CHAPTER III

Strichartz inequalities

III.1. Introduction

In view of Plancherel theorem and the Fourier representation formulas for the wave equation, it is natural to study consider the wave equation in $L^2(\mathbb{R}^N)$ or in L^2 based spaces such as the Sobolev spaces $\dot{H}^s$ considered in the preceding chapter. However, this is not sufficient for the study of nonlinear wave equations, since $\||f|^p\|_{L^2(\mathbb{R}^N)} = \|f\|_{L^{2p}}^{2p}$, the appearance of Lebesgue spaces L^q with $q \neq 2$ is unavoidable. A first way to deal with this issue is to use Sobolev inequalities. For example, if one wants to consider solutions in the energy spaces for the equation

$$\partial_t^2 u - \Delta u = u^3, \quad x \in \mathbb{R}^3, \tag{III.1.1}$$

the energy inequality will yields terms of the form[1] $\|u^3\|_{L^1(0,T;L^2)} = \|u\|^3_{L^3(0,T),L^6} \lesssim T\|u\|_{L^\infty(0,T,\dot{H}^1)}$, which is sufficient to prove the existence and uniqueness of finite energy solutions for (III.1.1). However this strategy will not work for higher order nonlinearities, and in particular the quintic one which we will focus on in several chapters of this course. In this chapter I will introduce the celebrated *Strichartz inequalities*, that use the dipsersive properties of the wave equation to improve over Sobolev type inequalities. This type of inequalities was introduced by Robert Strichartz in an article published in 1977 [**10**], and generalized later by several authors. See e.g. [**6**] or the book [**9**].

The original inequalities of Strichartz were formulated in terms of Lebesgue spaces $L^q(\mathbb{R}\times\mathbb{R}^N)$ on the whole space time $\mathbb{R}\times\mathbb{R}^N$. Having in minds applications to nonlinear wave equations, it is useful to consider more general spaces where the Lebesgue exponents in space and times are distinct. If I is an interval, we will define $L^p(I, L^q(\mathbb{R}^N))$ as the set of integrable function $f : I \mapsto L^q(\mathbb{R}^N)$ such that

$$\|u\|_{L^p(\mathbb{R},L^q(\mathbb{R}^N))} = \big\|\|u(t)\|_{L^q(\mathbb{R}^N)}\big\|_{L^p(\mathbb{R})_t} = \left(\int_{\mathbb{R}} \|u(t)\|^p_{L^q} dt\right)^{1/p}. \tag{III.1.2}$$

if finite (with the usual modification if $p = \infty$). The notion of integrable functions with values in a Banach space can be rigorously defined by the theory of Bochner's integration, see e.g. section 1.2 in the book [**3**]. An element of $L^p(I, L^q(\mathbb{R}^N))$ can be identified with a (class) of measurable function on $I \times \mathbb{R}^N$. With the identification, we can use the density of $C_0^\infty(\mathbb{R}^N)$ in $L^q(\mathbb{R}^N)$, $q < \infty$, to prove that $C_0^\infty(\mathbb{R}^N)$ is dense in $L^p(I, L^q)$ if q and p are finite (or if $q < \infty$, $q = \infty$ and I has finite length). Using this fact, we will only work on L^pL^q norms of smooth functions, for which the definition of (III.1.2) is clear. We will often use the generalized Hölder inequality in these spaces:

[1]See below for the notations $L^p(0, T; L^q)$

Proposition III.1.1. *Let $\theta \in [0,1]$, p, q, p_1, q_1, p_2, q_2 in $[1,\infty]$ with*

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

Let $f \in L^{p_1}L^{q_1}$ and $g \in L^{p_2}L^{q_2}$. Then $fg \in L^pL^q$ and

$$\|fg\|_{L^pL^q} \leq \|f\|_{L^{p_1}L^{q_1}} \|g\|_{L^{p_2}L^{q_2}}.$$

We will often write $L^p(I, L^q)$ instead of $L^p(I, L^q(\mathbb{R}^N))$ to lighten notations. When $I = \mathbb{R}$, we will also use the notation L^pL^q.

III.2. Statement of the estimate

The Strichartz inequalities in space dimension 3 with initial data in the energy space read as follows:

Theorem III.2.1. *Let $(u_0, u_1) \in (\dot{H}^1 \times L^2)(\mathbb{R}^3)$ and $f \in L^1(\mathbb{R} \times L^2(\mathbb{R}^3))$. Let*

$$u(t) = \cos(t|D|)u_0 + \frac{\sin(t|D|)u_1}{|D|} + \int_0^t \frac{\sin\big((t-s)|D|\big)}{|D|} f(s)ds. \tag{III.2.1}$$

Then for any (p,q) with $p > 2$,

$$\frac{1}{p} + \frac{3}{q} = \frac{1}{2}, \tag{III.2.2}$$

one has $u \in L^p(\mathbb{R}, L^q(\mathbb{R}^3))$ and

$$\|u\|_{L^p(\mathbb{R},L^q)} \leq C \left(\|(u_0,u_1)\|_{\dot{H}^1 \times L^2} + \|f\|_{L^1(\mathbb{R},L^2)} \right). \tag{III.2.3}$$

for a constant $C > 0$ depending only on p.

