^p \leq \|f\|_{\dot{B}^\sigma}^{p-2} \|f\|_{\dot{H}^\sigma}^2.$$

Notation II.2.5. Let φ be a function on $\mathbb{R}^N$. For $u \in \mathcal{S}'(\mathbb{R}^N)$, we denote

$$\varphi(D)u = \overline{F}\left(\varphi(\xi)\widehat{u}(\xi)\right).$$

The operator $\varphi(D)$ is called the Fourier multiplier.

The tempered distribution $\varphi(D)u$ is not well-defined for all functions φ and $u \in \mathcal{S}'$: we need $\varphi \widehat{u}$ to define a tempered distribution. This is for example the case if $\varphi \in L^\infty$ and $u \in \dot{H}^\sigma$ (in this case $\varphi(D)u \in \dot{H}^\sigma$), or if $\varphi \in \mathcal{P}(\mathbb{R}^N)$ (the space of C^∞ functions with slow growth).

Proof. We fix a parameter $A > 0$ and decompose f into a *high-frequency* part $f_{>A}$ and a *low-frequency* part $f_{<A}$:

$$f_{>A} = \overline{\mathcal{F}}\left(\mathbb{1}_{|\xi|>A}\widehat{f}(\xi)\right) = \mathbb{1}_{|D|>A}f, \quad f_{<A} = \mathbb{1}_{|D|<A}f = 1 - f.$$

Let $k(A)$ be the largest integer such that $2^{k(A)} \leq A$. By using the Cauchy-Schwarz inequality, then the fact that $\sigma < N/2$, we obtain:

$$|f_{<A}(x)| = \frac{1}{(2\pi)^N}\left|\int_{|\xi|<A} e^{ix\cdot\xi}\widehat{f}(\xi)d\xi\right| \leq \frac{1}{(2\pi)^N}\sum_{k\leq k(A)}\int_{2^k\leq|\xi|\leq 2^{k+1}}|\widehat{f}(\xi)|d\xi$$

$$\leq \frac{1}{(2\pi)^N}\sum_{k\leq k(A)} 2^{k(N/2-\sigma)}\left(\int_{2^k\leq|\xi|\leq 2^{k+1}}|\xi|^{2\sigma}|\widehat{f}(\xi)|^2 d\xi\right)^{1/2} \leq C_N A^{N/2-\sigma}\|f\|_{\dot{B}^\sigma},$$

where C_N depends only on the dimension N. Then we write (using Fubini's equality):

$$\|f\|_{L^p}^p = \int |f(x)|^p dx = \int_{\mathbb{R}^N} p\int_0^{|f(x)|}\lambda^{p-1}d\lambda dx$$
$$= p\int_0^{+\infty}\lambda^{p-1}\left|\left\{x\in\mathbb{R}^N \ : \ |f(x)|\geq\lambda\right\}\right|d\lambda.$$

Let $A(\lambda)$ be such that

$$C_N A(\lambda)^{\frac{N}{2}-\sigma}|f|\dot{B}^\sigma = \lambda/2.$$

For any x in $\mathbb{R}^N$,

$$|f< A(\lambda)(x)| \leq \frac{\lambda}{2}.$$

Thus $|f(x)| > \lambda \Longrightarrow |f_{>A(\lambda)}(x)| > \lambda/2$. Hence:

$$\|f\|_{L^p}^p \leq p\int_0^\infty \lambda^{p-1}\left|\left\{x\in\mathbb{R}^N \ : \ |f_{>A(\lambda)}(x)| > \lambda/2\right\}\right|d\lambda$$

By integrating $|f_{>A(\lambda)}|^2$ over the set $\left\{x\in\mathbb{R}^N \ : \ |f_{>A(\lambda)}(x)| > \lambda/2\right\}$, we get

$$\left|\left\{x\in\mathbb{R}^N \ : \ |f_{>A(\lambda)}(x)| > \lambda/2\right\}\right| \leq \frac{4}{\lambda^2}\|f_{>A(\lambda)}\|_{L^2}^2.$$

