Dynamics of semilinear wave equation
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Contents

CHAPTER I

Linear wave equation: classical theory

I.1. Presentation of the equation

The linear wave equation is the equation:

$$\partial_t^2 u - \Delta u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{LW}$$

where $N \geq 1$ is the spatial dimension (in this course, we will often assume $N = 3$), and

$$\Delta = \sum_{k=1}^{N} \frac{\partial^2}{\partial x_k^2}.$$

(We will use either the notations ∂_y or $\frac{\partial}{\partial y}$ for the derivative with respect to the variable $y \in \{t, x_1, \ldots, x_N\}$).

This is an evolution equation: we fix initial data at a certain time $t = t_0$, and are interested in the evolution of the equation over time t. Since the equation is of order 2, we actually fix an initial data for $\vec{u} = (u, \partial_t u)$:

$$\vec{u}_{\restriction t=t_0} = (u_0, u_1) \tag{I.1.1}$$

where (u_0, u_1) is to be taken in a certain functional space.

We will consider in this course initial data *with real values*. The passage to complex or vector values is immediate for most properties of the equation (LW) (by working coordinate by coordinate), but can induce drastic changes in the nonlinear case, if the nonlinearity mixes the components.

Equation (LW) is invariant under several obvious space-time transformations. If u is a solution, it is also the case of

$$\mu u(t - t_0, \lambda(Rx - x_0)),$$

where $\mu \in \mathbb{R}$, $t_0 \in \mathbb{R}$, $\lambda > 0$, $R \in \mathcal{O}_N(R)$, $x_0 \in \mathbb{R}^N$. It is in fact invariant under a larger group of linear transformations, the Lorentz group (cf Exercise I.8 p. 16).

As a consequence, we can limit ourselves, without loss of generality, to the case of an initial time $t_0 = 0$, i.e.

$$\vec{u}_{\restriction t=0} = (u_0, u_1) \tag{ID}$$

Furthermore, the equation is invariant under time inversion: if u is solution, it is also the case of $t \mapsto u(-t, x)$. It is thus a reversible equation.

We will also consider the equation with a force:

$$\partial_t^2 u - \Delta u = f, \tag{I.1.2}$$

(still with an initial condition of type (ID)), whose understanding will be crucial for the study of the nonlinear wave equation.

The Cauchy problem (LW), (ID) can be approached in at least 3 different ways:

- The classical approach which consists in finding an explicit formula to express the solution. It works when the initial data is sufficiently regular ($C^3 \times C^2$ in dimension 3 of space) and gives classical solutions (that is to say C^2 in (t,x) and satisfying (LW) in the sense of classical differentiation).
- The use of the Fourier transformation in space, which is very simple (once the Fourier transformation is known) and particularly effective in Sobolev spaces based on L^2 (which are natural spaces for the study of the equation due to the conservation of energy and other L^2-based quantities). This method allows to obtain weak solutions with degrees of regularity lower than the previous ones, and to use tools based on the Fourier transformation, which can be useful, for example, to prove certain dispersive properties of the equation.
- The "functional analysis" approach, by the theory of semi-groups, which gives the same type of solutions as the previous method.

In this chapter, we will detail the classical method, first by writing the explicit formula for solutions in dimension 1 of space, then in higher dimensions. We will study in the following chapter the equation in the energy space by the Fourier transformation. This chapter is partly based on Chapter 5 of the beautiful book by Folland on partial differential equations [**15**].

I.2. Explicit Formula in Dimension 1

In dimension 1, the equation (LW) can be written as:

$$(\partial_t^2 - \partial_x^2)u = 0, \tag{I.2.1}$$

which can be written $(\partial_t - \partial_x)(\partial_t + \partial_x)u = 0$. We thus make the change of variables $\eta = x + t$, $\xi = x - t$. Setting $v(\eta,\xi) = u\left(\frac{\eta-\xi}{2}, \frac{\eta+\xi}{2}\right)$, or $u(t,x) = v(t+x, t-x)$, we have:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 v}{\partial \eta^2} + \frac{\partial^2 v}{\partial \xi^2} + 2\frac{\partial^2 v}{\partial \xi \partial \eta},$$

and

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial \eta^2} + \frac{\partial^2 v}{\partial \xi^2} - 2\frac{\partial^2 v}{\partial \xi \partial \eta},$$

which gives:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -4\frac{\partial^2 v}{\partial \eta \partial \xi}.$$

Thus, we obtain:

$$\text{(LW)} \iff \frac{\partial^2 v}{\partial \eta \partial \xi} = 0.$$

Let u be a C^2 solution of (I.2.1), (ID). Thus, $u_1 \in C^1(\mathbb{R})$ and $u_0 \in C^2(\mathbb{R})$.

The equality $\frac{\partial^2 v}{\partial \eta \partial \xi} = 0$ shows that $\frac{\partial v}{\partial \xi}$ is a (class C^1) function $w(\xi)$ independent of η. Integrating with respect to ξ for fixed η, we deduce:

$$v(\eta,\xi) = \underbrace{\int_0^\xi w(\sigma)d\sigma}_{\varphi(\xi)} + \psi(\eta),$$

for a certain function ψ, necessarily C^2 since v is of class C^2 and w of class C^1. Thus, we necessarily have:

$$v(\eta,\xi) = \varphi(\xi) + \psi(\eta), \quad \varphi, \psi \in C^2(\mathbb{R}^2),$$

or equivalently:

$$u(t,x) = \varphi(x-t) + \psi(x+t). \tag{I.2.2}$$

Using the initial condition (ID), a direct calculation gives:

$$\psi(\eta) = \frac{1}{2}\int_0^\eta u_1(\sigma)d\sigma + \frac{1}{2}u_0(\eta) + c,$$

$$\varphi(\xi) = -\frac{1}{2}\int_0^\xi u_1(y)dy + \frac{1}{2}u_0(\xi) - c,$$

where $c \in \mathbb{R}$ (the choice of this constant is irrelevant). Hence, we deduce:

$$u(t,x) = \frac{1}{2}(u_0(x+t) + u_0(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} u_1(y)dy. \tag{I.2.3}$$

Conversely, it is easy to verify that formula (I.2.3) gives a C^2 solution of (I.2.1), (ID). Therefore, we have shown:

Proposition I.2.1. *Let $(u_0, u_1) \in C^2(\mathbb{R}) \times C^1(\mathbb{R})$. Then, there exists a unique solution $u \in C^2(\mathbb{R} \times \mathbb{R})$ of* (LW) *satisfying the initial condition* (ID). *This solution satisfies formula* (I.2.3).

On formula (I.2.2), we observe that a solution of the wave equation in dimension 1 is the sum of two waves: one, $\varphi(x-t)$, moving at speed 1 to the right, and the other $\psi(x+t)$, moving at the same speed to the left.[1]

It is also possible to obtain a formula for the equation with the right-hand side (I.1.2). We leave this as an exercise to the reader. Further on, we will provide a general method giving the solution of the equation with the right-hand side in terms of the equation without the right-hand side.

[1]Note that the equations (LW), (I.2.1) have been normalized, so that the speed of propagation is exactly 1.

We can see from formula (I.2.3) that $u(t,x)$ depends only on the values of (u_0, u_1) over $\Big[x-|t|, x+|t|\Big]$. This is a first example of "finite speed of propagation" which holds in all spatial dimensions.

I.3. Integral on the Sphere and Divergence Theorem

We denote $S^{N-1} = \{x \in \mathbb{R}^N,\ |x| = 1\}$, where $|\cdot|$ represents the Euclidean norm on $\mathbb{R}^N$:

$$|x|^2 = \sum_{j=1}^{N} x_j^2.$$

More generally, S_R^{N-1} will denote the sphere of radius R: $\{x \in \mathbb{R}^N,\ |x| = R\}$.

We denote $d\sigma$ as the volume element on one of these spheres. Thus, the integral of a function $f \in \mathcal{L}^1(S_R^{N-1})$ (i.e., a function integrable on S_R^{N-1}) is written as

$$\int_{S_R^{N-1}} f(y)d\sigma(y).$$

This integral can be calculated using spherical coordinates. In dimension 3, this writes:

$$\int_{S_R^2} f(y)d\sigma(y) = R^2 \int_0^{2\pi} \int_0^{\pi} f(R\sin\theta\cos\varphi, R\sin\theta\sin\varphi, R\cos\theta)\sin(\theta)d\theta d\varphi.$$

We denote $B_R^N(x_0)$ as the ball centered at x_0 with radius R:

$$B_R^N(x_0) = \Big\{x \in \mathbb{R}^N,\ |x-x_0| < R\Big\}$$

and simply $B_R^N = B_R^N(0)$.

We will use the following formulas:

Scaling:

$$\int_{S_R^{N-1}} f(y)d\sigma(y) = R^{N-1}\int_{S^{N-1}} f(Ry)d\sigma(y) \quad f \in \mathcal{L}^1(S_R^{N-1}).$$

Integral in radial coordinates: if $f \in \mathcal{L}^1(\{|x| \le R\})$,

$$\int_{B_R^N} f(x)dx = \int_0^R \int_{S_r^{N-1}} f(y)d\sigma(y)dr = \int_0^R \int_{S^{N-1}} f(r\omega)d\sigma(\omega)r^{N-1}dr$$

Divergence theorem: if $F \in C^1(\overline{B_R}, \mathbb{R}^N)$,

$$\int_{|x|\le R} \nabla\cdot F(x)dx = \int_{S_R^{N-1}} \frac{y}{|y|}\cdot F(y)d\sigma(y),$$

where $\nabla\cdot F = \sum_{j=1}^N \partial_{x_j}F_j$ is the divergence of the vector field F.

I.4. Energy density. Uniqueness and finite speed of propagation

Before giving an explicit formula for the wave equation in dimension 3, we prove a uniqueness result valid in any dimension:

THEOREM I.4.1. *Let* $(t_0, x_0) \in \mathbb{R}^{1+N}$, $t_1 > t_0$, $R > 0$. *We denote* $\Gamma = \Big\{(t,x) \in \mathbb{R}\times\mathbb{R}^N \ :\ t_0 \le t \le t_1,\ |x-x_0| \le R-|t-t_0|\Big\}$. *Let* $u \in C^2(\Gamma)$ *be a solution of* (LW) *on* Γ. *We suppose* $(u, \partial_t u)(t_0, x) = 0$ *for all* $x \in B_R(x_0)$. *Then* u *is identically zero on* Γ.

The proof of the theorem is based on a monotonicity law that has its own interest.

We define, for $(t,x) \in \Gamma$, the *density of energy* e_u as

$$e_u(t,x) = \frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{2}(\partial_t u(t,x))^2,$$

where $|\nabla u|^2 = \sum_{j=1}^N (\partial_{x_j}u)^2$, and we consider, for $t_0 \le t \le t_1$, the local energy

$$E_{\text{loc}}(t) = \int_{B_{R-(t-t_0)}(x_0)} e_u(t,x)dx = \int_{|x-x_0|<R-(t-t_0)} e_u(t,x)dx.$$

LEMMA I.4.2. *The function* E_{loc} *is decreasing on* $[t_0, t_1]$.

The lemma immediately implies Theorem I.4.1. Indeed, if $\vec{u}(t_0)$ vanishes on $B(x_0,R)$, then $E_{\mathrm{loc}}(t_0)=0$, and thus $E_{\mathrm{loc}}(t)=0$ for all $t\in[t_0,t_1]$, showing that u is zero on Γ.

Proof of Lemma I.4.2. We notice that

$$\frac{\partial e}{\partial t}=\sum_{j=1}^{N}\left(\partial_{x_j}u\partial_t\partial_{x_j}u+\partial_{x_j}^2u\partial_t u\right)=\sum_{j=1}^{N}\frac{\partial}{\partial x_j}\left(\partial_{x_j}u\partial_t u\right)=\nabla\cdot\left(\partial_t u\nabla u\right), \tag{I.4.1}$$

where $\nabla u=(\partial_{x_i}u)_{1\leq i\leq N}$ Without loss of generality, we can assume that $x_0=0$ and $t_0=0$. By the integration formula in radial coordinates,

$$E_{\mathrm{loc}}(t)=\int_0^{R-t}s^{N-1}\int_{S^{N-1}}e_u(t,s\omega)d\sigma(\omega)ds.$$

By differentiation under the integral sign, we get that E_{loc} is differentiable and

$$E_{\mathrm{loc}}'(t)=-(R-t)^{N-1}\int_{S^{N-1}}e_u(t,(R-t)\omega)d\sigma(\omega)+\int_{B_{R-t}^N}\frac{\partial e_u}{\partial t}(t,x)dx.$$

By formula (I.4.1), then the divergence formula

$$\int_{B_{R-t}^N}\frac{\partial e_u}{\partial t}(t,x)dx=\int_{B_{R-t}^N}\nabla\cdot\left(\partial_t u\nabla u\right)(t,x)dx=\int_{S_{R-t}^{N-1}}\frac{y}{|y|}\nabla u\partial_t u(t,y)d\sigma(y).$$

We thus have

$$E_{\mathrm{loc}}'(t)=-\int_{S_{R-t}^{N-1}}\left(\frac{1}{2}|\nabla u|^2+\frac{1}{2}(\partial_t u)^2-\frac{y}{|y|}\nabla u\partial_t u(t,y)\right)d\sigma(y)\leq-\frac{1}{2}\int_{S_{R-t}^{N-1}}\left(\frac{y}{|y|}\nabla u+\partial_t u(t,y)\right)^2d\sigma(y).$$

□

I.5. Explicit formulas.

This section is devoted to explicit formulas in space dimensions $N\geq 2$. In dimension $N=3$, we will show that for any initial data $(u_0,u_1)\in C^3\times C^2$, there exists a unique solution $u\in C^2(\mathbb{R}^{1+3})$ of (LW), (ID), and provide an explicit formula for this solution. We will also provide a formula in dimension $N=2$. We refer the reader to [**15**, Chapter 5B] for expressions of solutions when $N\geq 4$.

