CHAPTER V

# Examples of dynamics

## V.1. Scattering

### 1.a. Definition and characterization.

Definition V.1.1. The solution $u$ of (W5) is said to *scatter* in the future to a linear solution if $T_+(u) = +\infty$ and there exists $\vec{v}_0 \in \dot{\mathcal{H}}^1(\mathbb{R}^3)$ such that

$$\lim_{t\to\infty} \left\| \vec{S}_L(t)\vec{v}_0 - \vec{u}(t) \right\|_{\dot{\mathcal{H}}^1} = 0. \tag{V.1.1}$$

In the remainder of this section, we will simply say that a solution as in Definition V.1.1 scatters or is a scattering solution. We next give a characterization of scattering solutions:

Proposition V.1.2. *The solution $u$ of* (W5) *scatters if and only if $u \in L^5([0,T_+), L^{10})$, where $T_+$ is the maximal time of existence of $u$.*

Proof. Let $u$ be a solution such that $u \in L^5([0,T_+), L^{10})$. By the blow-up criterion, we already know that $T_+(u) = +\infty$. Let $\vec{v}_0 \in \dot{\mathcal{H}}^1$. Since $\vec{S}_L(t)$ conserves the $\dot{\mathcal{H}}^1$ norm, we have

$$\text{(V.1.1)} \iff \lim_{t\to\infty} \left\| \vec{v}_0 - \vec{S}_L(-t)\vec{u}(t) \right\|_{\dot{\mathcal{H}}^1} = 0.$$

We are thus reduced to prove that $\vec{S}_L(-t)\vec{u}_0(t)$ has a limit in $\dot{\mathcal{H}}^1$. Since $u$ is a solution in the sense of Definition IV.2.1, we have

$$\vec{S}_L(-t)\vec{u}(t) = \vec{u}_0 + \int_0^t \vec{S}_L(t-s)(0,u^5(s))ds.$$

Using $u \in L^5([t_0,+\infty, L^{10})$ and

$$\left\| S_L(-s)(0,u^5(s)) \right\|_{\dot{\mathcal{H}}^1} = \left\| u^5(s) \right\|_{L^2} = \|u(s)\|^5_{L^{10}},$$

we see that

$$\int_{t_0}^{\infty} \left\| \vec{S}_L(-s)(0,u^5(s)) \right\|_{\dot{\mathcal{H}}^s} ds = \|u\|^5_{L^5([0,\infty),L^{10})} < \infty.$$

Thus $\int_0^t \vec{S}_L(-s)(0,u^5(s))ds$ converges in $\dot{\mathcal{H}}^1$ as $t$ goes to $\infty$, which shows that $u$ scatters to a linear solution.

Next, we consider a solution $u$ of (W5) that scatters to a linear solution. Thus $T_+(u) = \infty$, and there exists $\vec{v}_0 \in \dot{\mathcal{H}}^1$ such that

$$\lim_{t\to\infty} \left\| \vec{u}(t) - \vec{S}_L(t)\vec{v}_0 \right\|_{\dot{\mathcal{H}}^1} = 0.$$

Fix $T \geq 0$ such that

$$\|S_L(T)\vec{v}_0\|_{L^5([t_0,\infty[,L^{10})} \leq \delta_0/2,$$

where $\delta_0$ is given by the local well-posedness theory (Theorem IV.3.1). Then, by Strichartz estimates

$$\|S_L(\cdot)\vec{u}(T)\|_{L^5([0,\infty[,L^{10})} \leq \|S_L(\cdot)\vec{v}_0\|_{L^5([T,\infty[,L^{10})} + C_S \left\|\vec{u}(T) - \vec{S}_L(T)\vec{v}_0\right\|_{\dot{\mathcal{H}}^1} \leq \delta_0$$

for large $T$. By Theorem IV.3.1 and the uniqueness Lemma IV.3.2, $u \in L^5([T,+\infty), L^{10})$ which concludes the proof. □

Combining Theorem IV.3.1, Strichartz estimates and Proposition V.1.2, we obtain:

Corollary V.1.3 (Small data scattering). *There exists a constant $\varepsilon > 0$ such that for all $\vec{u}_0 \in \dot{\mathcal{H}}^1$ with $\|\vec{u}_0\|_{\dot{\mathcal{H}}^1} \leq \varepsilon$, the solution of* (W5), (ID) *scatter in both time directions.*

Two natural questions arise:

**Existence of wave operators:** Given $\vec{v}_0 \in \dot{\mathcal{H}}^1$, does there exist a solution $u$ of (W5) with $T_+(u) = +\infty$ and

$$\lim_{t\to\infty} \left\|\vec{u}(t) - \vec{S}_L(t)\vec{v}_0\right\|_{\dot{\mathcal{H}}^1} = 0? \tag{V.1.2}$$

**Asymptotic completeness:** Do all solutions of (W5) scatter?

It turns that the answer to the first question is always positive, independently of the sign $\sigma$ in (W5). The asymptotic completeness is a much more delicate issue. We already know that it is not true in the focusing case $\sigma = 1$, since there exist solutions blowing-up in finite time (see Section IV.7). On the other hand, the asymptotic completeness holds in the defocusing case $\sigma = -1$. We will prove this fact for radial solutions. The general proof is more complicated (see [**3**]) but relies on the same type of arguments.