Combining with the Plancherel theorem, then Fubini's theorem, we obtain

$$\|f\|_{L^p}^p \leq \frac{4p}{(2\pi)^N}\int_0^\infty \lambda^{p-1}\int_{|\xi|>A(\lambda)}|\widehat{f}(\xi)|^2 d\xi d\lambda$$
$$= \frac{4p}{(2\pi)^N}\int_{\mathbb{R}^N}\left|\widehat{f}(\xi)\right|^2\int_0^{c(f,\xi)}\lambda^{p-3}d\lambda d\xi = C_{p,N}\int_{\mathbb{R}^N}\left|\widehat{f}(\xi)\right|^2 c(f,\xi)^{p-2}d\xi,$$

where $c(f,\xi) = 2C_N\|f\|_{\dot{B}^\sigma}|\xi|^{\frac{N}{2}-s}$, and $C_{p,N}$ depends only on N and p. It can be easily verified that $(\frac{N}{2}-s)(p-2) = 2s$, which proves the announced inequality. $\square$

We will focus more particularly on the case $s = 1$. According to the above, the Sobolev space $\dot{H}^1(\mathbb{R}^N)$, $N \geq 3$, is a Hilbert space, contained in $L^{\frac{2N}{N-2}}$, which can be defined as the closure of the space $\mathcal{S}(\mathbb{R}^N)$ under the norm $\dot{H}^1(\mathbb{R}^N)$. We can characterize this norm with the first-order partial derivatives of f. Indeed,

$$|f|_{\dot{H}^1}^2 = \frac{1}{(2\pi)^N}\int |\xi|^2 \big|\widehat{f}(\xi)\big|^2 d\xi = \sum_{j=1}^{N}\int \big|\xi_j \widehat{f}(\xi)\big|^2 d\xi,$$

which shows by Plancherel's theorem and formula (II.1.7)

$$\|f\|_{\dot{H}^1}^2 = \int |\nabla f(x)|^2 dx.$$

The attentive reader will have noticed that the space $\dot{H}^1(\mathbb{R}^N)$ is not the set of $\varphi \in \mathcal{S}'(\mathbb{R}^N)$ such that for all j, $\partial_{x_j}\varphi \in L^2(\mathbb{R}^N)$: indeed, constant functions are in this space, but not in $\dot{H}^1(\mathbb{R}^N)$. However, the density result of Theorem II.2.2 implies that $\dot{H}^1(\mathbb{R}^N)$ is the closure of $C_0^\infty(\mathbb{R}^N)$ under the norm $\|\cdot\|_{\dot{H}^1}^2$.

II.3. The Wave Equation in the Schwartz Space

Let $(u_0, u_1) \in \mathcal{S}(\mathbb{R}^N)$. We will write the solution u of (I.1.1), (I.1.3) using the Fourier transformation. We start with a formal calculation, assuming that $u(t) \in \mathcal{S}$ for all t (which we will prove later). We denote $\widehat{u}(t)$ as the Fourier transform of u with respect to the spatial variable, i.e.,

$$\widehat{u}(t,\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} u(t,x)dx.$$

Thus, we have

$$\widehat{\Delta u}(t,\xi) = -|\xi|^2\widehat{u}(t,\xi),$$

and the wave equation (I.1.1) is formally equivalent to the linear differential equation

$$\partial_t^2\widehat{u}(t,\xi) + |\xi|^2\widehat{u}(t,\xi),$$

where the variable ξ is considered as a parameter. The solution to this equation, with initial conditions $(\widehat{u}(0), \partial_t\widehat{u}(0)) = (u_0, u_1)$, yields

$$\widehat{u}(t,\xi) = \cos(t|\xi|)\widehat{u}_0(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\widehat{u}1(\xi),$$

or, with the previously introduced notation,

$$u(t) = \cos(t|D|)u_0 + \frac{\sin(t|D|)}{|D|}u_1. \tag{II.3.1}$$

THEOREM II.3.1. *Let $(u_0, u_1) \in \mathcal{S}(\mathbb{R}^N)^2$. Then u defined by* (II.3.1) *is an element of $C^\infty(\mathbb{R}\times\mathbb{R}^N)$. It is the unique C^2 solution of* (I.1.1), (I.1.3).