5.a. The radial case in dimension 3. When the initial conditions depend only on the variable $r=|x|$, the explicit formula is very simple.

We start by observing that if f depends only on the variable r, then the function f is C^2 as a function on $\mathbb{R}^3$ if and only if it is C^2 as a function of the variable r on $[0,\infty[$, and satisfies $\frac{df}{dr}(0)=0$. Moreover,

$$\Delta f=\frac{d^2f}{dr^2}+\frac{2}{r}\frac{df}{dr}$$

(cf Exercise I.1). We notice that we can rewrite this formula as

$$r\Delta f=\frac{d^2}{dr^2}(rf).$$

Now let u be a C^2 solution of (LW), (ID) with initial data (u_0,u_1) assumed to be radial. We also assume that for all t, $u(t)$ is a radial function. We will show a posteriori that this second assumption is a consequence of the assumption on the initial data. The previous formula gives

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial r^2}\right)(ru)=0.$$

The function $(t,r)\mapsto ru(t,r)$ is thus a solution of the wave equation in dimension 1, on $\mathbb{R}_t\times]0,\infty[$. To obtain a function on $\mathbb{R}^2$, we extend $ru(t,r)$ to an odd function:

$$v(t,y)=yu(t,|y|).$$

One can verify (using Exercise I.1) that v is of class C^2 on $\mathbb{R}^2$, and that

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial y^2}\right)v=0.$$

Formula (I.2.3) then gives:

$$v(t,y)=\frac{1}{2}(v_0(y+t)+v_0(y-t))+\frac{1}{2}\int_{y-t}^{y+t}v_1(\sigma)d\sigma,$$

where $(v_0,v_1)=\vec{v}_{\restriction t=0}$, thus

$$u(t,r)=\frac{1}{2r}\Big((r+t)u_0(|r+t|)+(r-t)u_0(|r-t|)\Big)+\frac{1}{2r}\int_{r-t}^{r+t}\sigma u_1(|\sigma|)d\sigma. \tag{I.5.1}$$

Notice that when $t>0$ (to fix ideas),

$$\int_{r-t}^{r+t}\sigma u_1(|\sigma|)d\sigma=\int_{|r-t|}^{r+t}\sigma u_1(|\sigma|)d\sigma.$$

The finite speed of propagation is satisfied: the solution $u(t,r)$ depends only on the initial condition (u_0,u_1) on the ball centered at r with radius $|t|$.

The formula (I.5.1) defines a function $u(t,r)$ of class C^2 outside the origin $x=0$, as soon as the initial conditions (u_0,u_1) have the expected regularity $C^2\times C^1$. However, there is a subtle phenomenon of loss of regularity of the solution u compared to the initial data at the origin : there exist data $(u_0,u_1)\in C^2\times C^1$ such that u, defined by formula (I.5.1), cannot be extended by a C^2 function up to $r=0$. Indeed, it can be checked that (at fixed t),

$$\lim_{r\to 0}u(t,r)=u_0(|t|)+|t|u_0'(|t|)+tu_1(|t|), \tag{I.5.2}$$

which shows that if (u_0,u_1) are $C^k\times C^{k-1}$ functions, then $u(t,0)$ is only C^{k-1} in general (see also Exercise I.2). We can interpret this phenomenon physically as follows: a singularity on the circle $r=r_0$ at the initial time 0 that travels at speed 1 towards the origin will concentrate at the origin at time $t=r_0$, causing a stronger singularity.

The limit (I.5.2) suggests a maximal loss of regularity of a derivative with respect to the initial data, which is indeed the case:

PROPOSITION I.5.1. *Let* $(u_0,u_1)\in(C^3\times C^2)(\mathbb{R}^3)$ *be radial functions. Then formula* (I.5.1) *extended by* $u(t,0)=u_0(t)+tu_0'(t)+tu_1(t)$, *defines a* C^2 *function on* $\mathbb{R}\times\mathbb{R}^3$, *radial with respect to the variable* x, *and satisfying* (LW), (ID).

The Proposition I.5.1 is left as an exercise to the reader. Combining with the uniqueness property (Theorem I.4.1), we obtain that (I.5.1) gives the unique solution of (LW) with initial data (u_0,u_1).

The formula (I.5.1) is remarkably simple. In higher space dimensions, we also have an explicit formula for radial solutions, which becomes more complicated as the dimension increases (see Exercise I.3). The loss of regularity observed in dimension 3 (and absent in dimension 1) increases with dimension, as the reader can verify on the formula obtained in Exercise I.3.

There is no simple formula in the radial case in even dimensions.

We also have explicit formulas (of course more complicated) without radiality assumptions, in all dimensions. We will explicitly state these formulas when $N=3$, then $N=2$.

5.b. General solutions in dimension 3: averaging over spheres. We start with computation in general dimension $N\geq 2$. If $f\in C^0(\mathbb{R}^N)$ and $r\in\mathbb{R}$, we let

$$(M_f)(r,x)=\frac{1}{\sigma_N}\int_{S^{N-1}}f(x+ry)d\sigma(y), \tag{I.5.3}$$

where σ_N is the surface of the sphere S^{N-1}. When $r>0$, it is the average of f over the sphere of radius r and center x, which can also be expressed as:

$$(M_f)(r,x)=\frac{1}{\sigma_N r^{N-1}}\int_{S_r^{N-1}}f(x+z)d\sigma(z)$$

We recall that $\sigma_3=4\pi$.

Note that in the definition of M_f, r and x are two independent variables (with contrast with the previous subsection where we had $r=|x|$).

Denote

$$\Delta_r=\frac{\partial^2}{\partial r^2}+\frac{N-1}{r}\frac{\partial}{\partial r}$$

the restriction of the Laplace operator on radial functions in dimension N. Then:

LEMMA I.5.2. *Let $f \in C^2(\mathbb{R}^N)$. Then $M_f \in C^2(\mathbb{R} \times \mathbb{R}^N)$ and, for $r > 0$,*

$$\Delta_r M_f = M_{\Delta f} = \Delta_x M_f.$$

PROOF. The fact that M_f is C^2 is a direct consequence of the theorem of differentiation under the integral sign.

We have

$$\frac{\partial}{\partial r} M_f(r,x) = \frac{1}{\sigma_N} \int_{S^{N-1}} \omega \cdot \nabla f(x + r\omega) d\sigma(\omega) = \frac{1}{\sigma_N} \int_{|y|<1} (r\Delta f)(x + ry) dy,$$

by the divergence formula. Thus

$$\frac{\partial}{\partial r} M_f(r,x) = \frac{1}{\sigma_N r^{N-1}} \int_0^r \int_{S^{N-1}} (\Delta f)(x + s\omega) d\sigma(\omega) s^{N-1} ds.$$

As a consequence,

$$\frac{\partial}{\partial r}\left(r^{N-1} \frac{\partial}{\partial r} M_f(r,x) \right) = \frac{r^{N-1}}{\sigma_N} \int_{S^{N-1}} (\Delta f)(x + r\omega) d\sigma(\omega).$$

This is exactly the equality $\Delta_r M_f = M_{\Delta f}$. The equality $\Delta_x M_f = M_{\Delta f}$ is immediate from the theorem of differentiation under the integral sign. □

Consider a function $u \in C^2(\mathbb{R} \times \mathbb{R}^N)$. By the Lemma, for all $x \in \mathbb{R}^N$,

$$\partial_t^2 M_{u(t)}(r,x) - \Delta_r M_{u(t)}(r,x) = M_{(\partial_t^2 u - \Delta u)(t)}(r,x), \quad t \in \mathbb{R},\ r > 0$$

where we have denoted by $u(t)$ the function $x \mapsto u(t,x)$, and used the fact that $\partial_t^2 M_{u(t)} = M_{\partial_t^2 u(t)}$.

Thus u is solution to the linear wave equation on $\mathbb{R} \times \mathbb{R}^N$ if and only if $M_{u(t)}(r,x)$ is, for all x, a solution of the linear wave equation in radial coordinates. In space dimension $N = 3$, we can deduce from the explicit formula obtained in the preceding subsection the following formula for the solutions of (LW) without symmetry:

THEOREM I.5.3. *Let $(u_0, u_1) \in C^3(\mathbb{R}^3) \times C^2(\mathbb{R}^3)$. Then the wave equation* (LW) *with initial conditions* (ID) *has a unique solution, given by the formula:*

$$u(t,x) = tM_{u_1}(t,x) + \frac{\partial}{\partial t}(tM_{u_0}(t,x)). \tag{I.5.4}$$

PROOF. Let u be a C^2 solution of (LW), (ID) with $N = 3$. Then for all x, $(t,r) \mapsto M_{u(t)}(r,x)$ is a C^3 solution of the radial wave equation in space dimension 3

$$(\partial_t^2 - \Delta_r) M_{u(t)}(r,x) = 0.$$

We fix t and x, and let $r > 0$ goes to 0. By the formula (I.5.2) of the trace of a radial solution at $r = 0$, we obtain

$$u(t,x) = \lim_{r \to 0} M_{u(t)}(r,x) = tv_1(t,x) + \frac{\partial}{\partial t}(tv_0(t,x)),$$

where $v_0(r,x) = M_{u(t)}(r,x)_{\restriction t=0}$ and $v_1(r,x) = \frac{\partial}{\partial t} M_{u(t)}(r,x)_{\restriction t=0}$. By the definition of $M_{u(t)}$ and differentiation under the integral sign, we see that

$$v_0(t,x) = M_{u_0}(t,x), \quad v_1(t,x) = M_{u_1}(t,x),$$

which proves that u must satisfies (I.5.4).

It remains to check that (I.5.4) yields a solution of (LW), (ID). Letting u be definied by this formula, we see that u is C^2, and that $u(0) = u_0$, $\partial_t u(0) = u_1$ (to check this last point, observe that $M_f(t,x)$ is an even function in t, and thus its first derivative in t vanishes at $t = 0$).

Furthermore, we have

$$\partial_t^2(tM_{u_1}(t,x)) = 2\partial_t M_{u_1}(t,x) + t\partial_t^2 M_{u_1}(t,x) = t\Delta_x M_{u_1}(t,x).$$

by Lemma I.5.2. This proves that $tM_{u_1}(t,x)$ satisfies (LW). Since $\partial_t^2 - \Delta$ and ∂_t commute, the same proof yields that $\frac{\partial}{\partial t}(tM_{u_0}(t,x))$ satisfies (LW), concluding the proof. □

Notice that we can rewrite the formula of the theorem as:

$$u(t,x) = tM_{u_1}(t,x) + M_{u_0}(t,x) + tM_{y \cdot \nabla u_0}(t,x). \tag{I.5.5}$$

The preceding method remains valid in any space dimension N: the only point where we have used that $N = 3$ is when we have applied the explicit formula for radial solutions of the wave equation in space dimension 3. In higher odd dimensions, similar (although more complicated) formulas for radial solutions exist (see Exercise I.1), and we can thus deduce a formula for general solutions of (LW). In even space dimension, there is no easy formula for radial solutions, and the explicit formula follows from the *method of descent*, which is described below in the case $N = 2$.

We now give two important consequences of the previous formula.

Corollary I.5.4 (Strong Huygens' principle). *The solution $u(t, x)$ depends only on the values of u_0, ∇u_0, and u_1 on the sphere centered at x and of radius $|t|$.*

Remark I.5.5. The strong Huygens' principle is a stronger version of the finite speed of propagation property, which states that $u(t, x)$ depends only on the values of (u_0, u_1) on the *ball* centered at x and of radius $|t|$. This principle remains valid in any odd dimension ≥ 3 (the number of derivatives of u_0 and u_1 in the statement increases with the dimension). In even dimension, solutions only satisfy the finite speed of propagation: see §5.c. In dimension 1, as shown by formula (I.2.3), only solutions that are even in time (i.e. with initial condition of the form $(u_0, 0)$) satisfy the strong Huygens' principle.

The second consequence of the explicity formula proved above is an estimate related to the dispersive nature of the wave equation. We will denote

$$\|\varphi\|_{\dot{W}^{s,p}} = \sup_{|\alpha|=s} \|\partial_x^\alpha \varphi\|_{L^p(\mathbb{R}^N)}. \tag{I.5.6}$$

We prove:

Theorem I.5.6 (Dispersion inequality). *Let $(u_0, u_1) \in (C^3 \times C^2)(\mathbb{R}^3)$, with compact support and u the solution of* (LW), (ID). *Then for all $t > 0$,*

$$\|u(t)\|_{L^\infty(\mathbb{R}^3)} \lesssim \frac{1}{t} \left(\|u_0\|_{\dot{W}^{2,1}} + \|u_1\|_{\dot{W}^{1,1}} \right).$$

Proof. By space translation invariance it is sufficient to bound $|u(t, 0)|$. We have

$$4\pi u(t, 0) = t \int_{S^2} u_1(ty) d\sigma(y) + \int_{S^2} u_0(ty) d\sigma(y) + t \int_{S^2} y \cdot \nabla u_0(ty) d\sigma(y).$$

By the divergence theorem (denoting by B^3 the unit ball of $\mathbb{R}^3$),

$$t \int_{S^2} u_1(ty) d\sigma(y) = t \int_{B^3} \nabla \cdot (y u_1(ty))\, dy = 3t \int_{B^3} u_1(ty) dy + t^2 \int_{B^3} y \cdot \nabla u_1(ty) dy. \tag{I.5.7}$$

We have

$$\left| \int_{B^3} y \cdot \nabla u_1(ty) dy \right| \leq \frac{1}{t^3} \int_{tB^3} |\nabla u_1(y)| dy \leq \frac{3}{t^3} \|u_1\|_{\dot{W}^{1,1}}, \tag{I.5.8}$$

and

$$\int_{B^3} |u_1(ty)| dy \leq t \int_{\mathbb{R}^3} |\partial_{x_1} u_1(ty)| dy \leq \frac{1}{t^2} \|u_1\|_{\dot{W}^{1,1}}, \tag{I.5.9}$$

where we have used the inequality $\int_{B^3} |\varphi| dx \lesssim \int_{\mathbb{R}^3} |\partial_{x_1} \varphi|$, that follows immediately from the formula

$$\varphi(x_1, x_2, x_3) = \int_{-\infty}^{x_1} \partial_{x_1} \varphi(s, x_2, x_3) ds.$$

Combining (I.5.7), (I.5.8) and (I.5.9), we obtain

$$\left| t \int_{S^2} u_1(ty) d\sigma(y) \right| \lesssim \frac{1}{t} \|u_1\|_{\dot{W}^{1,1}}. \tag{I.5.10}$$

By the same proof, using also the inequality $\int_{B^3} |\varphi| \lesssim \int_{\mathbb{R}^3} |\partial_{x_1} \partial_{x_2} \varphi|$, we have

$$\left| \int_{S^2} u_0(ty) d\sigma(y) \right| + \left| \int_{S^2} y \cdot \nabla u_0(ty) d\sigma(y) \right| \lesssim \frac{1}{t} \|u_0\|_{\dot{W}^{2,1}}. \tag{I.5.11}$$

This concludes the proof of the dispersion inequality. □

5.c. Dimension $1+2$. A solution u of equation (LW) with $N=2$ is also a solution of the same equation with $N=3$, constant with respect to the 3rd spatial coordinate. From Theorem I.5.3, one can derive an expression of u from the initial data. This strategy is called "descent method".