### 1.b. Existence of wave operators.

Theorem V.1.4. *Let $\vec{v}_0 \in \dot{\mathcal{H}}^1$. Then there exists a solution $u$ of* (W5) *with $T_+(u) = +\infty$ and such that* (V.1.2) *holds.*

Proof. Let $\vec{v}_0 \in \dot{\mathcal{H}}^1$. Let $u$ be a scattering solution of (W5) such that (V.1.2) holds. Letting $t \to \infty$ in the equality

$$\vec{S}_L(-t)\vec{u}(t) = \vec{u}_0 + \sigma \int_0^t \vec{S}_L(-s)\left(0, \sigma u^5(s)\right) ds,$$

we obtain

$$\vec{v}_0 = \vec{u}_0 + \sigma \int_0^\infty \vec{S}_L(-s)\left(0, \sigma u^5(s)\right) ds. \tag{V.1.3}$$

Note that the integral is convergent in $\dot{\mathcal{H}}^1$ by conservation of the energy for the linear wave equation and since $u \in L^5([0,\infty[, L^{10})$. In view of (V.1.3), we can rewrite Duhamel's formula as

$$u(t) = S_L(t)\vec{v}_0 - \sigma \int_t^\infty S_L(t-s)\left(0, u^5(s)\right) ds. \tag{V.1.4}$$

This shows that the problem of existence of wave operator can be interpreted as a Cauchy problem with initial data at time infinity. To solve this problem, we fix $t_0$ large such that

$$\|S_L(\cdot)\vec{v}_0\|_{L^5([t_0,\infty),L^{10})},$$

and we prove that the operator $A$ defined by

$$Av(t) = S_L(t)\vec{v}_0 - \sigma \int_t^{\infty} S_L(t-s)\left(0, v^5(s)\right) ds$$

is a contraction of the metric space $X$ defined by

$$X = \left\{ v \in L^5([t_0,\infty), L^{10}),\ \|v\|_{L^5([t_0,\infty),L^{10})} \leq 2\delta_0 \right\}$$

The details are very close to the ones of the proof of Theorem IV.3.1 and are left to the reader. □

**1.c. Asymptotic completeness in the defocusing case.** We next prove:

THEOREM V.1.5. *Let $u$ be a solution of* (W5) *with $\sigma = -1$ and radial initial data. Then $u$ scatters.*

We divide the proof (coming from [**13**] into a few Lemmas.

LEMMA V.1.6 (Morawetz inequality). *There exists $C > 0$ with the following property Let $u$ be a solution of* (W5). *Assume that $\vec{u}_0$ is radial, compactly supported and smooth. Let $E$ be the energy of $u$ and $I_{\max}$ its maximal interval of existence. Then*

$$\int_{I_{\max}} \int_{\mathbb{R}^3} \frac{1}{|x|} |u(t,x)|^6 dx dt \leq CE.$$

PROOF. By persistence of regularity and finite speed of propagation, the solution $u$ is $C^{\infty}$ on $I_{\max} \times \mathbb{R}^3$, and there exists $R > 0$ such that $|x| \leq R + |t|$ on the support of $u$.

Let

$$M(t) = \int_0^{\infty} \partial_t u(t,r) \partial_r u(t,r) r^2 dr + \int_0^{\infty} \partial_t u(t,r) u(t,r) r dr.$$

Then

$$M'(t) = \int_0^{\infty} \partial_t^2 u (u + r\partial_r u) r dr + \underbrace{\int_0^{\infty} \partial_t u \partial_r (\partial_t u) r^2 dr + \int_0^{\infty} (\partial_t u)^2 r dr}_{=0},$$

where we use a straightforward integration by parts to prove that the two last terms cancel each other. Using the equation, we have

$$\begin{aligned} M'(t) = \int_0^{\infty} &\left( \partial_r^2 u + \frac{2}{r} \partial_r u - u^5 \right) (u + r\partial_r u)\, r dr \\ &= \int_0^{\infty} \frac{1}{2} \frac{\partial}{\partial r} (u + r\partial_r u)^2\, dr - \int_0^{\infty} u^5 (u + r\partial_r u) r dr \\ = -\frac{1}{2} u^2(t,0) - \int_0^{\infty} u^6 r dr &- \int_0^{\infty} \frac{1}{6} \frac{\partial}{\partial r} u^6\, r^2 dr = -\frac{1}{2} u^2(t,0) - \frac{2}{3} \int_0^{+\infty} u^6 r dr. \end{aligned}$$

Next, we notice that $M(t) \lesssim E$. Indeed, this follows easily by the Cauchy-Schwarz inequality and Hardy's inequality

$$\int_0^{\infty} u^2 dr \leq 4 \int_0^{\infty} (\partial_r u)^2 r^2 dr, \tag{V.1.5}$$

which follows from Cauchy-Schwarz and the equality $\int_0^{\infty} \partial_r(u^2) r dr = -\int_0^{\infty} u^2 dr$. Integrating the bound $M'(t) \leq -\frac{2}{3} \int_0^{\infty} u^6 r dr$ between two times $a$ and $b$, with

$T_- < a < b < T_+$, and letting $b \to T_+$ and $a \to T_-$ we obtain the desired conclusion. $\square$

We next prove

LEMMA V.1.7 (Bound of the $L^8$ norm). *Let $u$ be a radial solution of* (W5) *with $\sigma = -1$. Then*

$$\|u\|_{L^8(I_{\max}\times\mathbb{R}^3)} \lesssim E^{1/4}. \tag{V.1.6}$$

PROOF.

STEP 1. We prove the bound when $\vec{u}_0$ is $C^\infty$, compactly supported. For this we use the Morawetz estimate of Lemma V.1.6 and the radial Sobolev inequality:

$$u^2(t,r) \leq \frac{1}{r}\int_r^\infty (\partial_\rho u(t,\rho))^2\rho^2 d\rho \lesssim \frac{1}{r}E. \tag{V.1.7}$$

This last inequality proved with the fundamental theorem of calculus and Cauchy-Schwarz inequality:

$$|u(r)| = \left|\int_r^\infty \partial_\rho u(\rho)d\rho\right| \leq \sqrt{\int_r^\infty (\partial_\rho u(\rho))^2\rho^2 d\rho}\sqrt{\int_r^\infty \rho^{-2}d\rho}.$$

Combining Lemma V.1.6 with (V.1.7), we obtain

$$\int_{I_{\max}} u^8(t,r)r^2 drdt \lesssim E\int_{I_{\max}}\int_0^\infty u^6(t,r)rdrdt \lesssim E^2,$$

which give (V.1.6) in this case.