PROOF. Uniqueness follows from Theorem I.4.1. Hence, it suffices to prove that u, defined by (II.3.1), is C^∞ and satisfies (I.1.1), (I.1.3). We have

$$u(t,x) = \frac{1}{(2\pi)^N}\int \mathbb{R}^N e^{ix\cdot\xi}\left(\cos(t|\xi|)\widehat{u}_0(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\widehat{u_1}(\xi)\right)d\xi.$$

By writing

$$\frac{\sin(t|\xi|)}{|\xi|} = t\sum_{k\geq 0}\frac{(-1)^k(t|\xi|)^{2k}}{(2k+1)!},$$

we see that it is a C^∞ function of (t,ξ). Moreover, $\frac{\left|\partial_t^j \sin(t|\xi|)\right|}{|\xi|} \leq |t||\xi|^j$. Similarly, $(t,\xi)\mapsto \cos(t|\xi|)$ is C^∞ and $\left|\partial_t^j \cos(t|\xi|)\right| \leq |\xi|^j$. Using the fact that $\widehat{u}_0$ and $\widehat{u}_1$ are elements of $\mathcal{S}(\mathbb{R}^N)$, by the theorem of differentiation under the integral sign, we obtain that u is C^∞ and satisfies (I.1.1). The Fourier inversion formula shows that u also satisfies the initial conditions (I.1.3). □

II.4. The wave equation in Sobolev spaces

4.a. The equation in general homogeneous Sobolev spaces. Let $(u_0,u_1)\in \dot{H}^\sigma\times\dot{H}^{\sigma-1}$, $\sigma<N/2$. We define as before u by (II.3.1). We also define the formal derivative of u with respect to time:

$$u'(t,x)=\cos(t|D|)u_1-|D|\sin(t|D|)u_0.$$

Then u and u' satisfy the following properties:

Claim II.4.1. *$u\in C^0(\mathbb{R},\dot{H}^\sigma)$, $u'\in C^0(\mathbb{R},\dot{H}^{\sigma-1})$, $u(0)=u_0$, $u'(0)=u_1$.*

Using that $\widehat{u}_0\in L^2(|\xi|^{2\sigma}d\xi)$ and $\widehat{u}_1\in L^2(|\xi|^{2\sigma-2}d\xi)$, it is easy to see that

$$\widehat{u}\in C^0(\mathbb{R},|\xi|^{2\sigma}),\quad \widehat{u'}\in C^0(\mathbb{R},|\xi|^{2\sigma-2}), \tag{II.4.1}$$

which yields the announced continuity property. The facts that $\widehat{u}(0)=u_0$ and $\widehat{u'}(0)=u_1$ follow immediately from the definition.

Claim II.4.2. $\forall t,\quad \|(u(t),u'(t))\|_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}}=\|(u_0,u_1\|_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}}.$

Indeed,

$$\begin{aligned}\int_{\mathbb{R}^N}|\widehat{u}(t,\xi)|^2|\xi|^{2\sigma}d\xi+\int_{\mathbb{R}^N}\widehat{u'}(t,\xi)|\xi|^{2\sigma-2}d\xi\\ =\int_{\mathbb{R}^N}\left|\cos(t|\xi|)\widehat{u}_0(\xi)+\frac{\sin(t|\xi|)}{|\xi|}\widehat{u}_1(\xi)\right|^2|\xi|^{2\sigma}d\xi\\ +\int_{\mathbb{R}^N}\Big|-|\xi|\sin(t|\xi|)\widehat{u}_0(\xi)+\cos(t|\xi|)\widehat{u}_1(\xi)\Big|^2|\xi|^{2\sigma-2}d\xi\\ =\int_{\mathbb{R}^N}\left(|\widehat{u}_0(\xi)|^2+|\widehat{u}_1(\xi)|^2|\xi|^{-2}\right)|\xi|^{2\sigma}d\xi,\end{aligned}$$

which gives the desired property.

Claim II.4.3. *Let $(u_{0,n},u_{1,n})\in(\mathcal{S}_0(\mathbb{R}^N))^2$ such that $(u_{0,n},u_{1,n})$ converges to (u_0,u_1) in $\dot{H}^\sigma\times\dot{H}^{\sigma-1}$. Let u_n be the solution of* (I.1.1) *with data $(u_{0,n},u_{1,n})$. Then*

$$\lim_{n\to\infty}\sup_{t\in\mathbb{R}}\|u_n(t)-u(t)\|_{\dot{H}^\sigma}+\|\partial_t u_n(t)-u'(t)\|_{\dot{H}^{\sigma-1}}=0.$$

It follows immediately from the preceding point, applied to $(u-u_n,u'-\partial_t u_n)$.