THEOREM I.5.7. *Let $(u_0,u_1)\in(C^3\times C^2)(\mathbb{R}^2)$. Then equation* (LW) *has a unique C^2 solution on $\mathbb{R}\times\mathbb{R}^2$, given by the formula*

$$u(t,x)=\frac{1}{2\pi}\left[\frac{\partial}{\partial_t}\left(t\int_{|y|\le 1}\frac{u_0(x+ty)}{\sqrt{1-|y|^2}}dy\right)+t\int_{|y|\le 1}\frac{u_1(x+ty)}{\sqrt{1-|y|^2}}dy\right]. \tag{I.5.12}$$

PROOF. Uniqueness follows from Theorem I.4.1. Moreover, as in the proof of Theorem I.5.3, the formula for even solutions in time (with initial condition $(u_0,0)$) can be easily deduced from the formula for odd solutions in time (with initial condition $(0,u_1)$). So we only consider this second case.

Let u be a C^2 solution of (LW) on $\mathbb{R}\times\mathbb{R}^2$, with initial data $(u,\partial_t u)(0)=(0,u_1)$, where $u_1\in C^2(\mathbb{R}^2)$. By Theorem I.5.3, considering u as a solution on $\mathbb{R}\times\mathbb{R}^3$, we obtain:

$$u(t,x_1,x_2)=\frac{t}{4\pi}\int_{S^2}\tilde{u}_1((x_1,x_2,0)+ty)d\sigma(y)dy,$$

where by definition $\tilde{u}_1(x_1,x_2,x_3)=u_1(x_1,x_2)$. Passing to spherical coordinates, we get

$$\begin{aligned}&\int_{S^2}\tilde{u}_1((x_1,x_2,0)+ty)d\sigma(y)\\ &=\int_0^{2\pi}\int_0^{\pi}u_1(x_1+t\sin\theta\cos\varphi,x_2+t\sin\theta\sin\varphi)\sin\theta d\theta d\varphi\\ &=2\int_0^{2\pi}\int_0^{\pi/2}u_1(x_1+t\sin\theta\cos\varphi,x_2+t\sin\theta\sin\varphi)\sin\theta d\theta d\varphi.\end{aligned}$$

The announced formula then follows from the change of variable $y_1=t\sin\theta\cos\varphi$, $y_2=t\sin\theta\sin\varphi$. ☐

It can be seen from the formula in Theorem I.5.7 that the strong Huygens principle is not verified in dimension $1+2$: the solution $u(t,x)$ depends on the values of the initial condition over the entire ball $B^2_{|t|}(x)$, not just on the sphere $\{y\in\mathbb{R}^2\ :\ |x-y|=|t|\}$.

I.6. Conservation Laws

The energy of a solution u on $\mathbb{R}\times\mathbb{R}^N$ is defined as:

$$E(\vec{u}(t))=\int_{\mathbb{R}^N}e_u(t,x)dx=\frac{1}{2}\int_{\mathbb{R}^N}\Big((\partial_t u(t,x))^2+|\nabla u(t)|^2\Big)dx.$$

This is the global version of the local energy considered in §I.4. The energy of a solution is conserved over time.

THEOREM I.6.1. *Let $u\in C^2(\mathbb{R}^{1+N})$ be a solution of* (LW), (ID). *Assume (u_0,u_1) has finite energy. Then for any t, $E(\vec{u}(t))$ is finite and $E(\vec{u}(t))=E(u_0,u_1)$.*

PROOF. One might be tempted to write

$$\frac{d}{dt}(E(\vec{u}(t)))=\int\partial_t e_u(t,x)dx=\int\nabla\cdot(\partial_t u\nabla u)dx=0,$$

but the last equality, obtained by integration by parts ignoring the "boundary" term (i.e., when $|x|\to\infty$) is purely formal. To justify the preceding calculation, we can use the decay of the local energy (Lemma I.4.2). For $R>0$, we define:

$$E_{<R}(\vec{u}(t))=\int_{|x|<R}e_u(t,x)dx.$$

Notice that this quantity is finite as soon as $u\in C^1(\mathbb{R}^{1+N})$. Fix $t>0$. By Lemma I.4.2, for any $R>t$,

$$E_{<R-t}(\vec{u}(t))\le E_{<R}(\vec{u}(0))\le E(u_0,u_1).$$

As we let R tend to $+\infty$, we obtain that $E(\vec{u}(t))$ is finite, and

$$E(\vec{u}(t))\le E(u_0,u_1).$$

Reversing the direction of time, we also obtain the inequality

$$E(u_0,u_1)\le E(\vec{u}(t)).$$

We have shown that the energy is conserved for $t \geq 0$. By applying this result to the solution $(t,x) \mapsto u(-t,x)$, we obtain energy conservation for $t \leq 0$, which concludes the proof. ☐

There exists another (vectorial) conserved quantity, the momentum, defined as

$$P(\vec{u}(t)) = \int \partial_t u(t,x) \nabla u(t,x) dx \in \mathbb{R}^N.$$

PROPOSITION I.6.2. *Let $u \in C^2(\mathbb{R}^{1+N})$ be a solution of* (LW) *with finite energy. Then*

$$\forall t \in \mathbb{R}, \quad P(\vec{u}(t)) = P(u_0, u_1).$$

The proof of this proposition is left as an exercise (see Exercise I.5).

I.7

I.8. Equation with a source term

We now consider the equation with a source term (I.1.2). We will express the solution of this equation in terms of the propagator of the free equation (LW). For $(u_0, u_1) \in C^3 \times C^2(\mathbb{R}^3)$, let $S_L(t)(u_0,u_1)$ denote the solution of (LW) with initial data (u_0,u_1) at $t=0$. We denote $S(t)u_1 = S_L(t)(0,u_1)$, so that

$$S_L(t)(u_0,u_1) = \frac{\partial}{\partial t}\left(S(t)u_0\right) + S(t)u_1.$$

For $u_1 \in C^2$, we recall that

$$(S(t)u_1)(x) = tM_{u_1}(t,x) = t\int_{S^2} u_1(x+ty)d\sigma(y).$$

THEOREM I.8.1 (Duhamel's Formula). *Let $(u_0,u_1) \in (C^2 \times C^3)(\mathbb{R}^3)$ and $f \in C^2(\mathbb{R}\times\mathbb{R}^3)$. Then the equation* (I.1.2), (ID) *has a unique C^2 solution, given by the formula:*

$$u(t) = S_L(t)(u_0,u_1) + \int_0^t S(t-s)f(s)ds.$$

REMARK I.8.2. The Duhamel term $\int_0^t S(t-s)f(s)ds$ can be explicited, see (I.8.1).

PROOF OF THEOREM I.8.1. Uniqueness follows immediately from Theorem I.4.1, since the difference of 2 solutions of (I.1.2) with the same source term f is a solution of (LW). For existence, taking into account Theorem I.5.3, it is sufficient to check that the function

$$U : (t,x) \mapsto \int_0^t S(t-s)f(s)ds$$

is C^2 and satisfies equation (I.1.2) with zero initial conditions.

We have:

$$U(t,x) = \frac{1}{4\pi}\int_0^t (t-s)\int_{S^2} f(s,x+(t-s)y)d\sigma(y)ds, \tag{I.8.1}$$

and the fact that U is C^2 follows from the theorem on differentiation under the integral sign.

Furthermore, using that $S(0)g = 0$ for any function g,

$$\frac{\partial U}{\partial t} = \int_0^t \frac{\partial}{\partial t}\Big(S(t-s)f(s)\Big)ds.$$

Upon further differentiation, we obtain

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial}{\partial t}\Big(S(t-s)f(s)\Big)_{\restriction s=t} + \int_0^t \frac{\partial^2}{\partial t^2}\Big(S(t-s)f(s)\Big)ds = f(t) + \int_0^t \Delta\Big(S(t-s)f(s)\Big)ds = f(t) + \Delta U.$$

where we used that $\frac{\partial}{\partial t}(S(t)g)_{\restriction t=0} = g$ for any function g of class C^2. ☐

REMARK I.8.3. Duhamel's formula is certainly not specific to dimension 3, as shown by the calculation leading to this formula, which is completely independent of dimension. The reader is invited to explicitly rewrite the solution of equation (I.1.2) when $N=1$ and $N=2$.

From Duhamel's formula, we deduce the energy inequality:

Proposition I.8.4. *Let u be a C^2 solution of* (I.1.2) *with $N = 3$ with initial data (u_0, u_1), such that $f \in C^2(\mathbb{R}^{1+3})$. Suppose furthermore that (u_0, u_1) has finite energy, and for all $T > 0$,*

$$\int_{[-T,+T]} \sqrt{\int_{\mathbb{R}^3} |f(t,x)|^2 dx} dt < \infty.$$

Then for all $t > 0$,

$$\sqrt{2E(\vec{u}(t))} \leq \sqrt{2E(u_0, u_1)} + \int_0^t \sqrt{\int_{\mathbb{R}^3} |f(s,x)|^2 dx} ds.$$

Proof. To lighten notations, we will denote:

$$\|\vec{u}(t)\|^2_{\dot{H}^1 \times L^2} = \int_{\mathbb{R}^3} |\nabla u(t,x)|^2 dx + \int_{\mathbb{R}^3} |\partial_t u(t,x)|^2 dx, \quad \|f\|_{L^1(I,L^2)} = \int_I \|f(t)\|_{L^2(\mathbb{R}^3)} dt$$

($\|\cdot\|_{\dot{H}^1}$ is the norm defining the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^3)$, see Section II.2 below). From Duhamel's formula and the conservation of energy for the free equation (LW), it suffices to verify that for all $t > 0$,

$$\left\| \left(\int_0^t S(t-s) f(s) ds, \partial_t \int_0^t S(t-s) f(s) ds \right) \right\|_{\dot{H}^1 \times L^2} \leq \|f\|_{L^1([0,t],L^2)} \tag{I.8.2}$$

By conservation of energy (Theorem I.6.1), we have

$$\Big\| \left(S(t-s)f(s), \partial_t (S(t-s)f(s))\right) \Big\|_{\dot{H}^1 \times L^2} = \|f(s)\|_{L^2},$$

which implies directly (I.8.2) □

I.9. Exercises

Exercise I.1. Let $f : \mathbb{R}^N \to \mathbb{R}$ ($N \geq 1$). Suppose f is radial (i.e. That it depends only on the variable $r = |x| = \sqrt{x_1^2 + x_2^2 + \ldots + x_N^2}$). Denote $f(x) = g(|x|)$, where $g : [0, \infty[\to \mathbb{R}$.

(1) Show that f is continuous on $\mathbb{R}^N$ if and only if g is continuous on $[0, \infty[$.
(2) Show that f is C^1 on $\mathbb{R}^N$ if and only if g is C^1 on $[0, \infty[$ and $g'(0) = 0$.
(3) Show that for any $k \geq 2$, f is C^k on $\mathbb{R}^N$ if and only if g is C^k on $\mathbb{R}^N$ and $g^{(j)}(0) = 0$ for all odd integers $j \leq k$.
(4) Assuming f is C^1, compute $\frac{\partial f}{\partial x_j}$ in terms of g', $j = 1, \ldots, N$. Compute $g'(r)$ in terms of ∇f.
(5) Assuming f is C^2 on $\mathbb{R}^N$, prove the formula

$$\Delta f(x) = g''(|x|) + \frac{N-1}{|x|} g'(|x|).$$

To lighten notation, we use the same notation (f) for functions f and g, and denote $g' = \frac{df}{dr}$, etc...

Exercise I.2 (Loss of regularity for the radial wave equation in dimension $1+3$). Let $k \geq 0$ and f be a *radial* function in $C^k(\mathbb{R}^3)$. Define a function u on $\mathbb{R} \times (\mathbb{R}^3 \setminus \{0\})$, radial with respect to the space variable, by

$$u(t,x) = \frac{1}{2r} \Big((r+t) f(|r+t|) + (r-t) f(|r-t|) \Big),$$

where $r = |x|$. Note that this defines a function of class C^k on $\mathbb{R} \times \left(\mathbb{R}^3 \setminus \{0\}\right)$.