We have focused on solutions with initial data $\dot{H}^1 \times L^2$ in space dimension 3, in view of application to the quintic wave equation in space dimension 3. Analogs of Theorem III.2.1 exist in all space dimensions $N \geq 2$, with data in $\dot{H}^\sigma \times \dot{H}^{\sigma-1}$, and different conditions on the right hand-side f. The condition (III.2.2) is necessary by the scaling. of the equation. For solutions in space dimension N with initial data in $\dot{H}^\sigma \times \dot{H}^{\sigma-1}$, it becomes

$$\frac{1}{p} + \frac{N}{q} = \frac{N}{2} - \sigma.$$

Let us mention that there is in general another condition on p and q (see the exercises below). This condition does not appear in Theorem III.2.1 as it is implied by the scaling condition (III.2.2).

Of particular interest is the case $\sigma = 1/2$ in space dimension 3, which was considered by R. Strichartz in his article [**10**], and which is useful to solve the cubic wave equation. We state this inequality and will leave some of the details of the proof to the reader:

Theorem III.2.2. *Let u be defined by* (III.2.1) *with*

$$(u_0, u_1) \in \dot{H}^{1/2}(\mathbb{R}^3) \times \dot{H}^{-1/2}(\mathbb{R}^3), \quad f \in L^{4/3}(\mathbb{R} \times \mathbb{R}^3).$$

Then $u \in L^4(\mathbb{R} \times \mathbb{R}^3)$ and

$$\|u\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \leq C \left(\|f\|_{L^{4/3}(\mathbb{R}\times\mathbb{R}^3)} + \|(u_0,u_1)\|_{\dot{H}^{1/2}\times H^{-1/2}} \right).$$

In the sequel of this chapter we will prove Theorem III.2.1 for $p \geq 4$, which will be sufficient for our applications to the nonlinear equations below.

We will use the following notations. If A and B are positive quantities, we will write $A \lesssim B$ when there exists a constant C, independent of the parameters, such that $A \leq CB$, and $A \equiv B$ when $A \lesssim B$ and $B \lesssim A$.

By the energy inequality and Sobolev embedding, we have for all t.

$$\|u(t)\|_{L^6} \lesssim \|u(t)\|_{\dot{H}^1} \lesssim \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} + \|f\|_{L^1(\mathbb{R}, L^2)},$$

which solves the case $p = \infty$, $q = 6$. Next, we notice that by Hölder inequality, if p and q satisfy (III.2.2) with $p \in (4, \infty)$, we have

$$\|u\|_{L^p L^q} \lesssim \|u\|^{1-\theta}_{L^\infty L^6} \|u\|^{\theta}_{L^4 L^{12}} \tag{III.2.4}$$

where $\theta = \frac{4}{p}$. Thus the inequality (III.2.3) for this pair (p, q) will follows from the same equality for $p = 4$, $q = 12$. We are just reduced to prove the estimate (III.2.3) for $p = 4$, $q = 12$. By density, we can assume $(u_0, u_1) \in \left(C_0^\infty(\mathbb{R}^3)\right)^2$, $f \in C_0^\infty(\mathbb{R} \times \mathbb{R}^3)$.

The inequality will follow from a dispersion inequality which is a quantitative version of the inequality $|u(t)| \lesssim 1$ obtained for compactly supported, smooth functions in Chapter I. This inequality is proved in Section III.3. To deduce the Strichartz inequality from the dispersion inequality many tooled one needs a few tools from harmonic analysis are needed. These tools, that include dyadic decomposition, Littlewood-Paley theory, interpolation of Lebesgue spaces, are recalled in Section III.4. Section III.6 is devoted to the end of the proof of Theorem III.2.1.

III.3. Dispersion inequality

For any function in $\mathcal{S}(\mathbb{R}^N)$, and $s \in \mathbb{N}$, we will denote

$$\|\varphi\|_{\dot{W}^{s,p}} = \sup_{|\alpha| = s} \|\partial_x^\alpha \varphi\|_{L^p(\mathbb{R}^N)}. \tag{III.3.1}$$

In this section we prove

THEOREM III.3.1. *Let $(u_0, u_1) \in \left(\mathcal{S}(\mathbb{R}^3)\right)^2$ and u the solution of* (I.1.1), (I.1.3). *Then for all $t > 0$,*

$$\|u(t)\|_{L^\infty(\mathbb{R}^3)} \lesssim \frac{1}{t} \left(\|u_0\|_{\dot{W}^{2,1}} + \|u_1\|_{\dot{W}^{1,1}}\right).$$

PROOF. By space translation invariance it is sufficient to bound for $|u(t, 0)|$. We have

$$u(t, 0) = t \int_{S^2} u_1(ty) d\sigma(y) + \int_{S^2} u_0(ty) d\sigma(y) + t \int_{S^2} y \cdot \nabla u_0(ty) d\sigma(y).$$

By the divergence theorem,

$$\begin{aligned} t \int_{S^2} u_1(ty) d\sigma(y) \\ = t \int_{B^3} \nabla \cdot (y u_1(ty))\, dy = 3t \int_{B^3} u_1(ty) dy + t^2 \int_{B^3} y \cdot \nabla u_1(ty) dy. \end{aligned} \tag{III.3.2}$$

