CHAPTER IV

Cauchy theory for the non-linear equation

In this chapter we will consider the nonlinear wave equation with a power-like nonlinearity

(IV.0.1) $$\partial_t u^2 - \Delta u = \sigma u^p,$$

on $I \times \mathbb{R}^N$, where N is an interval, where the power p is an integer ≥ 2 and σ is nonzero real parameter. Considering the unknown λu instead of u for a suitable choice of $\lambda > 0$, we see that we can assume

$$\sigma \in \{\pm 1\}.$$

We will briefly consider the general case, then restrict to the quintic case $p = 5$ in space dimension 3. We will also comment on the cubic case $p = 3$, in the same space dimension.

IV.1. Scaling invariance. Critical Sobolev space

Let u be a (nonzero) C^2 solution of (IV.0.1) on $(a, b) \times \mathbb{R}^N$, where $a < b$. Let $u_\lambda(t, x) = \lambda^\alpha u(\lambda t, \lambda x)$, where $\lambda > 0$ and $\alpha = \alpha(p, N)$ will be specified later. We have

$$\partial_t^2 u_\lambda - \Delta u_\lambda = \lambda^{\alpha+2-\alpha p} \sigma u_\lambda^p.$$

Thus, if $\alpha = \frac{2}{p-1}$, we see that u_λ is a solution of (IV.0.1) on $\left(\frac{a}{\lambda}, \frac{b}{\lambda}\right) \times \mathbb{R}^N$. We will assume that α has this particular value in the sequel, denoting

$$u_\lambda(t, x) = \lambda^{\frac{2}{p-1}} u(\lambda t, \lambda x).$$

Let

$$\dot{\mathcal{H}}^s = \dot{H}^s(\mathbb{R}^N) \times \dot{H}^{s-1}(\mathbb{R}^N).$$

The critical Sobolev exponent is by definition the unique s such that

$$\|\vec{u}_\lambda(0)\|_{\dot{\mathcal{H}}^{s_c}} = \|\vec{u}(0)\|_{\dot{\mathcal{H}}^{s_c}}.$$

Since by explicit computation

(IV.1.1) $$\|\vec{u}_\lambda(0)\|_{\dot{\mathcal{H}}^s} = \lambda^{\frac{2}{p-1}+s-N/2} \|\vec{u}(0)\|_{\dot{\mathcal{H}}^s}.$$

We see that

$$s_c = \frac{N}{2} - \frac{2}{p-1}.$$

We observe that s_c grows with p, and is always strictly smaller than $N/2$.

Consider a solution u of (IV.0.1) defined on a finite interval $[0, T[$. The corresponding solution u_λ is defined on $[0, T/\lambda[$. Growing λ has the effect of decreasing the time of existence. If $s > s_c$, the $\dot{\mathcal{H}}^s$ norm of $\vec{u}_\lambda(0)$ becomes larger. If $s < s_c$ it becomes smaller. Thus in this case the effect of scaling is to simultaneously decrease the norm of the initial data in $\dot{\mathcal{H}}^s$, $s < s_c$ and shrinking its interval of existence. This is contrary to the intuition that for smaller solutions, the effect of the nonlinearity is weaker, and the solution should behave in a linear way (and in particular has a long time of existence). This leads to an informal conjecture that s_c is a threshold for local well-posedness. It turns out that this conjecture is true for the wave equation: the equation (IV.0.1) is locally well-posed[1] in $\dot{\mathcal{H}}^s$ for $s \geq s_c$, and ill-posed if $\dot{\mathcal{H}}^s$ for $s < s_c$.

In this We will focus on the quintic case $p = 5$ in space dimension $N = 3$:

$$(\partial_t^2 - \Delta)u = \sigma u^5. \tag{W5}$$

In this case the critical Sobolev case is $\dot{\mathcal{H}}^1$, and the equation is called "energy critical". We will also sometimes consider the cubic equation

$$(\partial_t^2 - \Delta)u = \sigma u^3, \tag{W3}$$

in dimension $1 + 3$, for which $s_c = 1/2$. As usual, we will take initial data, say at $t = t_0$:

$$(u, \partial_t u)_{t=t_0} = (u_0, u_1). \tag{ID}$$

In all the sequel, we fix $N = 3$.

IV.2. Definition of solutions

As for the linear wave equation, the notion of classical (C^2) solution is too restrictive for the equation (W5), and we will define the following weaker notion of solution, based on Duhamel's formulation of the equation:

Definition IV.2.1. A *finite energy solution* of (W5), (ID) on an interval I with $t_0 \in I$ is a function $u \in L^5_{\mathrm{loc}}(I, L^{10})$ such that $\forall t \in I$,

$$u(t) = \cos\big((t - t_0)|D|\big)u_0 + \frac{\sin\big((t - t_0)|D|\big)}{|D|}u_1 + \int_{t_0}^{t} \frac{\sin\big((t - s)|D|\big)}{|D|}u^5(s)ds, \tag{IV.2.1}$$

where $(u_0, u_1) \in \dot{\mathcal{H}}^1$.

In the definition, by $u \in L^5_{\mathrm{loc}}(I, L^{10}(\mathbb{R}^3))$, we mean that $u \in L^5(J, L^{10})$ for any compact interval $J \subset I$.

Note that if u is a finite-energy solution in the above sense, one has $u^5 \in L^1_{\mathrm{loc}}(I, L^2(\mathbb{R}^3))$, and thus by Strichartz estimates,

$$\vec{u} \in C^0(I, \dot{\mathcal{H}}^1).$$

Also, by Chapter II, u satisfies the equation (W5) in the sense of distribution on $I \times \mathbb{R}^3$.

The solutions given by the Duhamel formula as in Definition IV.2.1 are called "strong" solutions in [**12**], by opposition to weaker notions of distributional solutions (that do not impose continuity in time) and classical solutions (that are C^2 and satisfy the equation in a classical sense). Note however that this terminology is not universal. For example the solutions of Definition IV.2.1 are called... "weak" solutions in [**10**].