Claim II.4.4. *One can identify u with a distribution on $\mathbb{R}\times\mathbb{R}^N$, and it satisfies the wave equation* (I.1.1) *in the distributional sense. Furthermore $u'=\partial_t u$ in the sense of distribution.*

We first give a "concrete" proof of these facts for the reader which is not familiar with the theory of distributions, assuming that σ is large enough so that the object considered are all functions on $\mathbb{R}\times\mathbb{R}^N$.

Let $\sigma \geq 0$. We let u_n be as in Claim II.4.3. Using that u_n is a C^∞ solution of (I.1.1) and integrating by parts, we obtain

$$\iint u_n(t,x)(\partial_t^2 - \Delta)\varphi dxdt = 0.$$

Using the Sobolev embedding $\dot{H}^\sigma \subset L^p$, $\frac{1}{p} = \frac{1}{2} - \frac{\sigma}{N}$, and the point (II.4.3), we see that

$$\lim_{n\to\infty} \|u - u_n\|_{L^p(K)} = 0,$$

for all compact K of $\mathbb{R}^N$. This implies

$$0 = \lim_{n\to\infty} \iint u_n(t,x)(\partial_t^2 - \Delta)\varphi dxdt = \lim_{n\to\infty} \iint u(t,x)(\partial_t^2 - \Delta)\varphi dxdt,$$

and thus

$$\forall \varphi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^N), \quad \iint u(\partial_t^2 - \Delta)\varphi dtdx = 0,$$

which is precisely the meaning of $\partial_t^2 u - \Delta u$ in the distributional sense.

Let $\sigma \geq 1$. The equality

$$\partial_t u_n = -|D|\sin(t|D|)u_{0,n} + \cos(t|D|)u_{1,n}.$$

holds by differentiation below the integral sign. By integration by parts,

$$\forall \varphi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^N), \quad \iint \partial_t u_n \varphi dtdx = -\iint u_n \partial_t \varphi dtdx,$$

Letting $n \to \infty$, we obtain

$$\forall \varphi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^N), \quad \iint u' \varphi dtdx = -\iint u \partial_t \varphi dtdx,$$

which means that $u' = \partial_t u$ in the distributional sense.

The proof for general σ is essentially the same, and can be skipped by the reader who is not familiar with distributions.

If $\varphi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^N)$ (the space of smooth functions with compact support on $\mathbb{R}\times\mathbb{R}^N$), one defines the action of u on $\mathcal{S}$ by

$$\langle u, \varphi\rangle = \int_{-\infty}^{+\infty} \langle u(t), \varphi(t)\rangle_{\mathcal{S}',\mathcal{S}}\, dt,$$

where $\varphi(t)$ is the function $t \mapsto \varphi(t,\cdot)$. It is a straightforward exercise to prove that u is well-defined and that is is a distribution on $\mathbb{R}\times\mathbb{R}^N$. The facts that u satisfies the wave equation in the distributional sense and that $u'(t) = \partial_t u(t)$ follow immediately from Claim II.4.3, that implies that $\lim u_n = u$ in the distributional sense, where u_n is a in Claim II.4.3. This last fact is an immediate consequence of Claim II.4.3.

In the sequel of the proof, we will use the formula (II.1.2) as the definition of the solution u of (I.1.1), (I.1.3) with $(u_0, u_1) \in \big(\mathcal{S}(\mathbb{R}^N)\big)^2$. The preceding claims show that such a u is a limit of smooth, classical solutions of (I.1.1), (I.1.3), and that it satisfies (I.1.1) in a weak sense. Also, we have

$$\partial_t u = -|D|\sin(t|D|)\widehat{u}_0 + \cos(t|D|)\widehat{u}_1$$

in the sense of distribution. In the sequel, we will always use the notation $\partial_t u$ to denote this quantity.