We have

$$\left| \int_{B^3} y \cdot \nabla u_1(ty) dy \right| \leq \frac{1}{t^3} \int_{tB^3} |\nabla u_1(y)| dy \leq \frac{3}{t^3} \|u_1\|_{\dot{W}^{1,1}}, \tag{III.3.3}$$

and

$$\int_{B^3} |u_1(ty)|dy \leq t \int_{\mathbb{R}^3} |\partial_{x_1} u_1(ty)|dy \leq \frac{1}{t^2} \|u_1\|_{\dot{W}^{1,1}}, \tag{III.3.4}$$

where we have used the Sobolev type inequality $\int_{B^3} |\varphi| dx \lesssim \int_{\mathbb{R}^3} |\partial_{x_1}\varphi|$, that follows immediately from the formula $\varphi(x_1, x_2, x_3) = \int_\infty \partial_{x_1}\varphi(t, x_2, x_3)dt$. Combining (III.3.2), (III.3.3) and (III.3.4), we obtain

$$\left| t \int_{S^2} u_1(ty) d\sigma(y) \right| \lesssim \frac{1}{t} \|u_1\|_{\dot{W}^{1,1}}. \tag{III.3.5}$$

By the same proof, using also the inequality $\int_{B^3} |\varphi| \lesssim \int_{\mathbb{R}^3} |\partial_{x_1}\partial_{x_2}\varphi|$, we have

$$\left| \int_{S^2} u_0(ty) d\sigma(y) \right| + \left| \int_{S^2} y \cdot \nabla u_0(ty) d\sigma(y) \right| \lesssim \frac{1}{t} \|u_0\|_{\dot{W}^{2,1}}. \tag{III.3.6}$$

This concludes the proof of the dispersion inequality. □

III.4. Some tools from harmonic analysis

We first recall an interpolation Theorem for a linear operator between L^p space.

Theorem III.4.1. *Let (X, μ), (Y, ν) be measure spaces. Let*

$$\theta \in]0, 1[, \quad (p_0, p_1, q_0, q_1, p, q) \in [1, \infty]^6$$

with

$$\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}, \quad \frac{1}{q} = \frac{\theta}{q_0} + \frac{1-\theta}{q_1}. \tag{III.4.1}$$

Let A be a linear operator defined on $L^{p_0}(X) + L^{p_1}(X)$ which is bounded from $L^{p_0}(X)$ to $L^{q_0}(X)$ and from $L^{p_1}(X)$ to $L^{q_1}(X)$. Then A is a bounded linear operator from $L^p(X)$ to $L^q(Y)$, and

$$\|A\|_{L^p(X)\to L^q(Y)} \leq \|A\|^{\theta}_{L^{p_0}(X)\to L^{q_0}(Y)} \|A\|^{1-\theta}_{L^{p_1}(X)\to L^{q_1}(Y)}.$$

In the theorem, $\|A\|_{E\to F}$ denotes the operator norm of the bounded operator $A : E \to F$, where E and F are Banach spaces.

We next recall Young's inequality for the convolution

Theorem III.4.2. *Let $f \in L^q(\mathbb{R}^N)$, $g \in L^r(\mathbb{R}^N)$ with $1/q + 1/r \geq 1$, and p defined by $\frac{1}{p} + 1 = \frac{1}{q} + \frac{1}{r}$. Then*

$$f * g(x) = \int f(x - y)g(y)dy$$

is defined for almost every $x \in \mathbb{R}^N$ and

$$\|f * g\|_{L^p} \leq \|f\|_{L^q} \|g\|_{L^r}, \tag{III.4.2}$$

Exercice III.1. Prove Young's inequality. Hint: start with the cases $(p, q) = (1, 1)$, $(p, q) = (\infty, 1)$, $(p, q) = (\infty, \infty)$ and use the interpolation theorem III.4.1.

When $N = 1$ and $\theta \in]0, 1[$, the function $t \mapsto 1/t^\theta$, is not in $L^{1/\theta}$ due to a logarithmic divergence at 0 and ∞. The Hardy-Littlewood-Sobolev inequality says that this function behaves as a $L^{1/\theta}$ function from the point of view of convolution. We will use this inequality in the particular case $\theta = 1/2$, $p = 4/3$, $q = 4$. We refer e.g. to [**1**, Theorem 1.7] for the proof.

THEOREM III.4.3 (Hardy Littlewood Sobolev). *Let $\theta \in]0,1[$, $(p,q) \in]1,\infty[^2$ satisfy*

$$\frac{1}{p}+\theta = 1+\frac{1}{q}.$$

Let $f \in L^p(\mathbb{R}^N)$. Let, for $t \in \mathbb{R}$,

$$g(t) = \int_{\mathbb{R}} f(s)\frac{1}{|t-s|^\theta}ds. \tag{III.4.3}$$

Then the integral defining g converges for almost every t, and

$$\|g\|_{L^q(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}.$$

We next give a few elements of Littlewood-Paley theory, which is a useful tool to study L^p spaces with $p \neq 2$ by Fourier transformation. What follows is by no mean a complete account on Littlewood-Paley theory: we will just state the needed results, and with sketches of proof. We refer to [**1**, Chapter 2] for a complete introduction to the subject.

We start with some inequalities on frequency localized function.