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[1]By "well-posed in X", we mean that there is existence and uniqueness of solutions with initial data in X and a reasonable stability theory. We will not give a more rigorous definition of local well-posedness. See e.g. Definition 3.4, Remark 3.5 of T. Tao's book [**12**]

We refer to Section 3.2 of [**12**] "What is a solution?", for a discussion on different types of solutions.

In the sequel, by "solution to (W5)" we will always mean (unless specified otherwise) a solution in the sense of Definition IV.2.1.

Exercice IV.1. Check that the definition of finite energy solutions above does not depend on the choice of the initial time. In other words, if u is a solution of (W5) on I and $t_1 \in I$, then for all $t \in I$,

$$u(t) = \cos\big((t-t_1)|D|\big)u(t_1) + \frac{\sin\big((t-t_1)|D|\big)}{|D|}\partial_t u(t_1) + \int_{t_1}^{t} \frac{\sin\big((t-s)|D|\big)}{|D|} u^5(s)ds.$$

IV.3. Existence and uniqueness

3.a. A local statement. We introduce the following notations: $\vec{u}_0 = (u_0, u_1)$

$$S_L(t)\vec{u}_0 = \cos(t|D|)\, u_0 + \frac{\sin(t|D|)}{|D|}u_1, \quad \vec{S}_L(t)\vec{u}_0 = \left(S_L(t)\vec{u}_0, \partial_t S_L(t)\vec{u}_0\right).$$

We start with the following local statement:

Theorem IV.3.1. *There exists $\delta_0 > 0$ with the following property. Let I be an interval with $t_0 \in I$. Let $\vec{u}_0 \in \dot{\mathcal{H}}^1$. Assume*

$$\|S_L(\cdot - t_0)\vec{u}_0\|_{L^5(I,L^{10})} = \delta \leq \delta_0. \tag{IV.3.1}$$

Then there exists is a unique solution u of (W5), (ID) *on I. Furthermore*

$$\sup_{t\in I} \left\|\vec{u}(t) - \vec{S}_L(t-t_0)\vec{u}_0\right\|_{\dot{\mathcal{H}}^1} + \|u - S_L(\cdot - t_0)\vec{u}_0\|_{L^5(I,L^{10})} \lesssim \delta^5. \tag{IV.3.2}$$

In the Theorem, $S_L(\cdot - t_0)\vec{u}_0$ denotes the map $t \mapsto S_L(t-t_0)\vec{u}_0$.

Theorem IV.3.1 has two important consequences:

Local well-posedness: Note that $(5, 10)$ is an admissible couple in dimension 3 (it satisfies (III.2.2)). By Theorem III.2.1, if $\vec{u}_0 \in \dot{\mathcal{H}}^1$, then $S_L(\cdot)\vec{u}_0 \in L^5(\mathbb{R}, L^{10}(\mathbb{R}^3))$. Thus if $T > 0$ is small enough, then

$$\|\vec{u}_0\|_{L^5([-T,+T],L^{10})} \leq \delta_0,$$

and Theorem IV.3.1 implies that there exists a solution to (W5), (ID) on $[-T, +T]$.

Small data global well-posedness: If $\vec{u}_0 \in \dot{\mathcal{H}}^1$ and $\|u_0\|_{\dot{H}^1} \leq \delta_0/C_S$, where C_S is the constant in the Strichartz inequality (III.2.3) with $p = 5$, $q = 10$, then $\|S_L(\cdot)\vec{u}_0\|_{L^5(\mathbb{R},L^{10})} \leq \delta_0$, and one can use Theorem IV.3.1 with $I = \mathbb{R}$. This shows that the corresponding solution u is globally defined, and that $u \in L^5(\mathbb{R}, L^{10})$.

Proof. Assume without generality that $t_0 = 0$. We use the Banach fixed point theorem, proving that the operator A, defined by

$$Av\,(t) = S_L(t)\vec{u}_0 + Bv\,(t), \quad Bv\,(t) = \int_0^t \frac{\sin\big((t-s)|D|\big)}{|D|} v^5(s)ds, \tag{IV.3.3}$$

is a contraction on X defined by

$$X = \left\{v \in L^5(I, L^{10}),\ \|v\|_{L^5(I,L^{10})} \leq 2\delta_0\right\}.$$

We first prove that A maps X into X. Indeed, If $v \in X$, then by Theorem III.2.1 (see Remark III.2.2),

$$\|Bv(t)\|_{L^5(I,L^{10})} \leq C_S \left\|v^5\right\|_{L^1(I,L^2)} \leq C_S \|v\|^5_{L^5(I,L^{10})} \leq C_S \delta_0^5 \leq \delta_0,$$

assuming $\delta_0 \leq C_S^{-1/4}$. Thus $Av \in X$.

We next prove that A is a contraction on X. Let $v, w \in X$. Using $w^5 - v^5 = (w-v)(w^4 + w^3 v + w^2 v^2 + w v^3 + v^4)$ and Young's inequality $ab \leq a^p/p + b^q/q$, $1/p + 1/q = 1$, one obtains

$$|v^5 - w^5| \leq \frac{5}{2} |v - w| (v^4 + w^4).$$

Combining with Hölder's inequality, we obtain

$$\text{(IV.3.4)} \quad \left\|v^5 - w^5\right\|_{L^1(I,L^2)} \leq \frac{5}{2} \|v - w\|_{L^5(I,L^{10})} \left(\|v\|^4_{L^5(I,L^{10})} + \|w\|^4_{L^5(I,L^{10})} \right).$$

By Strichartz estimates

$$\begin{aligned} \|Av - Aw\|_{L^5(I,L^{10})} &= \|Bv - Bw\|_{L^5(I,L^{10})} \\ &\leq C_S \left\|v^5 - w^5\right\|_{L^1(I,L^2)} \leq 5 C_S \|v - w\|_{L^5(I,L^{10})} \delta_0^4. \end{aligned}$$

If δ_0 is small enough ($\delta_0 = (10 C_S)^{-1/4}$ works), one has

$$\|Av - Aw\|_{L^5(I,L^{10})} \leq \frac{1}{2} \|v - w\|_{L^5(I,L^{10})}.$$

This shows that A is a contraction on X.