(1) Suppose that f is supported in the annulus $\{\frac{1}{2} \leq |x| \leq 2\}$ and is such that for $|\eta - 1| \leq 1/10$,

$$f(\eta) = \begin{cases} 2 - \eta & \text{if } \eta > 1 \\ \eta & \text{if } \eta < 1 \end{cases}.$$

Calculate $\lim_{r \to 0} u(t,r)$ when $t = 1$, $t > 1$, and $t < 1$ (close to 1). Conclude that u cannot be extended to a continuous function on $\mathbb{R} \times \mathbb{R}^3$.
(2) Similarly, give an example of a C^2 function f such that u cannot be extended to a C^2 function on $\mathbb{R} \times \mathbb{R}^3$.
(3) Assume f is C^3. Show that u defines a C^2 function on $\mathbb{R} \times \mathbb{R}^3$.
(4) Let g be a C^2 radial function on $\mathbb{R}^3$. Show that

$$u(t,r) = \frac{1}{2r} \int_{r-t}^{r+t} \sigma g(|\sigma|) d\sigma,$$

extends to a C^2 function on $\mathbb{R}^3$.

EXERCISE I.3 (Explicit solutions of the radial wave equation in odd space dimension). Let $N \geq 3$ be an odd integer, written as $N = 2k+1$. Let T_k be the operator defined by

$$T_k\phi = \left(r^{-1}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\phi(r)\right).$$

(1) Show that

$$T_k\varphi = \sum_{j=0}^{k-1} c_j r^{j+1}\phi^{(j)} r,$$

for some $c_j \in \mathbb{R}$. Determine c_0 and c_{k-1}.

(2) Show that for any function $\varphi \in C^{k+1}([0,+\infty[)$,

$$\frac{d^2}{dr^2}(T_k\varphi) = \left(r^{-1}\frac{d}{dr}\right)^k (r^{2k}\varphi'(r)).$$

Hint: You can start by verifying that the formula is true when $\varphi(r) = r^m$ for any integer m.

(3) Consider a solution $u(t,x)$ of the linear wave equation in space dimension N, radial with respect to the space variable. Suppose u is C^{k+1} on $\mathbb{R}^{1+N}$. Show prove

$$(\partial_t^2 - \partial_r^2)(T_k u) = 0.$$

Deduce an expression of $T_k u$ in terms of u_0 and u_1.

(4) Express $u(t,r)$ in terms of u_0 and u_1 when $N = 5$. What regularity of u_0 and u_1 is required for u to be C^2 on $\mathbb{R}^{1+5}$?

EXERCISE I.4. Let u be a solution of the wave equation (LW) in spce dimension $N \geq 3$, radial with respect to the space variable. Recall that $\Delta u = \frac{d^2}{dr^2} + \frac{N-1}{r}\frac{d}{dr}$. Suppose $u \in C^2(\mathbb{R}^{1+N})$, with compactly supported initial data. Let

$$v(t,r) = \int_r^\infty \rho\partial_t u(t,\rho)d\rho.$$

Show that v defines a radial solution, of class C^2, to the wave equation in space dimension $N-2$.

EXERCISE I.5 (Conservation of momentum). (1) Let u be a C^2 solution of (LW) on $\mathbb{R} \times \mathbb{R}^N$, and $j \in 1,\dots N$. Let $p_{j,u}(t,x) = \partial_{x_j}u(t,x)\partial_t u(t,x)$. Show

$$\frac{\partial p_{j,u}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x_j}\left((\partial_t u)^2 - |\nabla u|^2\right) + \nabla \cdot V,$$

where V is a C^1 vector field to be specified.

(2) Assume that (u_0, u_1) has finite energy. Justify that

$$P_j(\vec{u}(t)) = \int_{\mathbb{R}^N} p_{j,u}(t,x)dx$$

is defined for all times. Show that this quantity is independent of time. You can start by considering a local version of the momentum

$$\int_{[-R,R]^N} p_{j,u}(t,x)dx \text{ or } \int_{\mathbb{R}^N} p_{j,u}(t,x)\varphi\left(\frac{x}{R}\right)dx$$

then let R tend to $+\infty$. Here φ denotes a C^2 function with compact support equal to 1 in a neighborhood of the origin.

EXERCISE I.6. (1) Let $u_1 \in C^2(\mathbb{R}^3)$ such that

$$\forall t \geq 0,\ \forall x \in \mathbb{R}^3, \quad u_1(x) \geq 0.$$

Assume $u_0 = 0$. Let u be the corresponding solution of (LW). Prove

$$\forall t \geq 0,\ \forall x \in \mathbb{R}^3, \quad u(t,x) \geq 0.$$

(2) Suppose now $N = 1$ or $N = 2$. Let u be the solution of (LW), (ID), with $(u_0, u_1) \in C^3 \times C^2$ (if $N = 2$) or $C^2 \times C^1$ (if $N = 1$).

Show that if $u_1 \geq 0$ and $u_0 = 0$ then $u(t,x)$ has the sign of t for all x and $t \neq 0$.

When $N = 1$, give a weaker sufficient condition on (u_0, u_1) such that:

$$\forall t \geq 0,\ \forall x \in \mathbb{R}, \quad u(t,x) \geq 0.$$

Exercise I.7. Assume $N = 1$ or $N = 2$. Let u be a solution of (I.1.2), with $u_0 = u_1 = 0$, and f of class C^1 (if $N = 1$) or C^2 (if $N = 2$). Express u in terms of f.

Exercise I.8. The *Minkowski spacetime* of dimension N is the space $\mathbb{R}^{1+N}$, equipped with the quadratic form of signature $(1, N)$:

$$g(X) = x_0^2 - \sum_{j=1}^{N} x_j^2 = t^2 - |x|^2 = {}^tXJX,$$

where ${}^tX$ is the transpose of X,

$$X = (x_0, x_1, \dots, x_N),\ t = x_0,\ x = (x_1, \dots, x_N),$$

and $J = [J_{\mu,\nu}]_{0\leq\mu\ \nu\leq N}$ is the matrix such that $J_{0,0} = 1$, $J_{\ell,\ell} = -1$ if $\ell \in 1, \dots, N$, and $J_{\mu,\nu} = 0$ if $\mu \neq \nu$.

The Lorentz group $\mathrm{O}(1, N)$ is the group of real square matrices P of size $1 + N$ which leave the quadratic form g invariant, i.e., such that $g(PX) = g(X)$ for all X in $\mathbb{R}^{1+N}$. In other words, if P is a $(1 + N) \times (1 + N)$ matrix,

$$P \in \mathrm{O}(1, N) \iff {}^tPJP = J.$$

(1) Prove that a function v of class C^2 on $\mathbb{R}^{1+N}$ satisfies the wave equation (LW) if and only if $\mathrm{Tr}(Jv'') = 0$, where v'' is the Hessian matrix $\left[\partial_{x_\mu}\partial_{x_\nu}v\right]_{0\leq\mu,\nu\leq N}$.

(2) Let $P \in \mathrm{O}(1, N)$, $v \in C^2(\mathbb{R}^{1+N})$, and $w(X) = v(PX)$. Then

$$(\partial_t^2 - \Delta)v = 0 \iff (\partial_t^2 - \Delta)w = 0.$$

(3) Prove that the space rotations:

$$\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & R \end{bmatrix}, \quad R \in \mathrm{O}(N)$$

and the Lorentz boosts

$$\mathcal{R}\sigma = \begin{bmatrix} R_\sigma & \mathbf{0} \\ \mathbf{0} & I_{N-1}. \end{bmatrix}, \quad R_\sigma = \begin{bmatrix} \cosh(\sigma) & \sinh(\sigma) \\ \sinh(\sigma) & \cosh(\sigma) \end{bmatrix},$$

where I_{N-1} denotes the identity matrix $(N - 1) \times (N - 1)$ and $\sigma \in \mathbb{R}$ are Lorentz transformations. In these formulas, $\mathbf{0}$ always denotes the zero matrix of appropriate size.

CHAPTER II

The linear equation in Sobolev spaces

II.1. Reminders on the Fourier transform

Here, we recall the definition and basic properties of the Fourier transform on $\mathbb{R}^N$, in the most general framework possible, that of tempered distributions. We omit the proofs. For more details, one can consult, for example, the foundational writings of Laurent Schwartz [**29**], the course of Jean-Michel Bony [**5**], as well as [**2**, Section 1.2] for a quick introduction, and [**20**] for a more in-depth exposition (the first two references are in French).

We begin by introducing a notation: a *multi-index* is an element $\alpha = (\alpha_1, \ldots, \alpha_N)$ of $\mathbb{N}^N$. The order of α is $|\alpha| = \sum_{j=1}^N \alpha_j$. The derivative with respect to α of a function f of class $C^{|\alpha|}$ on $\mathbb{R}^N$ is then defined by:

$$\partial_x^\alpha f = \prod_{j=1}^N \partial_{x_j}^{\alpha_j} f.$$

1.a. Fourier Transform on $\mathcal{S}$.

Definition II.1.1. The Schwartz space $\mathcal{S}(\mathbb{R}^N)$ is the space of functions f of class C^∞ on $\mathbb{R}^N$ such that for every $p \in \mathbb{N}$,

$$N_p(f) := \sup_{x\in\mathbb{R}^N \ |\alpha|\leq p} (1+|x|)^p |\partial_x^\alpha f(x)| < \infty.$$

It can be observed that each N_p is a norm on $\mathcal{S}(\mathbb{R}^N)$, but N_p is not complete for any of these norms.

We equip $\mathcal{S}(\mathbb{R}^N)$ with the distance function

$$d(\varphi, \psi) = \sum_{p\geq 0} \frac{1}{2^p} \min\Big(N_p(\varphi - \psi), 1\Big). \tag{II.1.1}$$

Notice that $d(\varphi_n, \varphi)$ tends to 0 as n tends to infinity if and only if $N_p(\varphi_n - \varphi)$ tends to 0 for every p.

The metric space $(\mathcal{S}, d)$ is complete.[1]

The Fourier transform of an element φ of $\mathcal{S}$ is defined by the formula

$$\widehat{\varphi}(\xi) = \mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} \varphi(x) dx. \tag{II.1.2}$$

One easily checks that $\mathcal{F}$ is a continuous application from $\mathcal{S}$ into $\mathcal{S}$.

Fubini's theorem immediately implies the duality formula:

$$\int_{\mathbb{R}^N} \widehat{\varphi}(\xi)\psi(\xi) d\xi = \int_{\mathbb{R}^N} \varphi(x)\widehat{\psi}(x) dx, \tag{II.1.3}$$

for $\varphi, \psi \in \mathcal{S}(\mathbb{R}^N)$.

The Fourier transformation is a bijection of $\mathcal{S}$: by defining

$$\overline{\mathcal{F}}(\psi)(x) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{ix\cdot\xi} \psi(\xi) d\xi = \frac{1}{(2\pi)^N} \widehat{\psi}(-x), \tag{II.1.4}$$

we have the *Fourier inversion formula*: for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$,

$$\mathcal{F}\overline{\mathcal{F}}\varphi = \overline{\mathcal{F}}\mathcal{F}\varphi = \varphi. \tag{II.1.5}$$

[1]Such a vector space, equipped with a countable family of semi-norms, and which is complete as a metric space (where the distance function is defined as in (II.1.1)), is called a *Fréchet space*. It is a natural generalization of a Banach space when a unique norm is not sufficient to ensure completeness.

By combining the Fourier inversion formula (II.1.5) and the duality formula (II.1.3), we obtain the Plancherel theorem: for all φ, ψ in $\mathcal{S}$,

$$\int_{\mathbb{R}^N} \varphi(x)\overline{\psi}(x)dx = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} \widehat{\varphi}(\xi)\overline{\widehat{\psi}(\xi)}d\xi. \tag{II.1.6}$$

The Fourier transform exchanges multiplication by powers of x with differentiation. For all $\varphi \in \mathcal{S}(\mathbb{R}^N)$

$$\forall \alpha \in \mathbb{N}^N, \quad \mathcal{F}\partial_x^\alpha \varphi = i^{|\alpha|}\xi^\alpha \widehat{\varphi}(\xi), \quad \mathcal{F}(x^\alpha \varphi) = i^{|\alpha|}\partial_\xi^\alpha \widehat{\varphi}(\xi). \tag{II.1.7}$$

1.b. Fourier Transform of Tempered Distributions.

Definition II.1.2. The space $\mathcal{S}'(\mathbb{R}^N)$ of *tempered distributions* is the topological dual of $\mathcal{S}(\mathbb{R}^N)$, i.e., the vector space of continuous linear forms on $\mathcal{S}$.

In the definition, continuity must be interpreted in the sense of the topology induced by the distance d defined by (II.1.1). Using the definition of this topology, one sees that a linear form f on $\mathcal{S}$ is an element of $\mathcal{S}'$ if and only if:

$$\exists p \in \mathbb{N}, \quad \forall \varphi \in \mathcal{S}, \quad |\langle f, \varphi\rangle| \leq CN_p(\varphi).$$

We equip $\mathcal{S}'$ with the topology of pointwise convergence: a sequence $(f_n)_n$ of elements of $\mathcal{S}'$ converges to f in $\mathcal{S}'$ if and only if

$$\forall \varphi \in \mathcal{S}, \quad \lim_{n\to\infty} \langle f_n, \varphi\rangle = \langle f, \varphi\rangle .$$

Several function spaces continuously embed into $\mathcal{S}'(\mathbb{R}^N)$ in the following manner. If f is a measurable, locally integrable function on f such that

$$\forall R > 0, \quad \int_{|x|\leq R} |f(x)|dx \leq C(1+R)^C$$

for some constant $C > 0$, we define an element L_f of $\mathcal{S}'(\mathbb{R}^N)$ by

$$\langle L_f, \varphi\rangle = \int_{\mathbb{R}^N} f(x)\varphi(x)dx.$$

The preceding application is injective, i.e., L_f is null if and only if f is null almost everywhere on $\mathbb{R}^N$. We then identify f with the linear form L_f, also denoted f. The preceding identification allows us to consider $\mathcal{S}$, Lebesgue spaces $L^p(\mathbb{R}^N)$ $(1 \leq p \leq \infty)$, C_b^k (the space of C^k functions on $\mathbb{R}^N$ that are bounded along with all their derivatives up to order k) as subspaces of $\mathcal{S}'$.