THEOREM III.4.4 (Berstein-type estimates). *Let $\psi \in C_0^\infty(\mathbb{R}^N)$. Then if $1 \leq q \leq p \leq \infty$*

$$\forall f \in \mathcal{S}(\mathbb{R}^N), \quad \forall j \in \mathbb{Z}, \quad \forall \lambda > 0, \quad \|\psi(\lambda D)f\|_{L^p} \lesssim \lambda^{\left(\frac{N}{p}-\frac{N}{q}\right)}\Big\|f\Big\|_{L^q} \tag{III.4.4}$$

Assume furthermore $\psi(\xi) = 0$ for ξ close to 0. Then, if $s \in \mathbb{R}$ and $p \in [1,\infty]$,

$$\forall f \in \mathcal{S}(\mathbb{R}^N), \quad \forall j \in \mathbb{Z}, \quad \Big\||D|^s\psi(\lambda D)f\Big\|_{L^p} \approx \lambda^{-s}\Big\|\psi(\lambda D)f\Big\|_{L^p} \tag{III.4.5}$$

Moreover, if $s \in \mathbb{N}$,

$$\forall f \in \mathcal{S}(\mathbb{R}^N), \quad \forall j \in \mathbb{Z}, \quad \sup_{|\alpha|=s}\Big\|\partial_x^\alpha\left(\psi(\lambda D)f\right)\Big\|_{L^p} \approx \lambda^{-s}\Big\|\psi(\lambda D)f\Big\|_{L^p} \tag{III.4.6}$$

In the theorem, the implicit constants might depend on ψ, but of course not on f and $\lambda > 0$.

PROOF. *Step 1.*

We first prove (III.4.4) for $\lambda = 1$. We have

$$\psi(D)u = (\overline{\mathcal{F}}\psi) * u, \tag{III.4.7}$$

where $f * g$ is the convolution of f and g, defined by

$$(f*g)(x) = \int f(x-y)g(y)dy = \int f(y)g(x-y)dy.$$

This is a classical property of the Fourier transform, we can be checked by an explicit computation of $\mathcal{F}(\psi(D)u)$. Note that $\overline{\mathcal{F}}\psi \in \mathcal{S} \subset \bigcap_{1\leq p\leq\infty} L^p$. Using Young's inequality

we obtain that (III.4.4) holds for $\lambda = 1$, i.e. that there exists $C > 0$ such that

$$\forall f \in \mathcal{S}(\mathbb{R}^N), \quad \|\psi(D)f\|_{L^p} \leq \|f\|_{L^q}.$$

Step 2: rescaling. Denote by $T_\lambda u(x) = u(\lambda x)$. By a simple change of variable, one can prove

$$\Psi(D)(T_\lambda u) = T_\lambda\left(\psi(\lambda D)u\right)$$

Thus by Step 1,

$$\|T_\lambda\left(\psi(\lambda D)u\right)\|_{L^p} \lesssim \|T_\lambda u\|_{L^q}.$$

Since $\|T_\lambda f\|_{L^p} = \frac{1}{\lambda^{N/p}}\|f\|_{L^p}$, we obtain (III.4.4) for any $\lambda > 0$.
Step 3: proof of (III.4.5).

Let $\chi \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$, such that $\chi(\xi) = 1$ if $\xi \in \operatorname{supp}(\psi)$. Then

$$|D|^s\psi(\lambda D)u = |D|^s\chi(\lambda D)\psi(\lambda|D|)u = \frac{1}{\lambda^s}\Xi(\lambda D)\psi(\lambda D)u,$$

where $\Xi(\xi) = |\xi|^s\chi(\xi)$. Using (III.4.4) with $p = q$, we obtain

$$\big\||D|^s\psi(\lambda|D|)u\big\|_{L^p} \lesssim \frac{1}{\lambda^s}\big\|\psi(\lambda|D|)u\big\|_{L^p}. \tag{III.4.8}$$

Using (III.4.8), with s replaced by $-s$ and u replaced by $|D|^s\chi(\lambda D)u$, we obtain

$$\big\|\psi(\lambda|D|)u\big\|_{L^p} = \big\||D|^{-s}\psi(\lambda|D|)|D|^s u\big\|_{L^p} \lesssim \lambda^s\big\|\psi(\lambda|D|)|D|^s u\big\|_{L^p}.$$

This concludes the proof of (III.4.5).
Step 4: proof of (III.4.6). First, we have

$$\Big\|\psi(\lambda D)\partial_x^\alpha f\Big\|_{L^p} = \Big\|\partial_x^\alpha\chi(\lambda D)\psi(\lambda D)f\Big\|_{L^p} = \frac{1}{|\lambda|^{|\alpha|}}\Big\|\Xi_\alpha(\lambda D)\psi(\lambda D)f\Big\|_{L^p}, \tag{III.4.9}$$

where χ is as above and $\Xi_\alpha(\xi) = (i\xi)^\alpha\chi(\xi)$. The estimate $\lesssim$ in (III.4.6) then follows from Bernstein with $q = p$.

Next, if s is even, we have $|D|^s = (-\Delta)^{s/2}$, which shows that (III.4.5) implies the other estimate in (III.4.6).