Let u be the only fixed point of A in X. Since $u = Au$ and $u \in L^5(I, L^{10})$ we see that u is a solution of (W5) on I.[2] Using

$$u - S_L(\cdot)\vec{u}_0 = Bu,$$

and $\|Bu\|_{L^5(I,L^{10})} \leq \delta$, and Strichartz inequality, we obtain (IV.3.2). It remains to prove the uniqueness statement. From the contraction argument, we see that u is the unique solution of (W5) such that $\|u\|_{L^5(I,L^{10})} \leq \delta_0$. We prove a stronger statement, Lemma IV.3.2 below, that will conclude the proof. □

Lemma IV.3.2. *Let u, v be two solutions of* (W5) *on an interval I with $t_0 \in I$. Assume $\vec{u}(t_0) = \vec{v}(t_0)$. Then $u = v$.*

Proof. Assume again $t_0 = 0$ to simplify notations. Let $\delta_0 > 0$ be as in Theorem IV.3.1. We let $K = [a, b]$ be a compact subinterval of I such that $t_0 \in K$. We will prove that $u(t) = v(t)$ for $t \in K$. Since K is compact, we have by Definition IV.2.1,

$$u \in L^5(K, L^{10}), \quad v \in L^5(K, L^{10}).$$

We can thus divide K into p subintervals $[\tau_j, \tau_{j+1}]$, $0 \leq j \leq p-1$, with $\tau_0 < \tau_1 < \ldots < \tau_p$, such that

$$\forall j \in \{0, \ldots, J-1\}, \quad \max \left(\|u\|_{L^5([\tau_j,\tau_{j+1}],L^{10})}, \|v\|_{L^5([\tau_j,\tau_{j+1}],L^{10})} \right) \leq \delta_0.$$

Let j_0 be an index such that $0 \in [\tau_{j_0}, \tau_{j_0+1}]$. By the proof of Theorem III.2.1, with $I = [\tau_{j_0}, \tau_{j_0+1}]$, noting that u and v are in X, we obtain $u(t) = v(t)$ for $t \in [\tau_{j_0}, \tau_{j_0+1}]$. This implies

$$\vec{u}(\tau_{j_0}) = \vec{v}(\tau_{j_0}) \text{ and } \vec{u}(\tau_{j_0+1}) = \vec{v}(\tau_{j_0+1}).$$

[2] Recall that "solution" is to be taken in the sense of Definition IV.2.1.

We can then iterate the preceding arguments and the intervals $[\tau_j, \tau_{j+1}]$, $j = j_0+1$, $j = j_0+2$ until $j = J-1$, and $j = j_0-1$, $j = j_0-2$ until $j = 0$ to obtain that $u(t) = v(t)$ for $t \in K$, concluding the proof. □

3.b. Maximal solution. Using the above local existence theorem, we can now glue the solutions above to construct a maximal solution of (W5).

Corollary IV.3.3. *Let $\vec{u}_0 \in \dot{\mathcal{H}}^1$ and $t_0 \in \mathbb{R}$. Then there is a unique maximal solution of* (W5), (ID). *Denoting by $I_{\max} = (T_-, T_+)$ its interval of existence, we have the following* blow-up criteria*:*

$$T_+ < \infty \Longrightarrow u \notin L^5([t_0, T_+[, L^{10}), \quad T_- > -\infty \Longrightarrow u \notin L^5(]T_-, t_0], L^{10}). \tag{IV.3.5}$$

The phrase "maximal solution" in the theorem means that if v is another solution of (W5), (ID) defined on an interval I with $t_0 \in I$, then $I \subset I_{\max}$ and $u(t) = v(t)$ for all $t \in I$.

Proof. Let $\mathcal{J}$ be the set of all open intervals I such that $t_0 \in I$, and there exists a solution v of (IV.0.1), (ID) on I. Let

$$I_{\max} = \bigcup_{I \in \mathcal{H}} I.$$

By Theorem IV.3.1, $\mathcal{J}$ is nonempty. Thus $I_{\max}$ is an open interval containing t_0. If $t \in I_{\max}$, there exists an interval I and a solution v of (W5), (ID) on I. By the uniqueness Lemma IV.3.2, the value $v(t)$ does not depend on the choice of I. We denote by $u(t)$ this common value. Let K be a compact subinterval of $I_{\max}$. We next prove:

$$u \in L^5(K, L^{10}). \tag{IV.3.6}$$

Indeed, for all $t \in K$, there exist an open interval $I \in \mathcal{J}$ such that $t \in I$ and u is a solution of (W5) on I. This implies in particular that $u \in L^5([t-\varepsilon, t+\varepsilon], L^{10})$ if $\varepsilon = \varepsilon(t)$ is small enough. Using the compactness of K, we can cover K by a finite numbers of interval $]t-\varepsilon(t), t+\varepsilon(t)[$, and thus we obtain (IV.3.6).

If $t \in I_{\max}$, by the definition of $I_{\max}$ and the uniqueness Lemma IV.3.2, we have that

$$u(t) = S_L(t)\vec{u}_0 + \int_0^t \frac{\sin\big((t-s)|D|\big)}{|D|} u^5(s)ds,$$

which concludes the proof that u is a solution of (W5), (ID) on $I_{\max}$. The maximality of u is a direct consequence of the definition of $I_{\max}$ and Lemma IV.3.2. □

Let us mention that it is not possible to improve the blow-up criterion to

$$T_+ < \infty \Longrightarrow \limsup_{t\to\infty} \|\vec{u}(t)\|_{\dot{\mathcal{H}}^1} = +\infty.$$

Indeed, it was proved by Krieger, Schlag and Tataru [**8**] that there exist solutions of (W5) with $\sigma = 1$, with finite time of existence T_+ and such that

$$\limsup_{t\to T_+} \|\vec{u}(t)\|_{\dot{\mathcal{H}}^1} < \infty.$$

Exercice IV.2. Consider the cubic nonlinear wave equation (W3), (ID) with initial data (u_0, u_1) in the critical space $\dot{\mathcal{H}}^{1/2}$, in space dimension 3. Define a concept of "solution" for this equation analogous to the one of Definition IV.2.1. Prove the analogs of Theorem IV.3.1 and Corollary IV.3.3. *Hint*: use the $L^4(I\times\mathbb{R}^3)$

norm instead of the $L^5(I, L^{10}(\mathbb{R}^3))$ norm, and the Strichartz inequality of Theorem III.2.3.