STEP 2. To prove the bound for general solutions, we use a density argument. We consider a sequence of initial data $(\vec{u}_0^n)_n$ with $\vec{u}_0^n \in (C_0^\infty)^2$, radial, such that $\lim_n \vec{u}_0^n = \vec{u}_0$ in $\dot{H}^1$. Let $K \subset I_{\max}(\vec{u}_0)$ compact. By continuity of the flow (Theorem IV.5.1), $K \subset I_{\max}(\vec{u}_0^n)$ for large $n$ and

$$\lim_{n\to\infty} \|u^n - u\|_{L^\infty(K,L^6)} + \|u^n - u\|_{L^5(K,L^{10})} = 0.$$

Since Hölder inequality implies $L^5(K,L^{10}(\mathbb{R}^3)) \cap L^\infty(K,L^6(\mathbb{R}^3)) \subset L^8(K\times\mathbb{R}^3)$ with the bound

$$\|f\|^8_{L^8(K\times\mathbb{R}^3)} \leq \|f\|^3_{L^\infty(K,L^6)}\|f\|^5_{L^5(K,L^{10})},$$

we deduce

$$\lim_{n\to\infty} \left\|u^n - u^0\right\|_{L^8(K\times\mathbb{R}^3)} = 0. \tag{V.1.8}$$

By Step 1,

$$\|u^n\|^8_{L^8(K\times\mathbb{R}^3)} \lesssim (E(\vec{u}_0^n))^2 \underset{n\to\infty}{\longrightarrow} E^2,$$

which concludes the proof.

$\square$

We are now ready to end the proof of Theorem V.1.5

Proof of Theorem V.1.5. We fix $\vec{u}_0 \in \dot{\mathcal{H}}^1$. By Lemma V.1.7, we have $u \in L^8(I_{\max} \times \mathbb{R}^3)$. By Proposition V.1.2, and since $u \in L^5(K, L^{10})$ for all $K \in I_{\max}$ it is sufficient to prove $u \in L^5([\tau, T_+[, L^{10})$ for some $\tau \in I_{\max}$. We fix $\tau \in I_{\max}$ such that

$$\|u\|_{L^8([\tau,T_+[\times\mathbb{R}^3)} \leq \varepsilon, \tag{V.1.9}$$

$\varepsilon > 0$ small to be specified. For $t \in [\tau, T_+[$, we have by Hölder's inequality

$$\|u\|_{L^5([\tau,t],L^{10}(\mathbb{R}^3))} \leq \|u\|^{2/5}_{L^8([\tau,t]\times\mathbb{R}^3)} \|u\|^{3/5}_{L^4([\tau,t],L^{12}(\mathbb{R}^3))}. \tag{V.1.10}$$

Thus it is sufficent to prove $u \in L^4([\tau, T_+[, L^{12})$. For this we use Strichartz estimate, (V.1.10) and (V.1.9):

$$\|u\|_{L^4([\tau,t,L^{12})} \leq C_S \|\vec{u}(\tau)\|_{\dot{\mathcal{H}}^1} + C_S \|u\|^5_{L^5([t_0,t],L^{10})} \leq 2C_S\sqrt{E} + C_S\varepsilon^2 \|u\|^3_{L^4([t_0,t],L^{12})}.$$

We prove by a bootstrap argument:

$$\|u\|_{L^4([\tau,t],L^{12})} \leq 3C_S\sqrt{E}. \tag{V.1.11}$$

Indeed if (V.1.11) holds, for some $t$, we have

$$\|u\|_{L^4([\tau,t,L^{12})} \leq 2C_S\sqrt{E} + C_S\varepsilon^2(3C_S\sqrt{E})^3 \leq \frac{5}{2}C_S\sqrt{E},$$

where we have choose $\varepsilon$ so small that $\varepsilon^2(3C_S)^3 E \leq \frac{1}{2}$. This proves (V.1.11) by the intermediate value theorem.

By the same proof in a neighborhood of $T_-$, we obtain, $u \in L^5(I_{\max}, L^{10})$, which concludes the proof that $u$ scatters in both time directions. □

Exercice V.1. In the setting of Theorem V.1.5, prove

$$\|u\|_{L^4(\mathbb{R},L^{12})} \leq C\big(E(u_0, u_1)\big)^2. \tag{V.1.12}$$

## V.2. Stationary solutions and travelling waves

**2.a. Stationary solutions.** We are interested by stationary solutions of the equation (W5), i.e. nonzero, $\dot{H}^1$ solutions of the elliptic equation $-\Delta Q = \sigma Q^5$. In the defocusing case $\sigma = -1$, the equation is

$$-\Delta Q + Q^5 = 0,$$

to be interpreted in the sense of distribution on $\mathbb{R}^3$. This means

$$\forall \varphi \in C_0^\infty(\mathbb{R}^3), \quad \int \nabla Q \cdot \nabla \varphi + \int Q^5 \varphi = 0.$$

Approximation $Q$ by smooth, compactly supported functions, we obtain

$$\int |\nabla Q|^2 + \int Q^6 = 0,$$

which implies $Q = 0$ a.e. Thus in the defocusing case, the only nonstationary solution is the constant null solution. This was already known, since in this case, all solutions scatter and a scattering solution cannot be stationary since it is in $L^5(\mathbb{R}, L^{10})$.

We next consider the focusing case $\sigma = 1$. The equation is:

$$-\Delta Q = Q^5, \quad Q \in \dot{H}^1(\mathbb{R}^3). \tag{Ell}$$

Since $Q$ must be a solution of (W5) in the sense of definition IV.2.1, we can also assume $Q \in L^{10}$.

EXERCICE V.2. Prove that a solution of (Ell) with $Q \in L^{10}$ is $C^\infty$, and that it is bounded as well as all of its derivative. *Hint:* use the equation and Sobolev embeddings.