Examples of tempered distributions that are not functions are given by the (improperly named) Dirac delta function at a, denoted δ_a and defined by $\langle \delta_a, \varphi\rangle = \varphi(a)$, as well as the surface measure σ on the sphere S^{N-1}, defined by:

$$\langle \sigma, \varphi\rangle = \int_{S^{N-1}} \varphi(y)d\sigma(y).$$

By duality, several actions can be defined on the elements of $\mathcal{S}'$.

Differentiation. Let $\alpha \in \mathbb{N}^N$ and $f \in \mathcal{S}'$. The *derivative* of f of order α is the element ∂_x^α of $\mathcal{S}'$ defined by:

$$\forall \varphi \in \mathcal{S}, \quad \langle \partial_x^\alpha f, \varphi\rangle = (-1)^{|\alpha|}\langle f, \partial_x^\alpha \varphi\rangle .$$

The integration by parts formula shows that if $f \in C_b^{|\alpha|}$, its derivative of order α in the sense of distributions coincides with its derivative in the classical sense.

Multiplication by a Function. We denote by $\mathcal{P} = \mathcal{P}(\mathbb{R}^N)$ the space of C^∞ functions with *slow growth*, i.e., such that

$$\forall \alpha, \quad \exists M, C > 0 \quad \forall x \in \mathbb{R}^N, \quad |\partial_x^\alpha g(x)| \leq C(1+|x|)^M. \tag{II.1.8}$$

It is easy to check that the multiplication by an element of $\mathcal{P}$ defines a continuous mapping from $\mathcal{S}$ into $\mathcal{S}$. We then define, for $f \in \mathcal{S}'$ and $g \in \mathcal{P}$, the product fg by:

$$\langle fg, \varphi\rangle = \langle f, g\varphi\rangle .$$

The product fg is an element of $\mathcal{S}'$. Fixing $g \in \mathcal{P}$, $f \mapsto fg$ is a continuous mapping from $\mathcal{S}'$ into $\mathcal{S}'$.

Fourier Transform. We define the Fourier transform of an element f of $\mathcal{S}'$ by

$$\forall \varphi \in \mathcal{S}, \quad \left\langle \widehat{f}, \varphi \right\rangle = \langle f, \widehat{\varphi} \rangle$$

The duality formula (II.1.3) shows that if $f \in \mathcal{S}$, its Fourier transform according to formula (II.1.2) and its Fourier transform in the sense of $\mathcal{S}'$ coincide.

We recall that $L^1(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N)$ are identified with subspaces of $\mathcal{S}'(\mathbb{R}^N)$. The Fourier transform on $\mathcal{S}'$ thus applies to elements of these two spaces. On $L^1(\mathbb{R}^N)$, we recover the Fourier transform in the classical sense.

Proposition II.1.3 (Fourier Transform in L^1). *Let $f \in L^1(\mathbb{R}^N)$, and $\widehat{f}$ be its Fourier transform in $\mathcal{S}'$. Then $\widehat{f}$ can be identified with the continuous function given by the formula:*

$$\widehat{f}(\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x)dx.$$

The second proposition immediately follows from the Plancherel theorem:

Proposition II.1.4 (Fourier Transform in L^2). *Let $f \in L^2(\mathbb{R}^N)$ then $\widehat{f} \in L^2(\mathbb{R}^N)$ and*

$$\|f\|_{L^2} = \frac{1}{(2\pi)^{N/2}} \|\widehat{f}\|_{L^2}.$$

Indeed, the Fourier inversion formula in $\mathcal{S}'$ (see below) implies that $f \mapsto \frac{1}{(2\pi)^{N/2}}\hat{f}$ is an isometry of $L^2(\mathbb{R}^N)$. The properties of the Fourier transform on $\mathcal{S}$ are transmitted by duality to the Fourier transform:

- We define the inverse Fourier transform $\overline{F}$ of an element f of $\mathcal{S}'$ by

$$\left\langle \overline{F} f, \varphi \right\rangle = \left\langle f, \overline{F}\varphi \right\rangle.$$

  Then we have the Fourier inversion formula:

$$\forall f \in \mathcal{S}', \quad \overline{\mathcal{F}}\mathcal{F} f = \mathcal{F}\overline{\mathcal{F}} f = f.$$

- Property (II.1.7) remains valid for $\varphi \in \mathcal{S}'$.

II.2. Sobolev Spaces

2.a. Definition. (cf [**2**, Section 1.3]) We mainly focus on Sobolev spaces on $\mathbb{R}^N$, of Hilbert type (i.e. based on L^2 spaces). In this section, we consider homogeneous Sobolev spaces $\dot{H}^\sigma$. We refer to the exercise sheet for classical Sobolev spaces H^σ.

The Hilbertian Sobolev spaces on $\mathbb{R}^N$ are easily defined using the Fourier transform:

Definition II.2.1. Let $\sigma \in \mathbb{R}$. The Sobolev space $\dot{H}^\sigma(\mathbb{R}^N)$ is the set of $f \in \mathcal{S}'(\mathbb{R}^N)$ such that $\widehat{f}$ can be identified with a function in $L^1(K)$ for every compact set K, such that the following quantity is finite:

$$\|f\|^2_{\dot{H}^\sigma} = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} |\xi|^{2\sigma} |\widehat{f}(\xi)|^2 d\xi.$$

The space $\dot{H}^\sigma$, equipped with the inner product:

$$(f,g)_{\dot{H}^\sigma} = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} |\xi|^{2\sigma} \widehat{f}(\xi)\overline{\widehat{g}(\xi)} d\xi$$

is a pre-Hilbert space.

Theorem II.2.2. *The space $\dot{H}^\sigma(\mathbb{R}^N)$ is complete if and only if $\sigma < N/2$. In this case, the vector space $\mathcal{S}_0$ of functions in $\mathcal{S}$ whose Fourier transform vanishes in a neighborhood of 0 is dense in $\dot{H}^\sigma(\mathbb{R}^N)$.*

Note that $\dot{H}^0$ is exactly the space L^2.

2.b. Sobolev Inequalities. We have the following Sobolev embedding on $\mathbb{R}^N$.

THEOREM II.2.3. *Let $\sigma \in]0, N/2[$, and $p \in (2,\infty)$ such that $\frac{1}{p} = \frac{1}{2} - \frac{\sigma}{N}$. Then $\dot{H}^\sigma(\mathbb{R}^N)$ is contained in L^p with continuous embedding.*

The result is well-known. We give a proof based on the Fourier transform, which yields a slightly stronger result that we will use in Chapter VI.

By the density result in Theorem II.2.2, it suffices to show that there exists a constant $C > 0$ such that

$$\forall f \in \mathcal{S}(\mathbb{R}^N), \quad \|f\|_{L^p(\mathbb{R}^N)} \leq C\|f\|_{\dot{H}^\sigma(\mathbb{R}^N)}. \tag{II.2.1}$$

Let $f \in \mathcal{S}$. We denote[2]

$$\|f\|^2_{\dot{B}^\sigma} = \sup_{k\in\mathbb{Z}} \frac{1}{(2\pi)^N} \int_{2^k \leq |x| \leq 2^{k+1}} |\xi|^{2\sigma} |\widehat{f}(\xi)|^2 d\xi,$$

and observe that $\|f\|_{\dot{B}^\sigma} \leq \|f\|_{\dot{H}^\sigma}$. We will prove the following result, which implies (II.2.1):

THEOREM II.2.4 (Improved Sobolev Inequality). *Let σ and p be as in the previous theorem. Then there exists a constant $C > 0$ such that*

$$\forall f \in \mathcal{S}(\mathbb{R}^N), \quad \|f\|^p_{L^p} \leq \|f\|^{p-2}_{\dot{B}^\sigma} \|f\|^2_{\dot{H}^\sigma}.$$

NOTATION II.2.5. Let φ be a function on $\mathbb{R}^N$. For $u \in \mathcal{S}'(\mathbb{R}^N)$, we denote

$$\varphi(D)u = \overline{\mathcal{F}}\left(\varphi(\xi)\widehat{u}(\xi)\right).$$

The operator $\varphi(D)$ is called Fourier multiplier (with symbol φ).

The tempered distribution $\varphi(D)u$ is not well-defined for all functions φ and $u \in \mathcal{S}'$: we need $\varphi\,\widehat{u}$ to define a tempered distribution. This is for example the case if $\varphi \in L^\infty$ and $u \in \dot{H}^\sigma$ (in this case $\varphi(D)u \in \dot{H}^\sigma$), or if $\varphi \in \mathcal{P}(\mathbb{R}^N)$ (the space of C^∞ functions with slow growth i.e. that satisfy (II.1.8)).

PROOF. We use a method introduced by Chemin and Xu in [**8**]. We fix a parameter $A > 0$ and decompose f into a *high-frequency* part $f_{>A}$ and a *low-frequency* part $f_{<A}$:

$$f_{>A} = \overline{\mathcal{F}}\left(\mathbb{1}_{|\xi|>A}\widehat{f}(\xi)\right) = \mathbb{1}_{|D|>A} f, \quad f_{<A} = \mathbb{1}_{|D|<A} f = 1 - f.$$

Let $k(A)$ be the largest integer such that $2^{k(A)} \leq A$. By using the Cauchy-Schwarz inequality, then the fact that $\sigma < N/2$, we obtain:

$$|f_{<A}(x)| = \frac{1}{(2\pi)^N} \left| \int_{|\xi|<A} e^{ix\cdot\xi} \widehat{f}(\xi) d\xi \right| \leq \frac{1}{(2\pi)^N} \sum_{k\leq k(A)} \int_{2^k \leq |\xi| \leq 2^{k+1}} |\widehat{f}(\xi)| d\xi$$

$$\leq \frac{1}{(2\pi)^N} \sum_{k\leq k(A)} 2^{k(N/2-\sigma)} \left(\int_{2^k \leq |\xi| \leq 2^{k+1}} |\xi|^{2\sigma} |\widehat{f}(\xi)|^2 d\xi \right)^{1/2} \leq C_N A^{N/2-\sigma} \|f\|_{\dot{B}^\sigma},$$

where C_N depends only on the dimension N. Then we write (using Fubini's equality):

$$\|f\|^p_{L^p} = \int |f(x)|^p dx = \int_{\mathbb{R}^N} p \int_0^{|f(x)|} \lambda^{p-1} d\lambda dx = p \int_0^{+\infty} \lambda^{p-1} \left| \left\{ x \in \mathbb{R}^N \ : \ |f(x)| \geq \lambda \right\} \right| d\lambda,$$

where $|S|$ denotes the Lebesgue measure of the measurable subset S of $\mathbb{R}^N$. Let $A(\lambda)$ be such that

$$C_N A(\lambda)^{\frac{N}{2}-\sigma} \|f\|_{\dot{B}^\sigma} = \lambda/2.$$

For any x in $\mathbb{R}^N$,

$$|f_{<A(\lambda)}(x)| \leq \frac{\lambda}{2}.$$

Thus $|f(x)| > \lambda \Longrightarrow |f_{>A(\lambda)}(x)| > \lambda/2$. Hence:

$$\|f\|^p_{L^p} \leq p \int_0^\infty \lambda^{p-1} \left| \left\{ x \in \mathbb{R}^N \ : \ |f_{>A(\lambda)}(x)| > \lambda/2 \right\} \right| d\lambda$$

By integrating $|f_{>A(\lambda)}|^2$ over the set $\left\{ x \in \mathbb{R}^N \ : \ |f_{>A(\lambda)}(x)| > \lambda/2 \right\}$, we get

$$\left| \left\{ x \in \mathbb{R}^N \ : \ |f_{>A(\lambda)}(x)| > \lambda/2 \right\} \right| \leq \frac{4}{\lambda^2} \|f_{>A(\lambda)}\|^2_{L^2}.$$

[2]This norm defines the Besov space $\dot{B}^\sigma_{2,\infty}$. See [**2**, Section 2.3] for the definition of general Besov spaces.

Combining with the Plancherel theorem, then Fubini's theorem, we obtain

$$\|f\|_{L^p}^p \leq \frac{4p}{(2\pi)^N}\int_0^\infty \lambda^{p-3}\int_{|\xi|>A(\lambda)}|\widehat{f}(\xi)|^2 d\xi d\lambda$$

$$= \frac{4p}{(2\pi)^N}\int_{\mathbb{R}^N}\left|\widehat{f}(\xi)\right|^2\int_0^{c(f,\xi)}\lambda^{p-3}d\lambda d\xi = C_{p,N}\int_{\mathbb{R}^N}\left|\widehat{f}(\xi)\right|^2 c(f,\xi)^{p-2}d\xi,$$

where $c(f,\xi) = 2C_N\|f\|_{\dot{B}^\sigma}|\xi|^{\frac{N}{2}-s}$, and $C_{p,N}$ depends only on N and p. It can be easily verified that $(\frac{N}{2}-\sigma)(p-2) = 2\sigma$, which proves the announced inequality. □

We will focus more particularly on the case $s = 1$. According to the above, the Sobolev space $\dot{H}^1(\mathbb{R}^N)$, $N \geq 3$, is a Hilbert space, contained in $L^{\frac{2N}{N-2}}$, which can be defined as the closure of the space $\mathcal{S}(\mathbb{R}^N)$ (or $C_0^\infty(\mathbb{R}^N)$) for the $\dot{H}^1(\mathbb{R}^N)$-norm. We can characterize this norm with the first-order partial derivatives of f. Indeed,

$$\|f\|_{\dot{H}^1}^2 = \frac{1}{(2\pi)^N}\int |\xi|^2\left|\widehat{f}(\xi)\right|^2 d\xi = \sum_{j=1}^N\int\left|\xi_j\widehat{f}(\xi)\right|^2 d\xi,$$

which shows by Plancherel's theorem and formula (II.1.7)

$$\|f\|_{\dot{H}^1}^2 = \int |\nabla f(x)|^2 dx.$$

The attentive reader will have noticed that the space $\dot{H}^1(\mathbb{R}^N)$ is not the set of $\varphi \in \mathcal{S}'(\mathbb{R}^N)$ such that for all j, $\partial_{x_j}\varphi \in L^2(\mathbb{R}^N)$: indeed, nonzero constant functions are in this space, but not in $\dot{H}^1(\mathbb{R}^N)$ (the Fourier transform $\hat{c}$ of a nonzero constant function is the multiple of a Dirac function, which does not satisfies the assumption of local integrability in the definition of $\dot{H}^1$).