IV.4. Finite speed of Propagation

Remark IV.4.1. The proof of Theorem III.2.1 implies that if I is an interval, $t_0 \in I$, and u is a solution of (W5), (ID) on I such that $\|u\|_{L^5(I,L^{10})} \leq \delta_0/2$, then u is the limit, in $L^5(I, L^{10})$, of the sequence u^n defined by $u^0 = 0$, $u^n = Au^n$, where A is the operator defined in the proof. Indeed, by Strichartz estimates,

$$\|S_L(\cdot - t_0)\vec{u}_0\|_{L^5(I,L^{10})} \leq \|u\|_{L^5(I,L^{10})} + C_S\|u\|^5_{L^5(I,L^{10})} \leq \frac{\delta_0}{2} + C_S\delta_0^5/32 \leq \delta_0.$$

Thus $\vec{u}_0$ satisfies the assumption of Theorem III.2.1 and the conclusion follows from the fact that u is a fixed point of the contraction A.

This remark will be used at least twice in the rest of this course to obtain properties of the solution u. The first occurrence of this is in the proof of finite speed of propagation for the nonlinear equation:

Theorem IV.4.2. *Let $(t_0, x_0) \in \mathbb{R}^{1+3}$, $t_1 > t_0$, $R > 0$. We denote $\Gamma = \Big\{(t,x) \in \mathbb{R}\times\mathbb{R}^N \ : \ t_0 \leq t \leq t_1,\ |x - x_0| \leq R - |t - t_0|\Big\}$ Let u and v be two solutions of* (W5) *on $[t_0, t_1]$. We suppose $(u, \partial_t u)(t_0, x) = (v, \partial_t v)(t_0, x)$ for all $x \in B_R(x_0)$. Then $u(t, x) = v(t, x)$ for almost all $(t, x) \in \Gamma$.*

Proof. Dividing the interval $[t_0, t_1]$ into subintervals $[\tau_j, \tau_{j+1}]$, $0 \leq j \leq J-1$, $t_0 = \tau_0 < \tau_1 < \ldots < \tau_J$, such that

$$\forall j \in \{0, \ldots, J-1\}, \quad \max\left(\|u\|_{L^5([\tau_j,\tau_{j+1}],L^{10}(\mathbb{R}^3))}, \|v\|_{L^5([\tau_j,\tau_{j+1}],L^{10}(\mathbb{R}^3))}\right) \leq \delta_0/2,$$

we see that it is sufficient to prove the theorem with the additional assumption

$$\max\left(\|u\|_{L^5([t_0,t_1],L^{10}(\mathbb{R}^3))}, \|v\|_{L^5([t_0,t_1],L^{10}(\mathbb{R}^3))}\right) \leq \delta_0/2.$$

Thus $u = \lim_{n\to\infty} u^n$, $v = \lim_{n\to\infty} v^n$ in $L^5(I, L^{10})$, $I = [t_0, t_1]$, where u^n and v^n are defined by

$$u^0 = v^0 = 0, \quad u^{n+1} = Au_n, \quad v^{n+1} = \widetilde{A}v^n,$$

where A is as in the proof of Theorem IV.3.1 (see (IV.3.3)), and $\widetilde{A}$ is the analog of A for the initial data of v:

$$\widetilde{A}w(t) = S_L(t)\vec{v}(0) + \int_0^t \frac{\sin\big((t-s)|D|\big)}{|D|} w^5(s)ds.$$

(As usual, we assume $t_0 = 0$ to simplify notations).

We prove by induction on n that $u^n(t, x) = v^n(t, x)$ for almost every $(t, x) \in \Gamma$. This is true for $n = 0$, since $u^0 = v^0 = 0$.

Next, we assume that $u^n(t, x) = v^n(t, x)$ for almost every $(t, x) \in \Gamma$. We have

$$u^{n+1}(t) - v^{n+1}(t) = S_L(t)(\vec{u}(0) - \vec{v}(0)) + \int_0^t \frac{(\sin(t-s)|D|)}{|D|}\left(u^n(s) - v^n(s)\right)ds.$$

By finite speed of propagation for the linear wave equation and the assumption that $\vec{u}^0(x) = \vec{v}^0(x)$ for $|x - x_0| < R$, we obtain that $S_L(t)(\vec{u}(0) - \vec{v}(0)) = 0$ for almost all $(t, x) \in \Gamma$. On the other hand, if $s \in [0, t]$, the inductive hypothesis implies that

$u^n(s,x) = v^n(s,x)$ for $|x-x_0| < R-s$. Combining with finite speed of propagation, we see that

$$\frac{(\sin(t-s)|D|)}{|D|}\left(u^n(s) - v^n(s)\right) = 0$$

for almost every (t,x) with $|x-x_0| < R-s-(t-s) = R-t$, i.e. for almost every $(t,x) \in \Gamma$.

Thus $u^n = v^n$ almost everywhere on Γ. Passing to the limit, we obtain $u^n = v^n$ on Γ. □

IV.5. Stability

We now prove that the flow of the equation (W5) is continuous in $\dot{\mathcal{H}}^1$, i.e. that if the initial data of two solutions u and v are close in this space, then $\vec{u}(t)$ and $\vec{v}(t)$ are close for all times t in their domain of existence. In the statement, we must take into account the fact that the solutions u and v might not be global.