If s is odd, we write

$$\begin{aligned}\Big\|\psi(\lambda D)|D|^s f\Big\| = \|\psi(\lambda D)|D|^{s+1}\frac{1}{|D|}f\|_{L^p} &\lesssim \sup_{|\alpha|=s+1}\big\|\partial_x^\alpha|D|^{-1}\psi(\lambda D)f\big\|_{L^p}\\ &\approx \frac{1}{\lambda}\sup_{|\alpha|=s+1}\|\partial_x^\alpha\psi(\lambda D)f\|_{L^p},\end{aligned}$$

and we conclude with (III.4.9) that the inequality $\gtrsim$ in (III.4.6) holds in this case also. □

The Littlewood-Paley theory is based on a dyadic decomposition of a distribution $f \in \mathcal{S}'(\mathbb{R}^N)$. We fix once and for all a radial function $\varphi \in C_0^\infty(\mathbb{R}^N)$ with $\varphi(\xi) = 1$ if $|\xi| \leq 1/2$, and $\varphi(x) = 0$ if $|x| \geq 1$. We let

$$\Theta_j(\xi) = \varphi\left(\frac{\xi}{2^{j+1}}\right) - \varphi\left(\frac{\xi}{2^j}\right) = \Theta\left(\frac{\xi}{2^j}\right), \quad \Theta(\xi) = \varphi(\xi/2) - \varphi(\xi).$$

We have

$$\operatorname{supp}\Theta_j \subset \left\{2^{j-1} \leq |\xi| \leq 2^{j+1}\right\}, \quad \sum_{j=-\infty}^{+\infty}\Theta_j(\xi) = 1,\ (\xi \neq 0),$$

where the sum is, for any fixed ξ, a finite sum. We denote

$$\Delta_j f = \Theta_j(D),$$

so that (at least formally) $f = \sum_{j\in\mathbb{Z}}\Theta_j(D)f$ (*Dyadic decomposition of f in frequencies*). If $f \in \mathcal{S}_0$, it is easy to prove that this sum converges in $\mathcal{S}$.

We have the inequality

$$\frac{1}{2} \leq \sum_{j\in\mathbb{Z}}\Theta_j^2(\xi) \leq 1. \tag{III.4.10}$$

Exercice III.2. Prove (III.4.10).

Combining with Plancherel identity, it follows that if $f \in \mathcal{S}(\mathbb{R}^N)$,

$$\|f\|^2_{L^2(\mathbb{R}^N)} \approx \sum_{j\in\mathbb{Z}} \|\Delta_j f\|^2_{L^2(\mathbb{R}^N)}, \tag{III.4.11}$$

and more generally,

$$\|f\|^2_{\dot{H}^s} \approx \sum_{j\in\mathbb{Z}} \|\Delta_j |D|^s f\|^2_{L^2} \approx \sum_{j\in\mathbb{Z}} (2^{2j})^s \|\Delta_j f\|^2_{L^2}.$$

The situation is more complicated for $p \neq 2$. Nevertheless, we have the following estimates:

THEOREM III.4.5. *For all $p \in (1,2]$, for any $f \in \mathcal{S}$*

$$\sum_{j\in\mathbb{Z}} \|\Delta_j f\|^2_{L^p} \lesssim \|f\|^2_{L^p} \tag{III.4.12}$$

For all $p \in [2,\infty)$, for any $f \in L^p$,

$$\|f\|^2_{L^p} \lesssim \sum_{j\in\mathbb{Z}} \|\Delta_j f\|^2_{L^p}. \tag{III.4.13}$$

We omit the proof refering the interested reader to [**1**, Theorem 2.40].

EXERCICE III.3. Prove:

- For all $p \in [1,2]$, for any $f \in \mathcal{S}$

$$\|f\|^p_{L^p} \lesssim \sum_{j\in\mathbb{Z}} \|\Delta_j f\|^p_{L^p} \tag{III.4.14}$$

- For all $p \in [2,\infty]$, for any $f \in L^p$,

$$\sum_{j\ \in\mathcal{Z}} \|\Delta_j f\|^p_{L^p} \lesssim \|f\|^p_{L^p} \tag{III.4.15}$$

(with the usual modification if $p = \infty$).

Hint: Start with the cases $p = 1$ and $p = 2$ for (III.4.14) and $p = \infty$ and $p = 2$ for (III.4.15), then use an interpolation argument.

The two estimates of Exercise III.3 complete the estimates of Theorem III.4.5. The proofs are simpler than the proof of Theorem III.4.5, but we did not detail them here since we will not need them in the sequel.

Note that there is no perfect equivalence between the norm $\|f\|_{L^p}$ and a norm defined as a ℓ^q norm of the sequence $\left(\|\Delta_j f\|_{L^p}\right)_j$ if $p \neq 2$.

Let us mention that the quantities

$$\|f\|^q_{\dot{B}^0_{p,q}} = \sum_{j\in\mathbb{Z}} \|\Delta_j f\|^q_{L^p} \tag{III.4.16}$$

appearing in (III.4.12), (III.4.13), (III.4.14) and (III.4.15) defines the norm of the so-called Besov space $\dot{B}^0_{p,q}$. See Sections 2.3, 2.4 and 2.5 of [**1**] for more details on Besov spaces.

III.5. A Strichartz inequality for the half wave equation

It is sometimes useful to decompose the wave equation in two first-order equations in the time-variable. This is particularly the case when dealing with Fourier analysis tools. We thus introduced the half-wave equations

$$\partial_t u + i|D|u = 0, \quad \partial_t u - i|D|u = 0,$$

and their solutions (given in term of Fourier representations) $e^{-it|D|}\varphi$ and $e^{it|D|}\varphi$. Note that the solution to the usual wave equation (I.1.1), (I.1.3) is given by

$$2u(t) = e^{it|D|}u_0 + e^{-it|D|}u_0 + \frac{e^{it|D|}}{i|D|}u_1 - \frac{e^{-it|D|}}{i|D|}u_1$$

Note also that if $v(t) = e^{it|D|}\varphi$, then $e^{-it|D|}u_0 = v(-t)$, thus it is sufficient to consider only the solution $e^{it|D|}\varphi$. The function $e^{it|\xi|}$ is not smooth at $\xi = 0$, so that $e^{it|D|}$ does not map $\mathcal{S}(\mathbb{R}^N)$ to $\mathcal{S}(\mathbb{R}^N)$. However it maps $\mathcal{S}_0(\mathbb{R}^N)$ to $\mathcal{S}_0(\mathbb{R}^N)$ (where as before $\mathcal{S}_0(\mathbb{R}^N)$ is the space of functions φ in $\mathcal{S}(\mathbb{R}^N)$ such that $\hat{\varphi}$ is identically 0 in a neighborhood of the origin).