The density result of Theorem II.2.2 implies that $\dot{H}^1(\mathbb{R}^N)$ is the closure of $C_0^\infty(\mathbb{R}^N)$ for the norm $\|\cdot\|_{\dot{H}^1}^2$. An other characterization, using the Sobolev inequality, is given by

$$\dot{H}^1(\mathbb{R}^N) = \left\{f \in L^{\frac{2N}{N-2}}(\mathbb{R}^N),\ |\nabla f| \in L^2(\mathbb{R}^N)\right\}. \tag{II.2.2}$$

The proof of (II.2.2) is left to the reader.

II.3. The wave equation in the Schwartz space

Let $(u_0, u_1) \in \mathcal{S}(\mathbb{R}^N)$. We will write the solution u of (LW), (ID) using the Fourier transformation. We start with a formal calculation, assuming that $u(t) \in \mathcal{S}$ for all t (which we will prove later). We denote $\widehat{u}(t)$ as the Fourier transform of u with respect to the spatial variable, i.e.,

$$\widehat{u}(t,\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi}u(t,x)dx.$$

Thus, we have

$$\widehat{\Delta u}(t,\xi) = -|\xi|^2\widehat{u}(t,\xi),$$

and the wave equation (LW) is formally equivalent to the linear differential equation

$$\partial_t^2\widehat{u}(t,\xi) + |\xi|^2\widehat{u}(t,\xi) = 0,$$

where the variable ξ is considered as a parameter. The solution to this equation, with initial conditions $(\widehat{u}(0), \partial_t\widehat{u}(0)) = (u_0, u_1)$, yields

$$\widehat{u}(t,\xi) = \cos(t|\xi|)\widehat{u}_0(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\widehat{u}_1(\xi),$$

or, with the previously introduced notation,

$$u(t) = \cos(t|D|)u_0 + \frac{\sin(t|D|)}{|D|}u_1. \tag{II.3.1}$$

Theorem II.3.1. *Let* $(u_0, u_1) \in \mathcal{S}(\mathbb{R}^N)^2$. *Then* u *defined by* (II.3.1) *is an element of* $C^\infty(\mathbb{R}\times\mathbb{R}^N)$. *It is the unique* C^2 *solution of* (LW), (ID).

PROOF. Uniqueness follows from Theorem I.4.1. Hence, it suffices to prove that u, defined by (II.3.1), is C^∞ and satisfies (LW), (ID). We have

$$u(t,x) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{ix\cdot\xi}\left(\cos(t|\xi|)\widehat{u}_0(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\widehat{u_1}(\xi)\right)d\xi.$$

By writing

$$\frac{\sin(t|\xi|)}{|\xi|} = t\sum_{k\geq 0}\frac{(-1)^k(t|\xi|)^{2k}}{(2k+1)!},$$

we see that it is a C^∞ function of (t,ξ). Moreover, $\frac{\left|\partial_t^j \sin(t|\xi|)\right|}{|\xi|} \leq |t||\xi|^j$. Similarly, $(t,\xi)\mapsto \cos(t|\xi|)$ is C^∞ and $\left|\partial_t^j \cos(t|\xi|)\right| \leq |\xi|^j$. Using the fact that $\widehat{u}_0$ and $\widehat{u}_1$ are elements of $\mathcal{S}(\mathbb{R}^N)$, by the theorem of differentiation under the integral sign, we obtain that u is C^∞ and satisfies (LW). The Fourier inversion formula shows that u also satisfies the initial conditions (ID). $\square$

II.4. The wave equation in Sobolev spaces

4.a. The equation in general homogeneous Sobolev spaces. Let $(u_0,u_1)\in \dot{H}^\sigma\times\dot{H}^{\sigma-1}$, $\sigma<N/2$. We define as before u by (II.3.1). We also define the formal derivative of u with respect to time:

$$u'(t,x) = \cos(t|D|)u_1 - |D|\sin(t|D|)u_0.$$

Then u and u' satisfy the following properties:

CLAIM II.4.1. *$u\in C^0(\mathbb{R},\dot{H}^\sigma)$, $u'\in C^0(\mathbb{R},\dot{H}^{\sigma-1})$, $u(0)=u_0$, $u'(0)=u_1$. Furthermore*

$$\forall t_0\in\mathbb{R},\quad \lim_{t\to t_0}\frac{u(t)-u(t_0)}{t-t_0} = u'(t_0) \text{ in } \dot{H}^{\sigma-1}. \tag{II.4.1}$$

REMARK II.4.2. Formula (II.4.1) says that u' is the time derivative of u "in the sense of $\dot{H}^{\sigma-1}$". If we let $E^\sigma = \dot{H}^{\sigma-1}+\dot{H}^\sigma$, we have

$$u\in C^1(\mathbb{R},E^\sigma),\quad \partial_t u = u',$$

where $\partial_t u$ is the time derivative in the usual sense, for the E^σ-valued function u. We will prove below that we also have $u'=\partial_t u$ in the sense of distribution on $\mathbb{R}\times\mathbb{R}^N$. In the sequel, we will use the notation $\partial_t u$ instead of u'.

PROOF. Using that $\widehat{u}_0\in L^2(|\xi|^{2\sigma}d\xi)$ and $\widehat{u}_1\in L^2(|\xi|^{2\sigma-2}d\xi)$, it is easy to see that

$$\widehat{u}\in C^0(\mathbb{R},L^2(|\xi|^{2\sigma}d\xi)),\quad \widehat{u'}\in C^0(\mathbb{R},L^2(|\xi|^{2\sigma-2}d\xi)), \tag{II.4.2}$$

which yields the announced continuity property. The facts that $u(0)=u_0$ and $u'(0)=u_1$ follow immediately from the definition.

Let us prove (II.4.1). We have for almost all ξ:

$$\lim_{t\to t_0}\frac{\cos(t|\xi|)\hat{u}_0(\xi)-\cos(t_0|\xi|)\hat{u}_0(\xi)}{t-t_0} = -|\xi|\sin(t_0|\xi|)\hat{u}_0(\xi). \tag{II.4.3}$$

Furthermore letting

$$I(t,\xi) = \left|\frac{\cos(t|\xi|)\hat{u}_0(\xi)-\cos(t_0|\xi|)\hat{u}_0(\xi)}{t-t_0} + |\xi|\sin(t_0|\xi|)\hat{u}_0(\xi)\right|^2|\xi|^{2\sigma-2},$$

we have

$$|I(t,\xi)|\leq 2(1+|t_0|)|\hat{u}_0(\xi)|^2|\xi|^{2\sigma},$$

by the bounds $|\cos(a)-\cos(b)|\leq|a-b|$ and $|\sin a|\leq|a|$. By dominated convergence, we obtain that (II.4.3) is valid in the weighted space $L^2(\mathbb{R}^N,|\xi|^{2\sigma-2}d\xi)$. This means exactly

$$\lim_{t\to t_0}\frac{\cos(t|D|)u_0-\cos(t_0|D|)u_0}{t-t_0} = \sin(t_0|D|)u_0 \text{ in } H^{\sigma-1}.$$

Treating similarly the term $\frac{\sin(t|D|}{|D|}$, we obtain (II.4.1). $\square$

CLAIM II.4.3. $\forall t,\quad \|(u(t),u'(t))\|_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}} = \|(u_0,u_1)\|_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}}.$

PROOF.

$$\int_{\mathbb{R}^N} |\widehat{u}(t,\xi)|^2|\xi|^{2\sigma}d\xi + \int_{\mathbb{R}^N} \widehat{u'}(t,\xi)|\xi|^{2\sigma-2}d\xi$$
$$= \int_{\mathbb{R}^N} \left|\cos(t|\xi|)\widehat{u}_0(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\widehat{u}_1(\xi)\right|^2 |\xi|^{2\sigma}d\xi$$
$$+ \int_{\mathbb{R}^N} \Big| - |\xi|\sin(t|\xi|)\widehat{u}_0(\xi) + \cos(t|\xi|)\widehat{u}_1(\xi)\Big|^2 |\xi|^{2\sigma-2}d\xi$$
$$= \int_{\mathbb{R}^N} \left(|\widehat{u}_0(\xi)|^2 + |\widehat{u}_1(\xi)|^2|\xi|^{-2}\right)|\xi|^{2\sigma}d\xi,$$

which gives the desired property. □

Formula (II.4.1), we see that as functions in $C^0(\mathbb{R}, \dot{H}^{\sigma-2})$:

$$\partial_t^2 u = \Delta u,$$

where Δ is the Fourier multiplier $-|D|^2$ (which maps $\dot{H}^\sigma$ to $\dot{H}^{\sigma-2}$, and ∂_t has to be understood as the derivative in appropriate Sobolev spaces. We will next prove that this equation is also valid in the distributional sense in $\mathbb{R}\times\mathbb{R}^N$.

CLAIM II.4.4. *Let $(u_{0,n}, u_{1,n}) \in (\mathcal{S}_0(\mathbb{R}^N))^2$ such that $(u_{0,n}, u_{1,n})$ converges to (u_0, u_1) in $\dot{H}^\sigma \times \dot{H}^{\sigma-1}$. Let u_n be the solution of* (LW) *with data $(u_{0,n}, u_{1,n})$. Then*

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \|u_n(t) - u(t)\|_{\dot{H}^\sigma} + \|\partial_t u_n(t) - u'(t)\|_{\dot{H}^{\sigma-1}} = 0.$$

PROOF. It follows immediately from the preceding claim, applied to $(u - u_n, u' - \partial_t u_n)$. □

CLAIM II.4.5. *One can identify u with a distribution on $\mathbb{R}\times\mathbb{R}^N$, and it satisfies the wave equation* (LW) *in the distributional sense. Furthermore $u' = \partial_t u$ in the sense of distribution.*

PROOF. We first give a "concrete" proof of these facts for the reader which is not familiar with the theory of distributions, assuming that σ is large enough so that the object considered are all functions on $\mathbb{R}\times\mathbb{R}^N$.

Let $\sigma \geq 0$. We let u_n be as in Claim II.4.4. Using that u_n is a C^∞ solution of (LW) and integrating by parts, we obtain

$$\iint u_n(t,x)(\partial_t^2 - \Delta)\varphi dxdt = 0.$$

Using the Sobolev embedding $\dot{H}^\sigma \subset L^p$, $\frac{1}{p} = \frac{1}{2} - \frac{\sigma}{N}$, and the point (II.4.4), we see that

$$\lim_{n\to\infty} \|u - u_n\|_{L^p(K)} = 0,$$

for all compact K of $\mathbb{R}^N$. This implies

$$0 = \lim_{n\to\infty} \iint u_n(t,x)(\partial_t^2 - \Delta)\varphi dxdt = \lim_{n\to\infty} \iint u(t,x)(\partial_t^2 - \Delta)\varphi dxdt,$$

and thus

$$\forall \varphi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^N), \quad \iint u(\partial_t^2 - \Delta)\varphi dtdx = 0,$$

which is precisely the meaning of $\partial_t^2 u - \Delta u = 0$ in the distributional sense.

Let $\sigma \geq 1$. The equality

$$\partial_t u_n = -|D|\sin(t|D|)u_{0,n} + \cos(t|D|)u_{1,n}.$$

holds by differentiation under the integral sign. By integration by parts,

$$\forall \varphi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^N), \quad \iint \partial_t u_n \varphi dtdx = -\iint u_n \partial_t \varphi dtdx,$$

Letting $n\to\infty$, we obtain

$$\forall \varphi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^N), \quad \iint u' \varphi dtdx = -\iint u \partial_t \varphi dtdx,$$

which means that $u' = \partial_t u$ in the distributional sense.

The proof for general σ is essentially the same, and can be skipped by the reader who is not familiar with distributions.

If $\varphi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^N)$ (the space of smooth functions with compact support on $\mathbb{R} \times \mathbb{R}^N$), one defines the action of u on φ by

$$\langle u, \varphi \rangle = \int_{-\infty}^{+\infty} \langle u(t), \varphi(t) \rangle_{\mathcal{S}', \mathcal{S}} \, dt,$$

where $\varphi(t)$ is the function $t \mapsto \varphi(t, \cdot)$. It is a straightforward exercise to prove that u is well-defined and that is is a distribution on $\mathbb{R} \times \mathbb{R}^N$. The facts that u satisfies the wave equation in the distributional sense and that $u'(t) = \partial_t u(t)$ follow immediately from Claim II.4.4, that implies that $\lim u_n = u$ in the distributional sense, where u_n is a in Claim II.4.4. This last fact is an immediate consequence of Claim II.4.4. □

From now on, we will use the formula (II.1.2) as the definition of the solution u of (LW), (ID) with $(u_0, u_1) \in \dot{H}^\sigma \times \dot{H}^{\sigma-1}$. The preceding claims show that such a u is a limit of smooth, classical solutions of (LW), (ID), and that it satisfies (LW) in a weak sense.