THEOREM IV.5.1. *Let $t_0 \in \mathbb{R}$, $\vec{u}_0 = (u_0, u_1) \in \dot{\mathcal{H}}^1$. Let u be the solution of* (W5), (ID). *Let I be a compact interval such that $t_0 \in I \subset I_{\max}(\vec{u}_0)$. Let $(\vec{u}_0^k)_k$ be a sequence in $\dot{\mathcal{H}}^1$ such that $\lim_n \vec{u}_0^k = \vec{u}_0$ in $\dot{\mathcal{H}}^1$. Let u^k be the corresponding solutions. Then for large k, $I \subset I_{\max}(\vec{u}_0^k)$, and*

$$\lim_{k\to\infty}\left(\sup_{t\in I}\left\|\vec{u}^k(t) - \vec{v}^k(t)\right\|_{\dot{\mathcal{H}}^1} + \|u^k - v^k\|_{L^5(I,L^{10})}\right) = 0.$$

PROOF. We will consider $T > 0$ such that

$$\|u\|_{L^5([0,T],L^{10})} \leq \delta_0 \tag{IV.5.1}$$

(where δ_0 is a small parameter), and prove that $T^+(u^k) > T$ for large k and

$$\|u - u^k\|_{L^5([0,T],L^{10})} \sup_{0\leq t\leq T}\left\|\vec{u}(t) - \vec{u}^k(t)\right\|_{\dot{\mathcal{H}}^1} \underset{k\to\infty}{\longrightarrow} 0. \tag{IV.5.2}$$

The conclusion of the theorem will then follow by iteration, dividing as above the interval I into subintervals where the L^5L^{10} norm of u is small.

We have

$$u(t) - u^k(t) = S_L(t)(\vec{u}_0 - \vec{u}_0^k) + \int_0^t \frac{\sin\left((t-s)|D|\right)}{|D|}\left(u^5(s) - (u^k)^5(s)\right) ds. \tag{IV.5.3}$$

As in (IV.3.4), we have

$$\left\|u^5 - (u^k)^5\right\|_{L^1([0,t],L^2)} \leq \frac{5}{2}\left\|u - u^k\right\|_{L^5([0,t],L^{10})}\left(\|u\|^4_{L^5([0,t],L^{10})} + \left\|u^k\right\|^4_{L^5([0,t],L^{10})}\right).$$

Using the triangle inequality, and (IV.5.1), we deduce

$$\left\|u^5 - (u^k)^5\right\|_{L^1([0,t],L^2)} \leq \frac{5}{2}\left\|u - u^k\right\|_{L^5([0,t],L^{10})}\left(2\delta_0^4 + \left\|u - u^k\right\|^4_{L^5([0,t],L^{10})}\right).$$

Thus, by (IV.5.3) and Strichartz estimate, we have that for all $t \in [0,T]$

$$a_k(t) \leq C\left(\varepsilon_k + \delta_0^4 a_k(t) + a_k(t)^5\right),$$

where $a_k(t) = \left\|u - u^k\right\|_{L^5([0,t],L^{10})}$, $\varepsilon_k = \left\|\vec{u}_0 - \vec{u}_0^k\right\|_{\dot{\mathcal{H}}^1} \underset{k\to\infty}{\longrightarrow} 0$, and C is a constant. Taking δ_0 small (so that $C\delta_0^4 \leq 1/2$, we deduce

$$a_k(t) \leq 2C\varepsilon_k + 2Ca_k(t)^5. \tag{IV.5.4}$$

We temporarily fix k, large enough so that $2C(4C\varepsilon_k)^5 \leq C\varepsilon_k$, and prove

$$\forall t \in [0,T], \quad a_k(t) \leq 3C\varepsilon_k. \tag{IV.5.5}$$

Indeed, (IV.5.5) is true for small $t > 0$, since a is continuous and $a(0) = 0$. If (IV.5.5) does not hold, using again the continuity of a, we see that there exists a $t \in [0,T]$ such that $3C\varepsilon_k < a_k(t) \leq 4C\varepsilon_k$. By (IV.5.4), and the smallness of ε_k we see that $a_k(t) \leq 3C\varepsilon_k$. This is a contradiction, concluding the proof of (IV.5.5). This type of reasoning is called a *bootstrap argument.* By (IV.5.5),

$$\lim_{k\to\infty} a_k(T) = 0$$

Using (IV.5.3) and Strichartz estimate again, we deduce

$$\sup_{t\in[0,T]} \left\|\vec{u}(t) - u^k(t)\right\|_{\dot{\mathcal{H}}^1} \underset{k\to\infty}{\longrightarrow} 0,$$

which concludes the proof. □

IV.6. Persistence of regularity, conservation of the energy

The energy of a solution u of (W5) is defined as

$$E(\vec{u}(t)) = \frac{1}{2}\int (\partial_t u(t,x))^2 dx + \frac{1}{2}\int |\nabla u(t,x)|^2 dx - \frac{\sigma}{6}\int (u(t,x))^6 dx, \tag{IV.6.1}$$

where all integrals are taken over $\mathbb{R}^3$. Multiplying the equation (W5) by $\partial_t u(t,x)$, integrating on $\mathbb{R}^3$ and integrating by part, we would obtain that the derivative of the energy is 0, and thus that it is independent of time. However this computation is purely formal. To make it rigorous, we need to work on more regular solutions. The key ingredient for this is the *persistence of regularity* property:

Theorem IV.6.1. *Let $\vec{u}_0 = (u_0, u_1) \in \dot{\mathcal{H}}^1$, u be the solution of* (W5), (ID) *given by Corollary IV.3.3, and $I_{\max}$ its maximal interval of existence. Let $\ell \geq 2$ be an integer. Assume $\vec{u}_0 \in \dot{\mathcal{H}}^\ell$. Then*

$$\vec{u} \in C^0\left(I_{\max}, \dot{\mathcal{H}}^\ell\right), \quad \partial_t^2 u \in C^0\left(I_{\max}, \dot{H}^{\ell-2}\right) \tag{IV.6.2}$$

In particular, if $\ell \geq 4$, $u \in C^2\left(I_{\max} \times \mathbb{R}^3\right)$

Proof. We prove the result for $\ell = 2$. The proof for $\ell \geq 3$ is very close and left to the reader. As usual, we assume $t_0 = 0$. We note that the property of $\partial_t^2 u$ in (IV.6.2) follows from $u \in C^0(I_{\max}, \dot{H}^\ell)$, the equation $\partial_t^2 u = \Delta u + \sigma u^5$ and Sobolev embedding. We ar thus left to prove $\vec{u} \in C^0(I_{\max}\dot{\mathcal{H}}^\ell$.