In this Section, we will prove

PROPOSITION III.5.1. *There exists $C > 0$ such that*

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^N), \quad \left\| \frac{e^{it|D|}}{|D|}\varphi \right\|_{L^4(\mathbb{R},L^{12})} \lesssim \|\varphi\|_{L^2}. \tag{III.5.1}$$

PROOF. *Step 1: frequency-localized dispersion estimate.*

We will use the Littlewood-Paley decomposition of φ, $\varphi = \sum_{j\in\mathbb{Z}} \Delta_j \varphi$. In this step we prove the following frequency localized version of the dispersion inequality for the wave equation

$$\forall j, \quad \left\| \frac{e^{it|D|}}{|D|}\Delta_j\varphi \right\|_{L^\infty} \lesssim \frac{2^j}{t}\|\Delta_j\varphi\|_{L^1}. \tag{III.5.2}$$

We let $\varphi_j = \Delta_j\varphi$. By the dispersion inequality for the full wave equation and Theorem III.4.4, we have

$$\left\| \frac{\sin(t|D|)}{|D|}\varphi_j \right\|_{L^\infty} \lesssim \frac{1}{|t|}\|\varphi_j\|_{\dot{W}^{1,1}} \approx \frac{2^j}{|t|}\|\varphi_j\|_{L^1}$$

and

$$\|\cos(t|D|)\varphi_j\|_{L^\infty} \approx \frac{1}{2^j}\|\cos(t|D|)\varphi_j\|_{L^\infty} \lesssim \frac{1}{2^j t}\|\varphi_j\|_{\dot{W}^{2,1}} \approx \frac{2^j}{t}\|\varphi_j\|_{L^1}.$$

Step 2. A $L^4/L^{4/3}$ dispersion inequality

We next introduce $\widetilde{\Delta}_j f = \Delta_{j-1}f + \Delta_j f + \Delta_{j+1}f$. Noting that $\Theta_{j-1} + \Theta_j + \Theta_{j+1} = 1$ on the support of Θ_j, we see that $\widetilde{\Delta}_h\Delta_j f = \Delta_j f$. For fixed $t > 0$ and j, consider the operator $\frac{e^{it|D|}}{|D|}\widetilde{\Delta}_j$. By Step 1, it is a bounded operator from L^1 to L^∞, with norm $\lesssim 2^j/t$. By Plancherel and Theorem III.4.4, it is bounded from L^2 to L^2 with norm $\lesssim 2^{-j}$. Using the interpolation Theorem III.4.1, we obtain that $e^{it|D|}|D|^{-1}\widetilde{\Delta}_j$ is a bounded operator from $L^{4/3}$ to L^4 with operator norm $\lesssim t^{-1/2}$. Using that $\widetilde{\Delta}_j\Delta_j = \Delta_j$, we deduce

$$\left\| e^{it|D|}\frac{1}{|D|}\Delta_j\varphi \right\|_{L^4} \lesssim \frac{1}{|t|^{1/2}}\|\Delta_j\varphi\|_{L^{4/3}} \tag{III.5.3}$$

Step 3. A frequency localized Strichartz inequality.

Next, we consider the operator T_j defined by

$$(T_j\varphi)(t,x) = \left(e^{it|D|}|D|^{-1/2}\Delta_j\varphi\right)(x)$$

In this step we prove that T_j extends to a bounded operator from $L^2(\mathbb{R}^3)$ to $L^4(\mathbb{R}\times\mathbb{R}^3)$, with an operator norm that is independent of j, i.e.

$$\forall\varphi\in\mathcal{S}(\mathbb{R}^3),\quad \left\|e^{it|D|}|D|^{-1/2}\Delta_j\varphi\right\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|\varphi\|_{L^2}. \tag{III.5.4}$$

We will use a so-called TT^* argument to reduce the proof of (III.5.4) to the proof of the boundedness of an operator acting on functions on $\mathbb{R}\times\mathbb{R}^3$.