Let us mention that the assumption $Q \notin L^{10}$ is not necessary, and that it is possible (but not trivial), to prove that $Q$ satisfying (Ell) must be in $C^\infty \cap L^\infty$ (see [**22**]). Note that in this case, a simple elliptic regularity argument based on Sobolev inequalities does not work. Indeed, we have:

$$Q \in \dot{H}^1 \Longrightarrow Q \in L^6 \Longrightarrow \Delta Q = -Q^5 \in L^{6/5} \Longrightarrow \nabla Q \in L^2, \tag{V.2.1}$$

where we used the Sobolev embeddings $\dot{H}^1 \subset L^6$ and $\dot{W}^{2,6/5} \subset \dot{H}^1$. Of course (V.2.1) does not give any improvement on the regularity of $Q$.

The equation (Ell) has nonzero solutions. In the radial case, the solutions are completely classified.

THEOREM V.2.1. *Let*

$$W(x) = \frac{1}{\left(1 + \frac{|x|^2}{3}\right)^{1/2}}, \quad x \in \mathbb{R}^3 \tag{V.2.2}$$

*Then $W$ is a solution of* (Ell). *Furthermore the set of radial solutions of* (Ell) *is given by*

$$\Sigma = \{0\} \cup \left\{ \frac{\iota}{\lambda^{1/2}} W\left(\frac{\cdot}{\lambda}\right),\ \lambda > 0,\ \iota \in \{\pm 1\}\right\}.$$

It can be checked by explicit computations that $W$ is a solution of (Ell) . Since the equation is invariant by scaling and sign change, we obtain also that $\frac{\iota}{\lambda^{1/2}} W\left(\frac{\cdot}{\lambda}\right)$ is also a solution for any $\lambda > 0$, $\iota = 1$ or $-1$. The fact that these are the only radial solutions of (Ell) can be proved by ODE arguments. This will be a consequence of a stronger rigidity theorem below (see Theorem V.3.3) and we thus omit the proof.

Let us mention that $W$ is the maximizer for the Sobolev inequality on $\mathbb{R}^3$: $\|f\|_{L^6} \lesssim \|\nabla f\|_{L^2}$. Thas is, if $f \in \dot{H}^1$, one has

$$\int |f|^6 \leq C_s \left(\int |\nabla f|^2\right)^3, \quad C_s = \int |W|^6 \times \left(\int |\nabla W|^2\right)^{-3}, \tag{V.2.3}$$

with equality if and only if $f = 0$ or $f = W$, up to scaling, space translation and sign change. This was proved independtly by Aubin [**1**] and Talenti [**20**] in the mid 70's.

Much less is known about the equation (Ell) without symmetry assumption. Multiplying (Ell) by $Q$ and integrating by parts, we obtain

$$\int |\nabla Q|^2 = \int |Q|^6 = 3E(Q, 0).$$

In particular, the energy of a nonzero solution $Q$ of (Ell) (considered as a solution of (W5)) is positive. Combining with (V.2.3), obtain that the energy of any nonzero solution of (Ell) is greater or equal to $E(W, 0)$. The least-energy nonzero solution $W$ of (Ell) is sometimes called the *ground state* of (W5). It was proved by Ding in 1986 (see [**8**]) that one can also construct arbitrarily large solutions of (Ell).

**2.b. Travelling waves.** Travelling wave solutions of (W5) are by definitions solutions of the form $\varphi(x - \mathbf{c}t)$, where the speed $\mathbf{c} \in \mathbb{R}^3$ is fixed, and $\varphi \in \dot{H}^1$. Using the invariance of (W5) by rotation, we can assume $\mathbf{c} = (c, 0, 0)$, where $c \in \mathbb{R}$. We are thus lead to study solutions of (W5) of the form

$$u(t, x) = \varphi(x_1 - tc, x_2, x_3), \quad c \in \mathbb{R}, \quad \varphi \in \dot{H}^1. \tag{V.2.4}$$

These solutions can be deduced from solutions of the elliptic equation (Ell).

Theorem V.2.2. *Let $u$ be a nonzero solution of* (W5) *of the form* (V.2.4). *Then $\sigma = 1$, $|c| < 1$, and $Q$ defined by*

$$Q(x_1, x_2, x_3) = \varphi\left(x_1\sqrt{1 - c^2}, x_2, x_3\right) \tag{V.2.5}$$

*is a solution of* (Ell).

Remark V.2.3. Recall from (I.7.2) the definition of the Lorentz boost of a function $u : \mathbb{R}^4 \to \mathbb{R}$. One can check that the Lorentz boost of a $C^2$, global solution $u$ of (W5) is also a solution of (W5). The travelling waves are exactly given by applying Lorentz boosts to solutions of (Ell).

Proof of the Theorem. Let $u$ be a nonzero travelling wave solution.

The fact that $|c| < 1$ follows from finite speed of propagation. Indeed, arguing by contradiction, we consider a solution $u$ of (W5) of the form (V.2.4), with $c \geq 1$ (where we have assumed $c$ positive to fix ideas, the case $c \leq -1$ can be deduced by the transformation $x_1 \mapsto -x_1$).