Step 1.

We first consider a small $T > 0$. By the proof of Theorem IV.3.1, the restriction of u to $[-T,+T]$ is the limit, in $L^5([-T,+T), L^{10})$, of the sequence u^n defined as above by $u^0 = 0$, $u^{n+1} = Au^n$, where A is defined by (IV.3.3). Let $j \in \{1,2,3\}$. We have

$$\partial_{x_j}(u^{n+1}) = S_L(t)\vec{u}_0 + 5\int_0^t \frac{\sin\left((t-s)|D|\right)}{|D|}\left(u^n(s)\right)^4 \partial_{x_j} u^n(s) ds, \tag{IV.6.3}$$

where ∂_{x_j} is the distributional derivative with respect to x_j, and we have used the formula $\partial_{x_j}(v^5) = 5v^4\partial_{x_j}v$, which is valid for $v \in \dot{H}^2$ (this can be checked easily using that the functions in $\dot{H}^2$ are continuous[3].

We prove by induction on n that $(u^n, \partial_t u^n) \in C^0([-T,+T], \dot{H}^2)$ with

$$\text{(IV.6.4)} \qquad \sup_{-T\le t\le T} \|(u^n, \partial_t u^n)\|_{\dot{\mathcal{H}}^2} \le 2M, \quad \sup_{|\alpha|\le 1} \|\partial_x^\alpha u^n\|_{L^5([-T,+T],L^{10})} \le 2\delta_0,$$

where $M = \|\vec{u}_0\|_{\dot{\mathcal{H}}^2}$, and we have chosen T small enough, so that

$$\text{(IV.6.5)} \qquad \sup_{1\le j\le 3} \|S_L(\cdot)\partial_x^\alpha \vec{u}_0\|_{L^5([-T,+T],L^{10})} \le \delta_0.$$

The case $n = 0$ is trivial since $u^0 = 0$.

Next we assume that $u^n \in C^0([-T,+T], \dot{H}^2$ and satisfies (IV.6.4). Then by Strichartz estimates, the definition of u^{n+1}, the inductive hypothesis, (IV.6.5) and the smallness of δ_0:

$$\text{(IV.6.6)} \qquad \sup_{|\alpha|\le 1} \|\partial_x^\alpha u^n\|_{L^5([-T,+T],L^{10})} \le \delta_0 + C\delta_0^5 \le 2\delta_0.$$

The same argument, together with the definition of M yields that $(u^n(t), \partial_t u^n(t)) \in C^0([-T,+T], \dot{\mathcal{H}}^2)$ and

$$\text{(IV.6.7)} \qquad \sup_{-T\le t\le T} \|(u^n(t), \partial_t u^n(t))\|_{\dot{\mathcal{H}}^2} \le M + C\delta_0^5 \le 2M.$$

This shows that (IV.6.4) holds for all n as announced.

Step 2. Fixing $j \in \{1,2,3\}$ we will prove that $(\partial_{x_j} u^n)_n$ is a Cauchy sequence in $C^0([-T,+T], \dot{\mathcal{H}}^1)$ and $L^5([-T,+T], L^{10})$. Indeed,

$$\begin{aligned} &\left|(u^n)^4\partial_{x_j}u^n - (u^{n-1})^4\partial_{x_j}u^{n-1}\right| \\ &\qquad = \left|\left((u^n)^4 - (u^{n-1})^4\right)\partial_{x_j}u^n + \left(\partial_{x_j}u^{n-1} - \partial_{x_j}u^n\right)(u^n)^4\right| \\ &\lesssim \left|u^n - u^{n-1}\right|\left|\partial_{x_j}u^n\right|\left(|u^n|^3 + |u^{n-1}|^3\right) + \left|\partial_{x_j}u^{n-1} - \partial_{x_j}u^n\right|\left((u^n)^4 + (u^{n-1})^4\right). \end{aligned}$$

Which yields, by (IV.6.6)

$$\begin{aligned} &\left\|(u^n)^4\partial_{x_j}u^n - (u^{n-1})^4\partial_{x_j}u^{n-1}\right\|_{L^1([-T,+T],L^2)} \\ &\qquad \lesssim \delta_0^4\left\|\partial_{x_j}(u^n - u^{n-1})\right\|_{L^5([-T,+T],L^{10})} + \delta_0^4\left\|u^n - u^{n-1}\right\|_{L^5([-T,+T],L^{10})} \end{aligned}$$

By Strichartz estimates and the definition of u^n, letting

$$\begin{aligned} c_n &= \sup_{|\alpha\le 1} \left\|\partial_x^\alpha(u^n - u^{n-1})\right\|_{L^5([-T,+T],L^{10})} \\ d_n &= \sup_{-T\le t\le T} \|(u^n(t) - u^{n-1}(t), \partial_t u^n(t) - \partial_t u^{n-1}(t))\|_{\dot{\mathcal{H}}^2}, \end{aligned}$$

we obtain, for $n \ge 1$,

$$c_{n+1} \le \frac{1}{99}c_n, \quad d_{n+1} \le c_n.$$

[3]The reader which is not familiar with the theory of distribution can assume in all this proof that $\vec{u}^0 \in C^3 \times C^2$: all functions u^n will then by also of class C^2, and differentiation can be understood in the classical sense.