4.b. The wave equation in the energy space. Of particular interest for us is the case $s = 1$. We will call "finite energy solutions" the weak solutions with initial data $\dot{H}^1 \times L^2$ given by the preceding subsection in the case $s = 1$, $N \geq 3$. We will focus on the case $N = 3$. We note that if $(u_0, u_1) \in (C^3 \times C^2)(\mathbb{R}^3) \cap \left(\dot{H}^1 \times L^2\right)(\mathbb{R}^3)$, we have two ways of defining the solution u: by integrals on spheres, as in Theorem I.5.3, and using the Fourier transform, i.e. by formula (II.3.1). Let us prove that these two definitions coincide:

Proposition II.4.6. *Let $u \in C^2(\mathbb{R} \times \mathbb{R}^3)$ be a solution of* (LW), (ID)*. Assume furthermore $u_0 = u(0) \in \dot{H}^1$, $u_1 = \partial_t u(0) \in L^2$. Then*

$$u(t) = \cos(t|D|)u_0 + \frac{\sin(t|D|)}{|D|} u_1, \quad \partial_t u(t) = -|D| \sin(t|D|)u_0 + \cos(t|D|)u_1.$$

Proof. Let $(u_{0,n}, u_{1,n}) \in \left(\mathcal{S}(\mathbb{R}^N)\right)^2$ with

$$\lim_{n \to \infty} \|u_{0,n} - u_0\|_{\dot{H}^1} + \|u_{1,n} - u_1\|_{L^2} = 0.$$

Let u_n be the corresponding solution of (LW) given by (II.3.1) (note that by uniqueness it is also the solution given by Theorem I.5.3). Since $u - u_n$ is a C^2, finite energy solution of (LW), Theorem I.6.1 yields

$$\forall t, \quad \|u(t) - u_n(t)\|_{\dot{H}^1}^2 + \|\partial_t u(t) - \partial_t u_n(t)\|_{L^2}^2 = \|u_0 - u_{0,n}\|_{\dot{H}^1}^2 + \|u_1 - u_{1,n}\|_{L^2}^2,$$

which tends to 0 as n goes to infinity. This proves the result, since $u_n(t)$ converges to $\cos(t|D|)u_0 + \frac{\sin(t|D|)}{|D|} u_1$ in $\dot{H}^1(\mathbb{R}^3)$ and $\partial_t u_n(t)$ converges to $-|D| \sin(t|D|)u_0 + \cos(t|D|)u_1$ in L^2 by Claim II.4.4. □

Using the approximation of finite energy solutions by solutions with initial data in $\mathcal{S}$, we can transfer several results of Chapter I to general finite energy solutions. This is the case of the decay of energy on past wave cones, which imply finite speed of propagation. If u is a finite energy solution (in any dimension $N \geq 3$) and $R > 0$, $x_0 \in \mathbb{R}^N$, $t_0 \in \mathbb{R}$, we denote by

$$E_{\mathrm{loc}}(t) = \int_{|x - x_0| < R - |t - t_0|} e_u(t, x) dx.$$

Then

Theorem II.4.7. *$E_{\mathrm{loc}}(t)$ is nonincreasing for $t \geq t_0$.*

Proof. From Theorem I.4.1, this quantity is nonincreasing when $(u_0, u_1) \in \mathcal{S}$. Considering the approximation given by Claim II.4.4, we obviously have, as a consequence of this claim,

$$\forall t, \quad \lim_{n \to \infty} \int_{|x - x_0| < R - |t - t_0|} e_{u_n}(t, x) dx = \int_{|x - x_0| < R - |t - t_0|} e_u(t, x) dx.$$

This gives the desired monotonicity property. □

We note that for general finite energy solution the integration by parts used in the proof of Theorem I.4.1 is no longer valid (since the boundary terms are not always well-defined).

4.c. Equation with a source term. We next consider the wave equation with a source term (I.1.2). By linearity, it is sufficient to study the equation with zero initial data:

$$\partial_t^2 u - \Delta u = f, \quad \vec{u}_{\restriction t=0} = (0,0). \tag{II.4.4}$$

PROPOSITION II.4.8. *Assume $f \in C_c^\infty\left(\mathbb{R}^{1+N}\right)$. Then u defined by*

$$u(t) = \int_0^t \frac{\sin\left((t-s)|D|\right)}{|D|} f(s)ds \tag{II.4.5}$$

is a C^∞ solution of (II.4.4)*. It is unique (in the class of C^2 solutions of* (II.4.4)*).*

PROOF. The uniqueness follows as usual by Theorem I.4.1. It is thus sufficient to check that u defined by (II.4.5) is of class C^∞, and is a solution of (II.4.4). We consider the function F defined on $\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N$ by

$$F(t,s,x) = \left(\frac{\sin\left((t-s)|D|\right)}{|D|} f(s)\right)(x).$$

Thus by the Fourier inversion formula

$$F(t,s,x) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{ix\cdot\xi}\frac{\sin\left((t-s)|\xi|\right)}{|\xi|}\widehat{f}(s,\xi)d\xi,$$

where $\widehat{f}(s,\xi)$ is the Fourier transform, in the space variable, of the function $f(s)$:

$$\widehat{f}(s,\xi) = \int_{\mathbb{R}^N} e^{ix\cdot\xi} f(s,x)dx.$$

Using that $f \in C_c^\infty(\mathbb{R}^{N+1})$ we see that f is C^∞ and that

$$\forall j\in\mathbb{N},\ \forall \alpha \in \mathbb{N}^N,\ \forall p>0, \quad \exists C>0, \quad \forall x\in\mathbb{R}^N,\ \forall s\in\mathbb{R}, \quad \left|\partial_\xi^\alpha\partial_s^j \widehat{f}(s,\xi)\right| \leq C(1+|\xi|)^{-p}.$$

As a consequence, F is C^∞ with respect to the variable (s,t,x), one can differentiate under the integral sign and

$$\forall j\in\mathbb{N},\ \forall \alpha \in \mathbb{N}^N,\ \forall T>0, \quad \exists C>0, \quad \forall (t,s,x)\in[-T,T]^2\times\mathbb{R}^N, \quad \left|\partial_t^j\partial_x^\alpha F(t,s,x)\right| \leq C.$$

The desired result follows since by integration by parts in the ξ variable,

$$\Delta F(t,s,x) = -\frac{1}{(2\pi)^N}\int |\xi|^2 e^{ix\cdot\xi}\frac{\sin\left((t-s)|\xi|\right)}{|\xi|}\widehat{f}(s,\xi)d\xi$$

□

We note that the Duhamel formula (II.4.5) still makes sense when $f\in L^1(\mathbb{R},\dot{H}^{\sigma-1})$ (see Appendix A for the definition of this space), where σ is a fixed real number (assumed to be $< N/2$ for simplicity), and that it yields a function $u\in C^0(\mathbb{R},\dot{H}^\sigma)$, such that

$$\|\vec{u}(t)\|_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}} \leq \int_0^t \|f(s)\|_{\dot{H}^{\sigma-1}}ds. \tag{II.4.6}$$

Note that (II.4.6) is exactly the energy inequality proved in Chapter I when $\sigma=1$.

One can prove that in this case, $\partial_t u\in C^0(\mathbb{R},\dot{H}^{\sigma-1})$, where

$$\partial_t u = \int_0^t \cos\left((t-s)|D|\right)f(s)ds. \tag{II.4.7}$$

is the derivative in the sense of distribution (and as a derivative in $\dot{H}^{\sigma-1}$) and that $\partial_t^2 u - \Delta u = f$ in the sense distribution on $\mathbb{R}^{1+N}$ and in $H^{\sigma-2}(\mathbb{R}^N)$. For simplicity, we state this result in the case $\sigma=1$:

PROPOSITION II.4.9. *Let $f\in L^1(\mathbb{R},L^2(\mathbb{R}^N))$ and u be defined by* (II.4.5)*. Then $\partial_t u$ defined by* (II.4.7) *is the time derivative of u in L^2:*

$$\lim_{h\to 0}\left\|\frac{1}{h}(u(t+h)-u(t)) - \partial_t u(t)\right\|_{L^2} = 0.$$

Furthermore, $(u,\partial_t u)\in C^0(\mathbb{R},\dot{H}^1\times L^2)$ with $(u,\partial_t u)(0)=(0,0)$ and

$$\partial_t^2 u - \Delta u = f$$

in $\dot{H}^{-1}$ and in the sense of distribution in $\mathbb{R}\times\mathbb{R}^N$.

We omit the complete proof, which is similar to the proof of the analogous results in the preceding subsection. Let us mention that to prove that the equation is satisfied in the distributional sense, one can use that $C_c^\infty(\mathbb{R}\times\mathbb{R}^N)$ is dense in $L^1(\mathbb{R},L^2(\mathbb{R}^N))$. Thus there exists a sequence $(f_n)_n$ of $C_c^\infty(\mathbb{R}\times\mathbb{R}^N)$ functions such that

$$\lim_{n\to\infty}\|f_n-f\|_{L^1(\mathbb{R},L^2(\mathbb{R}^n))}=0.$$

The corresponding solution u_n of

$$\partial_t^2 u_n-\Delta u_n=f_n,$$

defined by

$$u_n(t)=\int_0^t\frac{\sin\big((t-s)|D|\big)}{|D|}f_n(s)ds$$

satisfies the assumptions of Proposition II.4.8. Furthermore, by the energy inequality (II.4.6), we have

$$\lim_{n\to\infty}\|(u_n-u,\partial_t u_n-\partial_t u)\|_{\dot{H}^1\times L^2}=0.$$

This implies that u satisfies $\partial_t^2 u-\Delta u=0$ in the distributional sense

As in the case of the free wave equation (LW) with nonzero initial data, we will take, in the sequel of this course, the formula (II.4.5) as a definition of the solution u of (II.4.4).

CHAPTER III

Strichartz inequalities

III.1. Introduction

In view of Plancherel theorem and the Fourier representation formulas for the wave equation, it is natural to study the wave equation in $L^2(\mathbb{R}^N)$ or in L^2 based spaces such as the Sobolev spaces $\dot{H}^s$ considered in the preceding chapter. However, this is not sufficient for the study of nonlinear wave equations. Indeed since $\||f|^p\|_{L^2(\mathbb{R}^N)} = \|f\|_{L^{2p}}^{2p}$, the appearance of Lebesgue spaces L^q with $q \neq 2$ is unavoidable. A first way to deal with this issue is to use Sobolev inequalities. For example, if one wants to consider solutions in the energy spaces for the equation

$$\partial_t^2 u - \Delta u = u^3, \quad x \in \mathbb{R}^3, \tag{III.1.1}$$

the energy inequality will yields terms of the form[1] $\|u^3\|_{L^1([0,T],L^2)} = \|u\|^3_{L^3([0,T],L^6)} \lesssim T\|u\|_{L^\infty([0,T],\dot{H}^1)}$, which is sufficient to prove the existence and uniqueness of finite energy solutions for (III.1.1). However this strategy will not work for higher order nonlinearities, and in particular the quintic one which we will focus on in several chapters of this course. In this chapter we will introduce the celebrated *Strichartz inequalities*, that use the dispersive properties of the wave equation to improve over Sobolev type inequalities. This type of inequalities was introduced by Robert Strichartz in an article published in 1977 [**32**], and generalized later by several authors. See e.g. [**18**] or the book [**30**].

The original inequalities of Strichartz were formulated in terms of Lebesgue spaces $L^q(\mathbb{R}\times\mathbb{R}^N)$ on the whole space time $\mathbb{R}\times\mathbb{R}^N$. Having in minds applications to nonlinear wave equations, it is useful to consider more general spaces $L^p(I, L^q(\mathbb{R}^N))$ where the Lebesgue exponents in space and times are distinct. We will often write $L^p(I, L^q)$ instead of $L^p(I, L^q(\mathbb{R}^N))$ to lighten notations. When $I = \mathbb{R}$, we will also use the notation L^pL^q. We will use the generalized Hölder inequality in these spaces:

Proposition III.1.1. *Let p, q, p_1, q_1, p_2, q_2 in $[1,\infty]$ with*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Let $f \in L^{p_1}L^{q_1}$ and $g \in L^{p_2}L^{q_2}$. Then $fg \in L^pL^q$ and

$$\|fg\|_{L^pL^q} \leq \|f\|_{L^{p_1}L^{q_1}}\|g\|_{L^{p_2}L^{q_2}}.$$

The proof of Proposition III.1.1, using the standard Hölder inequality, is left as an exercise to the reader. We will also use the following consequence of Proposition III.1.1 (whose proof is also left to the reader):

Corollary III.1.2. *Let $\theta \in [0,1]$, p, q, p_1, q_1, p_2, q_2 in $[1,\infty]$ with*

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

Let $f \in L^{p_1}L^{q_1} \cap L^{p_2}L^{q_2}$. Then $f \in L^pL^q$ and

$$\|f\|_{L^pL^q} \leq \|f\|^{\theta}_{L^{p_1}L^{q_1}}\|f\|^{1-\theta}_{L^{p_2}L^{q_2}}.$$

III.2. Statement of the estimate

The Strichartz inequalities in space dimension 3 with initial data in the energy space read as follows:

Theorem III.2.1. *Let $(u_0, u_1) \in (\dot{H}^1 \times L^2)(\mathbb{R}^3)$ and $f \in L^1(\mathbb{R}, L^2(\mathbb{R}^3))$. Let*

$$u(t) = \cos(t|D|)u_0 + \frac{\sin(t|D|)u_1}{|D|} + \int_0^t \frac{\sin\big((t-s)|D|\big)}{|D|} f(s)ds. \tag{III.2.1}$$

[1] See Appendix A for the notations $L^p(I, L^q)$

Then for any (p,q) *with* $p>2$,

$$\frac{1}{p}+\frac{3}{q}=\frac{1}{2}, \tag{III.2.2}$$

one has $u\in L^p(\mathbb{R},L^q(\mathbb{R}^3))$ *and*

$$\|u\|_{L^p(\mathbb{R},L^q)}\leq C\left(\|(u_0,u_1)\|_{\dot{H}^1\times L^2}+\|f\|_{L^1(\mathbb{R},L^2)}\right).$$

for a constant $C>0$ *depending only on* p.