We fix $L > 0$ such that

$$\int_{x_1 > L} |\nabla u_0|^2 + u_1^2 = \varepsilon^2, \tag{V.2.6}$$

where $\varepsilon > 0$ is small. Let $(v_0, v_1) \in \dot{\mathcal{H}}^1(\mathbb{R}^3)$ such that

$$(v_0, v_1)(x) = (u_0, u_1)(x), \quad \int |\nabla v_0|^2 + v_1^2 dx \leq 2\varepsilon^2. \tag{V.2.7}$$

(Defining $v_j$ for $j = 0, 1$ by $v_j(x) = u_j(2L - x)$ for $x \leq L$ would work for example). Let $v$ be the solution of (W5) with initial data $(v_0, v_1)$ at $t = 0$. By the small data theory (Theorem IV.3.1), $v \in L^5(\mathbb{R}, L^{10})$. By (V.2.7) and finite speed of propagation,

$$\forall t \geq 0, \quad \forall x \in \mathbb{R}^3, \quad x_1 \geq L + t \Longrightarrow v(t, x) = u(t, x) = \varphi(x_1 - ct, x_2, x_3).$$

Thus

$$\int_{\mathbb{R}^3} |v(t, x)|^{10} dx \geq \int_{x_1 \geq L + t} |\varphi(x_1 - ct, x_2, x_3)|^{10} dx \geq \int_{x_1 \geq L + (1-c)t} |\varphi(x)|^{10} dx \geq a,$$

where $a = \int_{x_1 \geq L} |\varphi|^{10} dx > 0$ by (V.2.6). This concludes the proof.

We thus have $c < 1$. In this case, it is easy to check, using (W5), that $Q$ defined by (V.2.5) satisfies $-\Delta Q = \sigma Q^5$. This implies since $Q$ is not identically 0, that $\sigma = 1$ and that $Q$ is solution to (Ell), which concludes the proof. $\square$

We will now consider exclusively the case $\sigma = 1$, i.e. the equation

$$\partial_t^2 u - \Delta u = u^5. \tag{W5f}$$

We have identified 3 types of solutions to (W5f).

(1) Solutions of the ordinary differential equation $y'' = y^5$, such as $\left(\frac{3}{4t^2}\right)^{\frac{1}{4}}$, that can be truncated to obtain finite time blow-up solutions with finite energy.
(2) Scattering solutions, that are global and asymptotically close to solutions of the linear wave equation, that move with the speed of light (1 in our normalization).
(3) Travelling wave solutions, with velocity $< 1$.

If we believe that the ODE solution will not play any role for the asymptotics of global solutions[1], we are lead to conjecture that this asymptotics will only be influenced by the travelling wave and linear solutions. Moreover, the different speeds of propagation would decouple asymptotically the linear and travelling wave dynamics. We will come back to a more precise form of this *resolution conjecture* in the next chapter.

In the sequel, we will focus on radial solutions, for which more things are known. Note that in this case, there is no travelling wave, and that the only nonzero solutions of (Ell) are given by the transforms of $W$.

We will identify a nondispersive property of solutions of (W5f), that turns out to characterize the stationary solutions of (W5f).

## V.3. Nonradiative solutions

**3.a. Definition and classification.** In order to study the dynamics of nonlinear dispersive equation, it is common to classify solutions that are "completely nondispersive" in a certain sense. These solutions tend to play an important role on the dynamics, and their classification is crucial for its understanding. We will give here the notion of "nonradiative solutions", that was introduced to prove the resolution into stationary solutions for radial solutions of (W5) (See Theorem **??** below and [**10**]).

Definition V.3.1. Let $u$ be a global solution of (W5) or of the linear wave equation (I.1.1). Let $R \in \mathbb{R}$ and $t_0 \in \mathbb{R}$. The solution $u$ is $(R, t_0)$-*nonradiative* when

$$\sum_{\pm} \lim_{t\to\pm\infty} \int_{|x|\geq R+|t-t_0|} e_u(t,x)dx = 0.$$

In the definition, we have used the notation $e_u(t,x) = \frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{2}(\partial_t u(t,x))^2$. To simplify notations, we will restrict without generality to the case $t_0 = 0$, and call the corresponding solutions $R$-nonradiative solutions. If $R = 0$, the solution with simply be called "nonradiative".

It is possible, using the explicit formulas of Chapter 1, to prove that the only 0-nonradiative solution of the linear wave equation (I.1.1) is the constant null solution. In the nonlinear case, using that the speed of travelling waves is always $< 1$, we see that travelling waves are also $R$-nonradiative solutions for all $R$. The rigidity conjecture for nonradiative solution says that this should be the only ones:

Conjecture V.3.2 (Rigidity conjecture for nonradiative solutions). *Let $u$ be a nonradiative solution of* (W5f). *Then $u$ is a travelling wave.*

[1]This belief is false in general for semi-linear wave equation, but turns out to be true in the energy-critical case. See [**11**] for an example of a global solution of the cubic wave equation which is asymptotically close to a solution of the corresponding ODE.

We prove this conjecture in the radial case:

THEOREM V.3.3 (Dynamical characterization of $W$). *Let $R_0 \geq 0$ and $u$ be a radial, $R_0$-nonradiative solution of* (W5f). *Then one of the following occurs:*

- *$u(t,x) = 0$ for $|x| > R_0 + |t|$.*
- *there exists $\lambda > 0$, $\iota \in \{\pm 1\}$,*

$$\forall |x| > R_0 + |t|, \quad u(t,x) = \iota W_\lambda(x),$$

*where*

$$W_\lambda(x) = \frac{1}{\lambda^{1/2}} W\left(\frac{x}{\lambda}\right).$$

REMARK V.3.4. In the case where $R_0 = 0$, we see that the theorem implies that $(u_0, u_1) = (\iota W_\lambda, 0)$, and thus that $u$ is the stationary solution $W_\lambda$. This implies the uniqueness part in Theorem V.2.1, since any solution of the elliptic equation (Ell) is also a nonradiative solution of (W5f).