This proves that $(u^n)_n$ is a Cauchy sequence, and thus has a limit, in $L^5([-T,+T],L^{10})$ and in $C^0([-T,+T],\dot{H}^2)$, and similarly that $(\partial_t u^n)_n$ has a limit in $C^0([-T,+T],\dot{H}^1)$. By uniqueness of limits (for example in L^1_{loc}), we obtain

$$\vec{u} \in C^0([-T,+T],\dot{\mathcal{H}}^2), \quad \forall j, \quad u \in L^5(\mathbb{R},L^{10}).$$

Step 3. Maximal interval of existence. We next consider

$$\tau_+ = \sup\left\{\tau < T_+, \quad \sup_{|\alpha|\leq 1} \|\partial_{x_j} u\|_{L^5([0,\tau],L^{10})} < \infty\right\},$$

where T_+ is the maximal time of existence of u as a $\dot{\mathcal{H}}^1$ solution, defined in Corollary IV.3.3. In this step we prove that $\tau_+ = T_+$.

Assume that $\tau_+ < T_+$. Thus $u \in L^5([0,\tau_+],L^{10})$. We let $\tau_0 \in [0,\tau_+[$ such that

$$\|u\|_{L^5([\tau_0,\tau_+],L^{10})} = \delta \ll 1.$$

Using Strichartz estimates and the formula

$$\partial_{x_j} u = S_L(t)\vec{u}_0 + 5\int_0^t \frac{\sin\big((t-s)|D|\big)}{|D|} (u(s))^4\, \partial_{x_j} u(s)ds,$$

we see that $\vec{u} \in C^0([0,\tau],\dot{\mathcal{H}}^2)$ for all $\tau < \tau_+$. As a consequence of Strichartz estimates, we also obtain, for all $\tau < \tau_+$,

$$\forall j, \quad \big\|\partial_{x_j} u\big\|_{L^5([\tau_0,\tau],L^{10})} \lesssim \|S_L(\cdot - \tau_0)\vec{u}(\tau_0)\|_{L^5([\tau_0,\tau],L^{10})} + C\delta^4 \|\partial_{x_j} u\|_{L^5([\tau_0,\tau],L^{10})}.$$

Hence for all $\tau < \tau_+$,

$$\forall j, \quad \big\|\partial_{x_j} u\big\|_{L^5([\tau_0,\tau],L^{10})} \lesssim \|S_L(\cdot - \tau_0)\vec{u}(\tau_0)\|_{L^5([\tau_0,\tau_+],L^{10})},$$

This shows that $\partial_{x_j} u \in L^5([0,\tau_+],L^{10})$. Using Strichartz estimates again, we see that $\vec{u} \in C^0([0,\tau_+],\dot{\mathcal{H}}^2)$. Thus $\vec{u}(\tau_+) \in \dot{\mathcal{H}}^2$, which is a contradiction with the definition of τ_+, since that by Steps 1 and 2, $\vec{u}$ is continuous, with values in $\dot{\mathcal{H}}^2$, close to τ_+.

This concludes the proof for $\ell = 2$. The proof for $\ell \geq 3$ is mostly identical, considering all $\partial_x^\alpha u$ with $|\alpha| \leq \ell - 1$ instead of $|\alpha| \leq 1$. □

EXERCICE IV.3. Prove that if $T_+ < \infty$, then

$$\lim_{t\to T_+} \|\vec{u}(t)\|_{\dot{\mathcal{H}}^2} = +\infty.$$

COROLLARY IV.6.2. *Let u be a solution with initial data $(u_0,u_1) \in (C_0^\infty(\mathbb{R}^3))^2$. Then the corresponding solution u of* (W5), (ID) *is in $C^\infty(I_{\max}\times\mathbb{R}^3)$, where $I_{\max} = I_{\max}(\vec{u}_0)$ is the maximal interval of existence of u.*

PROOF. The corollary follows immediately from Theorem IV.6.1, using that

$$C_0^\infty(\mathbb{R}^3) \subset \bigcap_{s\geq 1} \dot{\mathcal{H}}^s \subset C^\infty. \tag{IV.6.8}$$

□

EXERCICE IV.4. Prove (IV.6.8). *Hint*: use the Fourier representation of u:

$$u(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{ix\cdot\xi}\hat{u}(\xi).$$

We are now in position to prove rigorously the conservation of the energy:

THEOREM IV.6.3. *Let E be defined by* (IV.6.1). *Let $\vec{u}$ be a solution of* (W5). *Then the energy $E(\vec{u}(t))$ is independent of $t \in I_{\max}(u)$.*

PROOF. Let $t_0, t_1 \in I_{\max}(u)$. Let $\vec{u}_0^n = (u_0^n, u_1^n) \in \left(C_0^\infty(\mathbb{R}^3)\right)^2$ such that

$$\lim_{n\to\infty} \|\vec{u}_0^n - \vec{u}(t_0)\|_{\dot{H}^1} = 0. \tag{IV.6.9}$$

Let u^n be the solution of (W5) with initial data $u^n(0) = u_0^n$, $\partial_t u^n(0) = u_1^n$. By the stability theorem IV.5.1, $[t_0, t_1]$ is included in the maximal interval of existence of u^n for large n.

By Corollary IV.6.2, $u^n \in C^\infty([t_0, t_1] \times \mathbb{R}^3)$. Since it satisfies (IV.0.1) in the sense of distribution, it must also satisfy this equation in the classical sense. By finite speed of propagation $u^n(t)$ is a compactly supported function (in space) for all $t \in [t_0, t_1]$. We have

$$\int \partial_t^2 u^n \partial_t u^n - \int \Delta u^n \partial_t u^n - \sigma \int (u^n)^5 \partial_t u^n = 0$$

Since $\int \Delta u^n \partial_t u^n = \int \sum_{j=1,2,3} \partial_{x_j} u^n \partial_t \partial_{x_j} u^n$, we deduce

$$\frac{d}{dt} E(\vec{u}^n(t)) = 0, \quad t_0 \le t \le t_1.$$

Thus $E(\vec{u}^n(t_0)) = E(\vec{u}^n(t_1))$. Passing to the limit $n \to \infty$ and using Theorem IV.5.1, we deduce

$$E(\vec{u}(t_0)) = E(\vec{u}(t_1)),$$

concluding the proof. We have used that by the Sobolev embedding $\dot{H}^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$, the convergence in $\dot{H}^1$ implies the convergence in L^6. □

In the case $\sigma = -1$, all the terms in the definitions of the energy are positive, and we have

$$\vec{u}(t) \le 2E(\vec{u}(t)).$$

This implies that the $\dot{H}^1$ norm of any solution u of (W5) is bounded on its maximal interval of existence. This is not sufficient to ensure global existence. We will see however that in this case, all solutions are indeed global.