The inequality (III.5.4) is equivalent to the following statement:

$$\forall\varphi\in\mathcal{S}(\mathbb{R}^3),\quad \forall g\in C_0^\infty(\mathbb{R}\times\mathbb{R}^3),\quad \left|\iint (T_j\varphi)\overline{g}\,dxdt\right| \lesssim \|\varphi\|_{L^2(\mathbb{R}^3)}\|g\|_{L^{4/3}(\mathbb{R}\times\mathbb{R}^3)}.$$

Using Plancherel equality in the space variable for every $t\in\mathbb{R}$, we obtain

$$\iint (T_j\varphi)\overline{g}\,dxdt = \int \varphi(x)(T_j^*g)(x)dx,$$

where the (formal) adjoint T_j^* of T_j is defined by

$$T_j^*g(x) = \int_{\mathbb{R}} e^{-it|D|}|D|^{-1/2}\Delta_j g(t)dt.$$

We are thus reduced to prove

$$\forall g\in C_0^\infty(\mathbb{R}\times\mathbb{R}^3),\quad \left\|T_j^*g\right\|_{L^2(\mathbb{R}^3)} \lesssim \|g\|_{L^{4/3}(\mathbb{R}\times\mathbb{R}^3)}. \tag{III.5.5}$$

We have

$$\left\|T_j^*g\right\|_{L^2}^2 = \int_{\mathbb{R}^3} T_j^*g\overline{T_j^*g}\,dx = \iint_{\mathbb{R}\times\mathbb{R}^3} T_jT_j^*g\overline{g}\,dxdt, \tag{III.5.6}$$

and (III.5.5) would follow from the inequality

$$\left\|T_jT_j^*g\right\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|g\|_{L^{4/3}(\mathbb{R}\times\mathbb{R}^3)}. \tag{III.5.7}$$

We have

$$T_jT_j^*g(t,x) = \int_{\mathbb{R}} e^{i(t-s)|D|}|D|^{-1}\Delta_j g(s)ds.$$

Using the $L^4/L^{4/3}$ dispersion inequality of Step 2, we obtain at fixed t,

$$\left\|(T_jT_j^*g)(t)\right\|_{L^4(\mathbb{R}^3)} \lesssim \int_{\mathbb{R}} \frac{1}{|t-s|^{1/2}}\left\|\Delta_j g(s)\right\|_{L^{4/3}(\mathbb{R}^3)} ds$$

By Hardy Littlewood Sobolev inequality, we deduce

$$\|TT^*g\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|\Delta_j g\|_{L^{4/3}(\mathbb{R}\times\mathbb{R}^3)},$$

which yields (III.5.7) and thus concludes the proof of (III.5.4). Note that we can also localize the right-hand side in (III.5.4). Indeed, using $\widetilde{\Delta}_j = \Delta_{j+1}+\Delta_j+\Delta_{j-1}$, we see that (III.5.4) remains valid when Δ_j is replaced by $\widetilde{\Delta}_j$. Applying this inequality to $\Delta_j\varphi$, and using that $\widetilde{\Delta}_j\Delta_j = \Delta_j$, we obtain

$$\forall\varphi\in\mathcal{S}(\mathbb{R}^3),\quad \left\|e^{it|D|}|D|^{-1/2}\Delta_j\varphi\right\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|\Delta_j\varphi\|_{L^2}. \tag{III.5.8}$$

Step 5. The L^4L^{12} localized in frequency Strichartz inequality. We next prove

$$\left\|e^{it|D|}\Delta_j\varphi\right\|_{L^4(\mathbb{R},L^{12}(\mathbb{R}^3))} \lesssim \|\Delta_j\varphi\|_{\dot{H}^1}$$

Indeed by the Bernstein type's inequalities of Theorem III.4.4,

$$\begin{aligned}\left\|e^{it|D|}|D|^{-1}\Delta_j\varphi\right\|_{L^4(\mathbb{R},L^{12}(\mathbb{R}^3))} &\approx \frac{1}{2^{j/2}}\left\|e^{it|D|}|D|^{-1/2}\Delta_j\varphi\right\|_{L^4(\mathbb{R},L^{12}(\mathbb{R}^3))}\\ &\lesssim \left\|e^{it|D|}|D|^{-1/2}\Delta_j\varphi\right\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|\Delta_j\varphi\|_{L^2}.\end{aligned}$$

Step 6. Summing up the frequencies.

In this step, we conclude the proof of Proposition III.5.1, by summing up the estimate of Step 5 with respect to j. We fix $\varphi \in \mathcal{S}_0(\mathbb{R}^3)$. We have

$$\sum_{j\in\mathbb{Z}}\left\|e^{it|D|}\Delta_j\varphi\right\|^2_{L^4L^{12}} \lesssim \sum_{j\in\mathbb{Z}}\left\||D|\Delta_j\varphi\right\|^2_{L^2}. \tag{III.5.9}$$

The right-hand side is $\approx \|\varphi\|^2_{\dot{H}^1}$ by Plancherel equality (see (III.4.11)). We must prove that the left-hand side dominates $\left\|e^{it|D|}\varphi\right\|_{L^4L^{12}}$. Let $u = e^{it|D|\varphi}$ and $u_j = \Delta_j u$. By Minkowski inequality (i.e. the triangle inequality for the $L^2(\mathbb{R})$ norm), we see that

$$\sum_{j\in\mathbb{Z}}\|u_j\|^2_{L^4L^{12}} = \sum_{j\in\mathbb{Z}}\left\|\|u_j(t)\|^2_{L^{12}(\mathbb{R}^3)}\right\|_{L^2(\mathbb{R})} \geq \left\|\sum_{j\in\mathbb{Z}}\|u_j(t)\|^2_{L^{12}}\right\|_{L^2(\mathbb{R})}$$