**3.b. A lower bound of the exterior energy for the linear equation.** The proof of Theorem V.3.3 is based on its (quantitative) analog for the linear equation (I.1.1):

PROPOSITION V.3.5. *Let $R \geq 0$. Let $(u_0, u_1) \in \dot{\mathcal{H}}^1$ and $u_L(t) = S_L(t)(u_0, u_1)$. Then*

$$\sum_{\pm} \lim_{t \to \pm\infty} \int_{R+|t|}^{+\infty} e_u(t,r) r^2 \, dr = \frac{1}{2} \int_R^{+\infty} (\partial_r(r u_0))^2 + r^2 u_1^2 \, dr. \tag{V.3.1}$$

The right-hand side of (V.3.1) can be compared to the $\dot{\mathcal{H}}^1(\{|x| > R\})$-norm by a simple integration by parts. Indeed, if $R > 0$, we have, for any radial $\dot{H}^1$ function $f$ on $\mathbb{R}^3$,

$$\int_R^\infty \Big(\partial_r(r f(r))\Big)^2 dr = \int_R^\infty \Big(\partial_r f(r)\Big)^2 r^2 dr - R f(R)^2. \tag{V.3.2}$$

When $R = 0$, the boundary term vanishes and we have

$$\int_0^\infty \Big(\partial_r(r f(r))\Big)^2 dr = \int_0^\infty \Big(\partial_r f(r)\Big)^2 r^2 dr.$$

The formula (V.3.1) reads

$$\sum_{\pm} \lim_{t \to \pm\infty} \int_{|x| \geq |t|}^{+\infty} e_u(t,x) dx = \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla u_0(x)|^2 + u_1^2(x)\right) dx. \tag{V.3.3}$$

Let us mention that (V.3.3) remains valid without the assumption that $(u_0, u_1)$ is radial, and can be proved with the explicit formulas of Theorem I.5.2. It is still valid in any *odd* space dimension, as proved in [**9**], but not in *even* space dimension, even for radial solutions (see [**7**]).

Investigating (V.3.1), we see that the only radial $R$-nonradiative solutions of (W5) are the solutions that are equal to $\ell/r$ for $r > R + |t|$, (where $\ell \in \mathbb{R}$). Since $\ell/r$ is not in $\dot{H}^1(\mathbb{R}^3)$, we also obtain that 0 is the only 0-nonradiative solution.

PROOF OF PROPOSITION V.3.5. This follows from the explicit formula for radial, 3D solutions (see (I.5.1)),

$$u(t,r) = \frac{1}{r}(\varphi(r+t) - \varphi(t-r)), \tag{V.3.4}$$

where

$$\varphi(\eta)=\frac{1}{2}\eta u_0(|\eta|)+\frac{1}{2}\int_0^{\eta}\sigma u_1(|\sigma|)d\sigma. \tag{V.3.5}$$

Using this formula, we see that

$$\Big(\partial_r(ru)\Big)^2+\Big(\partial_t(ru)\Big)^2=2(\varphi'(r+t))^2+2(\varphi'(t-r))^2.$$

This gives

$$\sum_{\pm}\lim_{t\to\pm\infty}\int_{R+|t|}^{+\infty}|\partial_{t,r}(ru_L(t,r))|^2\,dx=2\int_R^{\infty}\left(\varphi'(\eta)\right)^2d\eta+2\int_{-\infty}^{-R}\left(\varphi'(\eta)\right)^2d\eta$$
$$=\int_R^{+\infty}\left(\partial_r(ru_0)\right)^2+r^2u_1^2\,dr.$$

Using (V.3.2) we obtain (V.3.1). Indeed by by the formula (V.3.4), we have

$$\lim_{t\to\infty}(R+t)u^2(t,R+T)=0,$$

since $|\varphi(\eta)|/\sqrt{|\eta|}$ goes to 0 as $|\eta|\to\infty$. □

**3.c. Proof of the rigidity result.** We next prove the rigidity Theorem V.3.3. The proof takes several steps.

STEP 1. Let $(u_0,u_1)\in\dot{\mathcal{H}}^1$ be as in Theorem V.3.3. Let $\varepsilon>0$ be a small parameter to be specified. In all the proof we fix $R_\varepsilon\geq R_0$ such that

$$\int_{R_\varepsilon}^{+\infty}\left((\partial_r u_0)^2+u_1^2\right)r^2\,dr\leq\varepsilon^2. \tag{V.3.6}$$

In this step, we prove

$$\forall R\geq R_\varepsilon,\quad\int_R^{+\infty}\left(\partial_r(ru_0)\right)^2+r^2u_1^2\,dr\leq CR^5u_0^{10}(R). \tag{V.3.7}$$

Let $R\geq R_\varepsilon$. We define the radial functions $v_0\in\dot{H}^1(\mathbb{R}^3)$, $v_1\in L^2(\mathbb{R}^3)$ as follows:

$$\begin{cases}(v_0,v_1)(r)=(u_0,u_1)(r) & \text{if } r>R\\ (v_0,v_1)(r)=(u_0(R),0) & \text{if } r\in(0,R).\end{cases} \tag{V.3.8}$$

We let $v(t)$ be the solution of (W5f) with initial data $(v_0,v_1)$, and $v_L(t,r)=S_L(t)(v_0,v_1)$ be the corresponding solution to the free wave equation. We note that by final speed of propagation

$$v(t,r)=u(t,r),\quad r>R+|t|.$$

By the small data theory, since $\varepsilon$ is small,

$$\sup_{t\in\mathbb{R}}\|\vec{v}(t)-\vec{v}_L(t)\|_{\dot{H}^1\times L^2}\leq C\left\|(v_0,v_1)\right\|^5_{\dot{H}^1\times L^2}. \tag{V.3.9}$$

By Proposition V.3.5,

$$\sum_{\pm}\lim_{t\to\pm\infty}\int_{R+|t|}^{+\infty}|\partial_{t,r}(v_L(t,r))|^2r^2\,dr\geq\int_R^{+\infty}\left(\partial_r(ru_0)\right)^2+u_1^2\,dr. \tag{V.3.10}$$

By (V.3.9), and finite speed of propagation

$$\int_{R+|t|}^{+\infty} \left|\partial_{t,r}(v_L(t,r)) - \partial_{t,r}(u(t,r))\right|^2 r^2\,dr \leq C\left(\int_R^{+\infty} ((\partial_r u_0)^2 + u_1^2) r^2 dr\right)^5.$$

Combining with (V.3.10) and using that the solution is $R$-nonradiative, we obtain

$$\int_R^{+\infty} (\partial_r(ru_0))^2 + r^2 u_1^2\,dr \leq C\left(\int_R^{+\infty} \left((\partial_r u_0)^2 + u_1^2\right) r^2 dr\right)^5.$$

With the integration by parts formula (V.3.2) and the smallness of $\varepsilon$, we deduce (V.3.7).