DEFINITION IV.6.4. The equation (W5) or the corresponding nonlinearity is called *defocusing* (or repulsive) when $\sigma = -1$ and *focusing* when $\sigma = 1$.

Let us mention that we can also construct classical solutions of (W5) (or of any equation of the form (IV.0.1) with $p \in \mathbb{N}$, $p \ge 2$, in space dimension 3), without Strichartz estimates, using the representation formulas of Chapter 1 and a fixed point argument. These solutions coincide with the finite energy solutions of Definition IV.2.1 when $\vec{u}_0 \in C_0^3(\mathbb{R}^3) \times C_0^2(\mathbb{R}^3)$ for example. This is an alternative approach to obtain Corollary IV.6.2. We refer to [**10**, Section I.5] for the details.

One can also prove persistence of regularity of the cubic wave equation:

THEOREM IV.6.5. *Let u be the solution of* (W3), (ID) *with initial data $\vec{u}_0 \in \dot{\mathcal{H}}^{1/2} \cap \dot{\mathcal{H}}^k$, for $k \ge 1$ such that $2k$ is an integer*[4]. *Then $\vec{u} \in C^0\left(I_{\max}, \dot{\mathcal{H}}^k\right)$, where $I_{\max}$ is the maximal interval of existence of u. Furthermore the energy of u:*

$$\frac{1}{2}\int (\partial_t u(t,x))^2 dx + \frac{1}{2}\int |\nabla u(t,x)|^2 dx - \frac{\sigma}{4}\int |u(t,x)|^4 dx$$

[4]See Exercise IV.2

is conserved.

Exercice IV.5. Prove Theorem IV.6.5. *Hint*: for the case $k = 1$, one can use the Hölder-type inequality

$$\|u^3\|_{L^1L^2} \leq \|u\|_{L^4L^4} \|u\|^2_{L^{8/3}L^8}$$

and the fact that $(8,8)$ is a $\dot{\mathcal{H}}^1$ Strichartz admissible couple.'

IV.7. Blow-up in finite time

In the focusing case $\sigma = 1$, there exists solutions blowing-up in finite time:

Theorem IV.7.1. *Let $T > 0$. There exists a solution u of* (W5)*, with C^∞, compactly supported initial data $\vec{u}_0$ at $t = 0$, such that $T_+(\vec{u}_0) = T$.*

Proof. By scaling invariance, it is sufficient to construct one solution of (W5) blowing-up in finite time, with compactly supported, smooth initial data.

Let Y be a solution of the ODE $Y'' = Y^5$ defined on $[0,1[$, and blowing-up at $t = 1$. For example $Y(t) = c(1-t)^{-1/2}$, where $\frac{3}{4} = c^4$. Note that Y is a solution of (W5) (in the classical sense).

Let $\varphi \in C_0^\infty(\mathbb{R}^3)$ such that $\varphi(x) = 1$ for $|x| \leq 2$. Let u be the solution of (I.1.1) with initial data $(\varphi Y(0), \varphi Y'(0))$. Let T_+ be the maximal time of existence of u. By finite speed of propagation,

$$u(t,x) = Y(t), \quad |x| \leq 2 - t, \quad t \in [0, T_+[.$$

If $T_+ > 1$, we have

$$\int_0^1 \left(\int_{|x|\leq 1} u^{10}(t,x)dx \right)^{1/2} dt = c^5 \int_0^1 \frac{1}{(1-t)^{5/2}} dt = +\infty,$$

a contradiction with the fact that u must be in $L^5([0,1], L^{10})$. Thus $T_+ \leq 1$, concluding the proof. □

The preceding proof is not completely rigorous: we have used finite speed of propagation for the equation (W5) outside of the framework of Theorem IV.4.2, since Y is not a solution of (W5) in the sense of Definition IV.2.1. We thus need the analog of IV.7.2 for classical solutions:

Theorem IV.7.2. *Let $(t_0, x_0) \in \mathbb{R}^{1+3}$, $t_1 > t_0$, $R > 0$. We denote $\Gamma = \Big\{(t,x) \in \mathbb{R} \times \mathbb{R}^N \ : \ t_0 \leq t \leq t_1, \ |x - x_0| \leq R - |t - t_0| \Big\}$ Let $u, v \in C^2(\Gamma)$ be two classical solutions of* (W5) *on Γ. We suppose $(u, \partial_t u)(t_0, x) = (v, \partial_t v)(t_0, x)$ for all $x \in B_R(x_0)$. Then $u(t,x) = v(t,x)$ all $(t,x) \in \Gamma$.*

We leave the proof of Theorem IV.7.2 as an exercise to the reader:

Exercice IV.6. Let u and v be as in Theorem IV.7.2. Assume $t_0 = 0$, $x_0 = 0$. Let

$$V(t) = \frac{1}{2} \int_{|x|<R-t} (u(t,x) - v(t,x))^2 dx + \frac{1}{2} \int_{|x|<R-t} (\partial_t u(t,x) - \partial_t v(t,x))^2 dx + \frac{1}{2} \sum_{j=1}^{3} \int_{|x|<R-t} (\partial_{x_j} u(t,x) - \partial_{x_j} v(t,x))^2 dx.$$

(1) Prove that $V'(t) \leq CV(t)$ for $t \in [t_0, t_1]$.

(2) Prove that $V(t) = 0$ for all $t \in [t_0, t_1]$.

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