By Theorem III.4.5, at fixed t,

$$\|u(t)\|^2_{L^{12}} \lesssim \sum_{j\in\mathbb{Z}}\|u_j(t)\|^2_{L^{12}}.$$

This shows

$$\sum_{j\in\mathbb{Z}}\|u_j\|^2_{L^4L^{12}} \gtrsim \left\|\|u(t)\|^2_{L^{12}(\mathbb{R}^3)}\right\|_{L^2(\mathbb{R})} = \|u\|^{1/2}_{L^4L^{12}},$$

which together with (III.5.9) concludes the proof of Proposition III.5.1. □

Remark III.5.2. An alternative, somehow simpler approach is to sum up over j the frequency localized dispersion inequality of Step 2 of the preceding proof. Using Theorem III.4.5, one obtains a $L^4/L^{4/3}$ dispersion inequality for the half-wave equation:

$$\left\|e^{it|D|}|D|^{-1}\varphi\right\|_{L^4} \lesssim \frac{1}{|t|^{1/2}}\|\varphi\|_{L^{4/3}}.$$

It is then possible to forget about frequency cut-off and run the preceding arguments to obtain Strichartz inequalities for the half-wave equation directly.

III.6. Proof of the Strichartz estimate for the full wave equation

We are now ready to prove Theorem III.2.1. We can treat separately the terms

$$u_L(t) = \cos(t|D|)u_0 + \frac{\sin(t|D|)u_1}{|D|}$$

and

$$(Bf)(t) = \int_0^t \frac{\sin\big((t-s)|D|\big)}{|D|}f(s)ds. \tag{III.6.1}$$

Using that $\cos(t|D|) = \frac{1}{2}\left(e^{it|D|} + e^{-it|D|}\right)$, $\sin(t|D|) = \frac{1}{2i}\left(e^{it|D|} - e^{-it|D|}\right)$, we obtain immediately from Proposition III.5.1

$$\|u_L\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|u_0\|_{\dot{H}^1} + \|u_1\|_{L^2}.$$

The other term is more delicate. We first consider

$$u_a(t) = \int_0^\infty \frac{e^{i(t-s)|D|}}{|D|} f(s)ds = e^{it|D|}F, \quad F = \int_0^\infty \frac{e^{-is|D|}}{|D|} f(s)ds$$

and

$$u_b(t) = \int_0^\infty \frac{e^{-i(t-s)|D|}}{|D|} f(s)ds$$

Using that $e^{-is|D|}/|D|$ is a bounded operator from L^2 to $\dot{H}^1$, we obtain that $F \in \dot{H}^1$ with

$$\|F\|_{\dot{H}^1} \lesssim \|f\|_{L^1(\mathbb{R},L^2(\mathbb{R}^3))}.$$

By the Strichartz estimate for the half-wave equation, Proposition III.5.1, we deduce

$$\|u_a\|_{L^4(\mathbb{R},L^{12}(\mathbb{R}^3))} \lesssim \|f\|_{L^1(\mathbb{R},L^2(\mathbb{R}^3))}.$$

Similarly

$$\|u_b\|_{L^4(\mathbb{R},L^{12}(\mathbb{R}^3))} \lesssim \|f\|_{L^1(\mathbb{R},L^2(\mathbb{R}^3))}.$$

Combining, we obtain

$$\|Af\|_{L^4(\mathbb{R},L^{12}(\mathbb{R}^3))} \lesssim \|f\|_{L^1(\mathbb{R},L^2(\mathbb{R}^3))}, \tag{III.6.2}$$

where A is the operator defined by

$$Af(t) = \int_0^\infty \frac{\sin\left((t-s)|D|\right)}{|D|} f(s)ds.$$

Note that Af is analogous to Bf defined above, the only difference between the two being that the integral defining Af is on $[0,\infty)$, whereas the integral defining Bf is on $[0,t[$. An important functional analysis result, due to Michael Christ and Alexander Kiselev [**4**], shows that the boundedness of A implies the boundedness of the operator B. We state this result in a version that was proposed by Christopher Sogge:

LEMMA III.6.1. *Let X and Y be Banach spaces. Let $1 \le p < q \le \infty$. Let K a continuous function from $\mathbb{R}^2$ to the space of bounded linear operators from X to Y. Let*

$$(Af)(t) = \int_{-\infty}^{\infty} K(t,\tau)f(\tau)d\tau,$$

and assume that A is a bounded operator from $L^p(\mathbb{R},X)$ to $L^q(\mathbb{R},Y)$, with operator norm C. Define the operator B by

$$(Bf)(t) = \int_{-\infty}^{t} K(t,\tau)f(\tau)d\tau.$$

Then B extends to a bounded operator from $L^p(\mathbb{R},X)$ to $L^q(\mathbb{R},Y)$, with operator norm $\le \frac{2C\theta^2}{1-\theta}$, where $\theta = 2^{\frac{1}{q}-\frac{1}{p}}$.

Applying Christ and Kiselev Lemma to

$$K(t,\tau) = \mathbb{1}_{\tau>0} \frac{\sin\big((t-\tau)|D|\big)}{|D|} \chi(\varepsilon|D|), \tag{III.6.3}$$

where $\chi \in C_0^\infty(\mathbb{R}^3)$ is equal to 1 close to 0, one obtains

$$\forall \varepsilon > 0, \quad \forall f \in L^1(\mathbb{R}, L^2), \quad \|\chi(\varepsilon D) Bf\|_{L^4 L^{12}} \lesssim \|f\|_{L^1 L^2},$$

where Bf is as in (III.6.1). Letting $\varepsilon \to 0$ we obtain the desired result.

Exercice III.4. Justify this last argument.

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