STEP 2. In this step we prove that there exists $\ell \in \mathbb{R}$ and $C > 0$ such that for large $R$,

$$\left|u_0(r) - \frac{\ell}{r}\right| \leq \frac{C}{r^3}, \quad \int_r^{+\infty} \rho^2 u_1(\rho)\,d\rho \leq \frac{C}{r^5}. \tag{V.3.11}$$

First fix $R$ and $R'$ such that $R_\varepsilon \leq R \leq R' \leq 2R$. Letting $\zeta_0(r) = ru_0(r)$, we have, using Cauchy-Schwarz, then Step 1

$$|\zeta_0(R) - \zeta_0(R')| \leq \int_R^{R'} |\partial_r \zeta_0(r)|\,dr \leq \sqrt{R}\sqrt{\int_R^{R'} (\partial_r \zeta_0)^2 dr} \leq \frac{1}{R^2}\zeta_0^5(R). \tag{V.3.12}$$

Since by the definition (V.3.6) of $R_\varepsilon$ and the integration by parts formula (V.3.2) one has

$$\frac{1}{R}\zeta_0^2(R) \leq \varepsilon, \tag{V.3.13}$$

we deduce from (V.3.12):

$$|\zeta_0(R) - \zeta_0(R')| \leq \varepsilon^2 \zeta_0(R), \qquad R_\varepsilon \leq R \leq R' \leq 2R. \tag{V.3.14}$$

Let $\alpha = \log_2(1+\varepsilon)$, so that $2^\alpha = (1+\varepsilon)$. By (V.3.14), for all $k$, $|\zeta_0(2^{k+1}R_\varepsilon)| \leq 2^\alpha |\zeta_0(2^k R_\varepsilon)|$. Thus the sequence $\left(\frac{|\zeta_0(2^k R_\varepsilon)|}{(2^k)^\alpha}\right)_{k\geq 0}$ is nonincreasing. This implies that $|\zeta_0(2^k R_\varepsilon)| \lesssim \left(2^k R_\varepsilon\right)^\alpha$, for $k \geq 0$ and thus, using (V.3.14) again,

$$|\zeta_0(R)| \lesssim R^\alpha.$$

We can take $\varepsilon$ small enough, so that $\alpha \leq 1/5$. The inequality (V.3.12) yields

$$|\zeta_0(R) - \zeta_0(R')| \lesssim \frac{1}{R}, \quad R_\varepsilon \leq R \leq R' \leq 2R. \tag{V.3.15}$$

This shows that $\sum_{k\geq 0} \left|\zeta_0(2^k R) - \zeta_0(2^{k+1}R_\varepsilon)\right| < \infty$, and thus that $\left(\zeta_0(2^k R_\varepsilon)\right)_k$ has a limit $\ell$ as $k \to \infty$. By (V.3.15)

$$\lim_{R\to\infty} \zeta_0(R) = \ell.$$

This implies that $\zeta_0$ is bounded. The inequality (V.3.12) then yields

$$\forall k \geq 0, \quad \forall R \geq R_\varepsilon, \quad \left|\zeta_0\left(2^k R\right) - \zeta_0\left(2^{k+1}R\right)\right| \lesssim \frac{1}{2^{2k}R^2}.$$

Summing over $k \geq 0$ and using the triangle inequality, we obtain,

$$|\zeta_0(R) - \ell| \lesssim \frac{1}{R^2},$$

which is the first inequality in (V.3.11). Combining with Step 1, we obtain the second inequality in (V.3.11).

STEP 3. In this step, we assume $\ell = 0$ and prove that $(u_0, u_1) \equiv (0, 0)$. Indeed by (V.3.14), if $R \geq R_\varepsilon$ and $k \in \mathbb{N}$,

$$\left|\zeta_0\left(2^{k+1}R\right)\right| \geq (1 - C\varepsilon^2)\left|\zeta_0\left(2^k R\right)\right|.$$

Hence by induction on $k$,

$$\left|\zeta_0\left(2^k R\right)\right| \geq (1 - C\varepsilon^2)^k |\zeta_0(R)|.$$

Since by the preceding step and the assumption $\ell = 0$, $|\zeta_0(2^k R)| \lesssim 1/(2^k R)^2$, we deduce, chosing $\varepsilon$ small enough and letting $k \to \infty$ that $\zeta_0(R) = 0$. Combining with (V.3.7) we deduce

$$R \geq R_\varepsilon \Longrightarrow \int_R^{+\infty} (\partial_r \zeta_0)^2 + u_1^2(r)\, dr = 0,$$

that is $u_0(r)$ and $u_1(r)$ are 0 for almost every $r \geq R_\varepsilon$. Going back to the definition of $R_\varepsilon$ we see that we can choose any $R_\varepsilon > R_0$, which concludes this step.