4.b. The wave equation in the energy space. Of particular interest for us is the case $s=1$. We will call "finite energy solutions" the weak solutions with initial data $\dot{H}^1 \times L^2$ given by the preceding subsection in the case $s=1$, $N \geq 3$. We will focus on the case $N=3$. We note that if $(u_0, u_1) \in (C^3 \times C^2)(\mathbb{R}^3) \cap \left(\dot{H}^1 \times L^2\right)(\mathbb{R}^3)$, we have two ways of defining the solution u: by integrals on spheres, as in Theorem I.5.2, and using the Fourier transform, i.e. by formula (II.3.1). Let us prove that these two definitions coincide:

PROPOSITION II.4.5. *Let* $u \in C^2(\mathbb{R} \times \mathbb{R}^3)$ *be a solution of* (I.1.1), (I.1.3). *Assume furthermore* $u_0 = u(0) \in \dot{H}^1$, $u_1 = \partial_t u(0) \in L^2$. *Then*

$$u(t) = \cos(t|D|)u_0 + \frac{\sin(t|D|)}{|D|}u_1, \quad \partial_t u(t) = -|D|\sin(t|D|)u_0 + \cos(t|D|)u_1.$$

PROOF. Let $(u_{0,n}, u_{1,n}) \in \left(\mathcal{S}(\mathbb{R}^N)\right)^2$ with

$$\lim_{n\to\infty} \|u_{0,n} - u_0\|_{\dot{H}^1} + \|u_{1,n} - u_1\|_{L^2} = 0.$$

Let u_n be the corresponding solution of (I.1.1) given by (II.3.1) (note that by uniqueness it is also the solution given by Theorem I.5.2). Since $u - u_n$ is a C^2, finite energy solution of (I.1.1), Theorem I.6.1 yields

$$\forall t, \quad \|u(t) - u_n(t)\|^2_{\dot{H}^1} + \|\partial_t u(t) - \partial_t u_n(t)\|^2_{L^2} = \|u_0 - u_{0,n}\|^2_{\dot{H}^1} + \|u_1 - u_{1,n}\|^2_{L^2},$$

which tends to 0 as n goes to infinity. This proof the result, since $u_n(t)$ converges to $\cos(t|D|)u_0 + \frac{\sin(t|D|)}{|D|}u_1$ in $\dot{H}^1(\mathbb{R}^3)$ and $\partial_t u_n(t)$ converges to $-|D|\sin(t|D|)u_0 + \cos(t|D|)u_1$ in L^2 by Claim II.4.3. □

Using the approximation of finite energy solutions by solutions with initial data in $\mathcal{S}$, we can transfer several results of Chapter I to general finite energy solutions. This is the case of the decay of energy on past wave cones, which imply finite speed of propagation. If u is a finite energy solution (in any dimension $N \geq 3$) and $R > 0$, $x_0 \in \mathbb{R}^N$, $t_0 \in \mathbb{R}$, we denote by

$$E_{\text{loc}}(t) = \int_{|x-x_0|<R-|t-t_0|} e_u(t,x)dx.$$

Then

THEOREM II.4.6. $E_{\text{loc}}(t)$ *is nonincreasing for* $t \geq t_0$.

PROOF. It follows immediately from Theorem I.4.1 the fact that this quantity is nonincreasing when $(u_0, u_1) \in \mathcal{S}$, and that for the approximation given by Claim II.4.3, we obviously have, as a consequence of this claim,

$$\forall t, \quad \lim_{n\to\infty} \int_{|x-x_0|<R-|t-t_0|} e_{u_n}(t,x)dx = \int_{|x-x_0|<R-|t-t_0|} e_u(t,x)dx$$

□

We note that for general finite energy solution the integration by parts used in the proof of Theorem I.4.1 is no longer valid (since the boundary terms are not always well-defined).

4.c. Equation with a source term. We next consider the wave equation with a source term (I.1.4). By linearity, it is sufficient to study the equation with zero initial data:

$$\partial_t^2 u - \Delta u = f, \quad \vec{u}_{\restriction t=0} = (0,0). \tag{II.4.2}$$

PROPOSITION II.4.7. *Assume* $f \in C^0\left(\mathbb{R}, \mathcal{S}(\mathbb{R}^N)\right)$. *Then* u *defined by*

$$u(t) = \int_0^s \frac{\sin\left((t-s)|D|\right)}{|D|} f(s)ds \tag{II.4.3}$$

is the unique solution of (II.4.2).

PROOF. The uniqueness follows as usual by Theorem I.4.1. It is thus sufficient to check that u defined by (II.4.3) is of class C^2, and is a solution of (II.4.2). We consider F the function defined on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$ by

$$F(t,s,x) = \left(\frac{\sin\big((t-s)|D|\big)}{|D|} f(s)\right)(x).$$

Thus

$$F(t,s,x) = \frac{1}{(2\pi)^N} \int e^{ix\cdot\xi} \frac{\sin\big((t-s)|\xi|\big)}{|\xi|} \widehat{f}(s,\xi)d\xi$$

Using that $\widehat{f} \in C^0\Big(\mathbb{R}, \mathcal{S}(\mathbb{R}^N)\Big)$, it is easy to check that F is continuous and C^∞ with respect to the variable (t,x), and that one can differentiate below the integral sign. The result follows since by integration by parts in the ξ variable,

$$\Delta F(t,s,x) = -\frac{1}{(2\pi)^N} \int |\xi|^2 e^{ix\cdot\xi} \frac{\sin\big((t-s)|\xi|\big)}{|\xi|} \widehat{f}(s,\xi)d\xi$$

□

We note that Duhamel formula (II.4.3) is still valid when $f \in L^1([-T,+T], \dot{H}^{\sigma-1})$ for all T, where σ is a fixed real number (assumed to be $< N/2$ for simplicity), and that it yields a function $u \in C^0(\mathbb{R}, \dot{H}^\sigma)$ with $\partial_t u \in C^0(\mathbb{R}, \dot{H}^{\sigma-1})$,

$$\partial_t u = \int_0^t \cos\big((t-s)|D|\big) f(s)ds, \tag{II.4.4}$$

in the sense of distribution, and such that

$$\|\vec{u}(t)\|_{\dot{H}^\sigma \times \dot{H}^{\sigma-1}} \leq \int_0^t \|f(s)\|_{\dot{H}^{\sigma-1}} ds. \tag{II.4.5}$$

Note that (II.4.5) is exactly the energy inequality proved in Chapter I when $\sigma = 1$.

We can approximate f by a sequence of functions (f_n) with

$$f_n \in C^0(\mathbb{R}, \mathcal{S}), \quad \forall t, \quad \lim_{n\to\infty} \int_{-T}^{+T} \|f(s) - f_n(s)\|_{\dot{H}^{\sigma-1}} ds = 0.$$

The corresponding solutions u_n defined by

$$u_n(t) = \int_0^t \frac{\sin\big((t-s)|D|\big)}{|D|} f_n(s)ds$$

are C^2 solutions of (II.4.2) and satisfy

$$\sup_{-T\leq t\leq T} \|\vec{u}_n(t) - \vec{u}(t)\|_{\dot{H}^\sigma \times \dot{H}^{\sigma-1}} \underset{n\to\infty}{\longrightarrow} 0. \tag{II.4.6}$$

As in the case of the free wave equation, this proves that u satisfies (I.1.1) in the sense of distribution. In this situation, we will take the formula (II.4.3) as a definition of the solution u of (I.1.1).

EXERCICE II.1. Assume that $\sigma = 1$. Let f defined on $\mathbb{R} \times \mathbb{R}^N$, such that $f \in L^1([-T, +T, L^2(\mathbb{R}^N))$. Prove that there exists a sequence of functions $f_n \in C_0^\infty(\mathbb{R} \times \mathbb{R}^N)$ such that

$$\forall T > 0, \quad \lim_{n\to\infty} \|f_n - f\|_{L^1([-T,+T],L^2(\mathbb{R}^N))} = 0.$$

EXERCICE II.2. Let u be a C^2 solution of (I.1.1) for some $f \in C^0(\mathbb{R} \times \mathbb{R}^N)$. Assume that $f \in L^1([-T, +T], L^2(\mathbb{R}^N))$ for all $T > 0$. Show that u satisfies (II.4.3).