STEP 4. We next assume $\ell \neq 0$. To fix ideas, we assume that $\ell$ is positive. By the definition (V.2.2) of $W$ and the definition of $W_\lambda$ we have, for $\lambda > 0$

$$W_\lambda(r) = \frac{\sqrt{3\lambda}}{r} + \mathcal{O}\left(\frac{1}{r^3}\right), \quad r \to \infty.$$

We choose $\lambda > 0$ such that $\sqrt{3\lambda} = \ell$ so that

$$\left|W_\lambda(r) - \frac{\ell}{r}\right| \lesssim \frac{1}{r^3} \tag{V.3.16}$$

for large $r$. In this step we prove that $(u_0 - W_\lambda, u_1)$ has compact support. Let $f = u - W_\lambda$. Then

$$\begin{cases} \partial_t^2 f - \Delta f = D_\lambda(f) := \displaystyle\sum_{k=1}^{5} \binom{5}{k} W_\lambda^{5-k} f^k. \\ \vec{f}_{\restriction t=0} = (f_0, f_1) := (u_0 - W_\lambda, u_1)\,, \end{cases} \tag{V.3.17}$$

For $\varepsilon > 0$ small, we fix $R'_\varepsilon \gg 1$ such that

$$\int_{R'_\varepsilon}^{+\infty} \left(|\partial_r f_0(r)|^2 + |f_1(r)|^2\right) r^2 dr \leq \varepsilon^2 \tag{V.3.18}$$

$$\int_{\mathbb{R}} \left(\int_{R'_\varepsilon + |t|}^{+\infty} W_\lambda^{10}(r) r^2\, dr\right)^{\frac{1}{2}} dt \leq \varepsilon^5. \tag{V.3.19}$$

Let $f_L$ be the solution of $\partial_t^2 f_L = \Delta f_L$ with

$$\vec{f}_{L\restriction t=0} = (\tilde{f}_0, \tilde{f}_1),$$

where $(\tilde{f}_0, \tilde{f}_1)$ coincides with $(f_0, f_1)$ for $r > R'_\varepsilon$ and is defined as in (V.3.8). Using (V.3.17) and the assumptions (V.3.18) and (V.3.19) on $R'_\varepsilon$, we obtain

$$\sup_{t \in \mathbb{R}} \left\| 1\!\!1_{\{|x| > |t| + R'_\varepsilon\}} \left|\nabla_{t,x}(\tilde{f}(t) - \tilde{f}_L(t))\right| \right\|_{L^2} \lesssim \varepsilon^4 \left\|(\tilde{f}_0, \tilde{f}_1)\right\|_{\dot{H}^1 \times L^2}. \tag{V.3.20}$$

Let $R \geq R'_\varepsilon$. Using that by Proposition V.3.5,

$$\sum_{\pm} \lim_{t \to \pm\infty} \int_R^{+\infty} \left(\partial_{t,r}\left(\tilde{f}_L(t,r)\right)\right)^2 r^2 dr \gtrsim \int_R^{+\infty} \left(\left(\partial_r(r\tilde{f})\right)^2 + r^2 \tilde{f}_1^2\right) dr,$$

and since

$$\sum_{\pm} \lim_{t\to\pm\infty} \int_R^{+\infty} \left(\partial_{t,r}\big(\tilde{f}(t,r)\big)\right)^2 r^2 dr = 0,$$

we deduce from (V.3.20)

$$\varepsilon^8 \int_R^{+\infty} \left((\partial_r f_0)^2 + f_1^2\right) r^2 dr \gtrsim \int_R^{+\infty} \left(\big(\partial_r(rf_0)\big)^2 + r^2 f_1^2\right) dr,$$

and thus

$$\varepsilon^8 R f_0^2(R) \gtrsim \int_R^{\infty} \left(\big(\partial_r(rf_0)\big)^2 + r^2 f_1^2\right) dr. \tag{V.3.21}$$

Letting $g_0 = rf_0$, we deduce by Cauchy-Schwarz that for $R \geq R'_\varepsilon$, $k \in \mathbb{N}$,

$$\left|g_0\left(2^{k+1}R\right) - g_0\left(2^k R\right)\right| \lesssim \int_{2^k R}^{2^{k+1}R} |\partial_r g_0| dr \lesssim \varepsilon^4 \left|g_0(2^k R)\right|.$$

This yields by an easy induction $|g_0(2^k R)| \geq \left(1 - C\varepsilon^4\right)^k |g_0(R)|$, where $C > 0$ is a constant which is independent of $\varepsilon$. Since by Step 2,

$$\frac{C}{\left(2^k R\right)^2} \geq \left|g_0\left(2^k R\right)\right|,$$

we obtain choosing $\varepsilon$ small enough that $g_0(R) = 0$ for large $R$. Combining with (V.3.21), we deduce that $(f_0(r), f_1(r)) = 0$ a.e. for large $R$, concluding this step.

STEP 5. In this step we still assume $\ell \neq 0$ and conclude the proof. We let

$$\rho = \inf\left\{R > R_0 \ : \ \int_R^{+\infty} \left((\partial_r f_0)^2 + f_1^2\right) r^2 dr = 0\right\}$$

and prove that $\rho = R_0$ i.e.that $u_0(r) = W_\lambda(r)$ for $r > R_0$.

We argue by contradiction, assuming $\rho > R_0$. By the preceding step and finite speed of propagation, the essential support of $f$ is included in $\{r \leq \rho + |t|\}$. Thus $f$ is solution of

$$\begin{cases} \partial_t^2 f - \Delta f = \mathbb{1}_{\{|x| \leq \rho + |t|\}} D_\lambda(f). \\ \vec{f}_{\restriction t=0} = (f_0, f_1) := (u_0 - W_\lambda, u_1), \end{cases}$$

Fix $R''_\varepsilon \in (1, \rho)$ such that,

$$\int_{R''_\varepsilon}^{+\infty} \left(|\partial_r f_0(r)|^2 + |f_1(r)|^2\right) r^2 dr \leq \varepsilon^2$$

$$\int_{\mathbb{R}} \left(\int_{R''_\varepsilon + |t|}^{\rho+|t|} W_\lambda^{10}(r) r^2\, dr\right)^{\frac{1}{2}} dt \leq \varepsilon^5.$$

The same argument as in the preceding step, replacing $R'_\varepsilon$ by $R''_\varepsilon$, yields that $(f_0, f_1) = 0$ for almost every $r > R''_\varepsilon$, which contradicts the definition of $\rho$. The proof is complete.

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