Remark III.2.2. If I is an interval with $0\in I$, $f\in L^1(I,L^2(\mathbb{R}^N))$, and u satisfies (III.2.1) for $t\in I$, then $u\in L^p(I,L^q(\mathbb{R}^3))$ and

$$\|u\|_{L^p(I,L^q)}\leq C\left(\|(u_0,u_1)\|_{\dot{H}^1\times L^2}+\|f\|_{L^1(I,L^2)}\right). \tag{III.2.3}$$

This follows immediately from the Theorem, extending f by $f(t)=0$ if $t\notin I$.

Remark III.2.3. We recall that in the setting of Theorem III.2.1, we also have $\vec{u}\in C^0\left(\mathbb{R},\dot{H}^1\times L^2\right)$, and the energy inequality

$$\|\vec{u}(T)\|_{\dot{H}^1\times L^2}\leq\|\vec{u}(0)\|_{\dot{H}^1\times L^2}+\|f\|_{L^1([0,T],L^2)},$$

for any $T>0$, which can be easily checked using the space Fourier transform of formula (III.2.1)

The Strichartz estimates express two phenomena, both related to the dispersive properties of the wave equation:

(1) *Local gain of regularity in space:* if u is as in Theorem III.2.1 and $q\in(6,\infty)$, then for almost every t in $\mathbb{R}$, $u(t)$ is in $L^q(\mathbb{R}^N)$. This is better, in terms of local space regularity, that the condition $u(t)\in L^6(\mathbb{R}^N)$ given by the energy inequality and the Sobolev embedding $\dot{H}^1(\mathbb{R}^3)\subset L^6(\mathbb{R}^3)$.
(2) *Decay for large times:* for (p,q) satisfying (III.2.2) with $p>2$, the L^q norm of $u(t)$, is, as a function of t, an element of $L^p(\mathbb{R})$. This is a property of decay in time when $t\to\pm\infty$ which completes the dispersion inequality of Theorem I.5.6.

We have focused on solutions with initial data $\dot{H}^1\times L^2$ in space dimension 3, in view of application to the quintic wave equation in space dimension 3. Analogs of Theorem III.2.1 exist in all space dimensions $N\geq 2$, with more general assumptions on the initial data (u_0,u_1) and the right hand-side f. The condition (III.2.2) is necessary by the scaling of the equation. For solutions in space dimension N with initial data in $\dot{H}^\sigma\times\dot{H}^{\sigma-1}$, it becomes

$$\frac{1}{p}+\frac{N}{q}=\frac{N}{2}-\sigma.$$

In this context, assuming $f=0$ to simplify, one has, with the additional condition

$$\frac{1}{p}+\frac{N-1}{2q}<\frac{N-1}{4}, \tag{III.2.4}$$

we have the Strichartz estimate:

$$\|u\|_{L^p(\mathbb{R},L^q)}\lesssim\|(u_0,u_1)\|_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}}.$$

Note that (III.2.4) does not appear in Theorem III.2.1 as it is implied by the scaling condition (III.2.2).

We refer to [**18, 30, 22**] for the proof of general Strichartz estimates that imply Theorem III.2.1. See also [**2**, Section 8.3].

To give the general idea of the proof of Theorem III.2.1, we will detail part of this proof in the case $p\geq 4$ in the remainder of this chapter. The main ingredient is a $L^4/L^{4/3}$ dispersion inequality for the half-wave equation which is a consequence of the L^∞/L^1 dispersion inequality (Theorem I.5.6) proved in Chapter I. In this chapter, we will assume the $L^4/L^{4/3}$ dispersion inequality and standard functional analysis results and prove Theorem III.2.1. The proof of the $L^4/L^{4/3}$ inequality is given in Appendix B together with elements of Littlewood-Paley theory that are needed for this proof.

We will use the following notations. If A and B are positive quantities, we will write $A\lesssim B$ when there exists a constant C, independent of the parameters, such that $A\leq CB$, and $A\equiv B$ when $A\lesssim B$ and $B\lesssim A$.

By the energy inequality and Sobolev embedding, we have for all t.

$$\|u(t)\|_{L^6}\lesssim\|u(t)\|_{\dot{H}^1}\lesssim\|(u_0,u_1)\|_{\dot{H}^1\times L^2}+\|f\|_{L^1(\mathbb{R},L^2)},$$

which solves the case $p=\infty$, $q=6$. Next, we notice that by Hölder inequality, if p and q satisfy (III.2.2) with $p\in(4,\infty)$, we have

$$\|u\|_{L^pL^q}\lesssim\|u\|^{1-\theta}_{L^\infty L^6}\|u\|^{\theta}_{L^4L^{12}} \tag{III.2.5}$$

where $\theta = \frac{4}{p}$. Thus the inequality (III.2.3) for this pair (p,q) will follows from the same equality for $p=4$, $q=12$. We are just reduced to prove the estimate (III.2.3) for $p=4$, $q=12$. By density, we can assume $(u_0,u_1) \in \left(C_0^\infty(\mathbb{R}^3)\right)^2$, $f \in C_0^\infty(\mathbb{R}\times\mathbb{R}^3)$.

The proof of Theorem III.2.1 is divided in two sections. In Section III.3, we prove a Strichartz inequality for the "half-wave equation", which is an order 1 equation related to the wave equation. Section III.4 is devoted to the end of the proof of Theorem III.2.1.

III.3. A Strichartz inequality for the half wave equation

It is sometimes useful to decompose the wave equation in two first-order equations in the time-variable. This is particularly the case when dealing with Fourier analysis tools. We thus introduce the half-wave equations

$$\partial_t u + i|D|u = 0, \quad \partial_t u - i|D|u = 0,$$

and their solutions (given in term of Fourier representations) $e^{-it|D|}\varphi$ and $e^{it|D|}\varphi$. Note that the solution to the usual wave equation (LW), (ID) is given by

$$2u(t) = e^{it|D|}u_0 + e^{-it|D|}u_0 + \frac{e^{it|D|}}{i|D|}u_1 - \frac{e^{-it|D|}}{i|D|}u_1$$

Note also that if $v(t) = e^{it|D|}\varphi$, then $e^{-it|D|}u_0 = v(-t)$, thus it is sufficient to consider only the solution $e^{it|D|}\varphi$. The function $e^{it|\xi|}$ is not smooth at $\xi = 0$, so that $e^{it|D|}$ does not map $\mathcal{S}(\mathbb{R}^N)$ to $\mathcal{S}(\mathbb{R}^N)$. However it maps $\mathcal{S}_0(\mathbb{R}^N)$ to $\mathcal{S}_0(\mathbb{R}^N)$ (where as before $\mathcal{S}_0(\mathbb{R}^N)$ is the space of functions φ in $\mathcal{S}(\mathbb{R}^N)$ such that $\hat{\varphi}$ is identically 0 in a neighborhood of the origin).

In this section, we sketch the proof of the following inequality:

Proposition III.3.1. *There exists $C>0$ such that*

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^3), \quad \left\| \frac{e^{i\cdot|D|}}{|D|}\varphi \right\|_{L^4(\mathbb{R},L^{12})} \lesssim \|\varphi\|_{L^2}, \tag{III.3.1}$$

where as usual $e^{i\cdot|D|}\varphi$ denotes $(t,x) \mapsto \left(e^{it|D|}\varphi\right)(x)$.

Proposition III.3.1 is a consequence of the dispersion inequality prove in Theorem I.5.6 which we recall:

$$\left\| \frac{\sin(t|D|)}{|D|} f \right\|_{L^\infty} \lesssim \frac{1}{|t|}\|f\|_{\dot{W}^{1,1}}, \quad \|\cos(t|D|)g\|_{L^\infty} \lesssim \frac{1}{|t|}\|g\|_{\dot{W}^{2,1}}. \tag{III.3.2}$$

Writing $e^{it|D|} = \cos(t|D|) + i\sin(t|D|)$, we obtain (using that $\left\||D|^{-1}f\right\|_{\dot{W}^{2,1}} \approx \|f\|_{\dot{W}^{1,1}} \approx \left\||D|f\right\|_{L^1}$, which can be proved rigorously):

$$\left\| \frac{e^{i(t|D|)}}{|D|}\varphi \right\|_{L^\infty} \lesssim \frac{1}{|t|}\left\||D|\varphi\right\|_{L^1}, \quad \varphi \in \mathcal{S}_0(\mathbb{R}^3). \tag{III.3.3}$$

Moreover, Plancherel equality gives:

$$\left\| \frac{e^{i(t|D|)}}{|D|}\varphi \right\|_{L^2} = \left\| \frac{1}{|D|}\varphi \right\|_{L^2}, \quad \varphi \in \mathcal{S}_0(\mathbb{R}^3). \tag{III.3.4}$$

A formal interpolation between the two estimates (III.3.3) and (III.3.4) gives:

$$\left\| e^{it|D|}\frac{1}{|D|}\varphi \right\|_{L^4} \lesssim \frac{1}{|t|^{1/2}}\|\varphi\|_{L^{4/3}}. \tag{III.3.5}$$

It is indeed possible to prove (III.3.5) from (III.3.2) and (III.3.4), using Littlewood-Paley theory and the Riesz-Thorin interpolation theorem. We postpone this proof to Appendix B for the interested reader. In the remainder of this Chapter, we will prove Proposition III.3.1, then Theorem III.2.1 assuming (III.3.5). For the proof of Proposition III.3.1, we will need the Hardy-Littlewood-Sobolev inequality, which we will use in the particular case $\theta = 1/2$, $p = 4/3$, $q = 4$:

Theorem III.3.2 (Hardy Littlewood Sobolev). *Let $\theta \in]0,1[$, $(p,q) \in]1,\infty[^2$ satisfy*

$$\frac{1}{p} + \theta = 1 + \frac{1}{q}.$$

Let $f \in L^p(\mathbb{R})$. Let, for $t \in \mathbb{R}$,

$$g(t) = \int_{\mathbb{R}} f(s)\frac{1}{|t-s|^\theta}ds. \tag{III.3.6}$$

Then the integral defining g converges for almost every t, and

$$\|g\|_{L^q(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}.$$

We refer e.g. to [**2**, Theorem 1.7] for the proof.

Remark III.3.3. The function $t \mapsto 1/t^\theta$, is not in $L^{1/\theta}$ due to a logarithmic divergence at 0 and ∞. Theorem III.3.2 says that the convolution by this function has the same boundedness properties (given by Young's Inequality, see Theorem B.2.1 in the appendix), in terms of L^p spaces, as a $L^{1/\theta}$ function.

Proof of Proposition III.3.1. We consider the operator T defined by

$$(T\varphi)(t,x) = \left(e^{it|D|}|D|^{-1/2}\varphi\right)(x)$$

We will prove that T extends to a bounded operator from $L^2(\mathbb{R}^3)$ to $L^4(\mathbb{R}\times\mathbb{R}^3)$, with an operator norm that is independent of j, i.e.

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^3), \quad \left\|e^{it|D|}|D|^{-1/2}\varphi\right\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|\varphi\|_{L^2}. \tag{III.3.7}$$

Note that the inequality (III.3.1) follows from (III.3.7) (applied to $|D|\varphi$) and the Sobolev inequality

$$\forall f \in \mathcal{S}, \quad \|f\|_{L^{12}(\mathbb{R}^3)} \lesssim \left\||D|^{1/2}f\right\|_{L^4(\mathbb{R}^3)}, \tag{III.3.8}$$

see e.g. [**19**, Theorem 6.2.4].

See also section B.4 for an alternative proof of (III.3.1) based on Littlewood-Paley theory and that does not use the Sobolev inequality (III.3.8).

To prove (III.3.7), we will use a so-called TT^* argument to reduce the proof of (III.3.7) to the proof of the boundedness of an operator acting on functions on $\mathbb{R}\times\mathbb{R}^3$.

The inequality (III.3.7) is equivalent to the following statement:

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^3), \quad \forall g \in C_0^\infty(\mathbb{R}\times\mathbb{R}^3), \quad \left|\iint (T\varphi)\overline{g}dxdt\right| \lesssim \|\varphi\|_{L^2(\mathbb{R}^3)}\|g\|_{L^{4/3}(\mathbb{R}\times\mathbb{R}^3)}.$$

Using Plancherel equality in the space variable for every $t\in\mathbb{R}$, we obtain

$$\iint (T\varphi)\overline{g}dxdt = \int \varphi(x)(T^*g)(x)dx,$$

where the (formal) adjoint T^* of T is defined by

$$T^*g(x) = \int_{\mathbb{R}} e^{-it|D|}|D|^{-1/2}g(t)dt.$$

We are thus reduced to prove

$$\forall g \in C_0^\infty(\mathbb{R}\times\mathbb{R}^3), \quad \|T^*g\|_{L^2(\mathbb{R}^3)} \lesssim \|g\|_{L^{4/3}(\mathbb{R}\times\mathbb{R}^3)}. \tag{III.3.9}$$

We have

$$\|T^*g\|_{L^2}^2 = \int_{\mathbb{R}^3} T^*g\overline{T^*g}dx = \iint_{\mathbb{R}\times\mathbb{R}^3} TT^*g\overline{g}dxdt, \tag{III.3.10}$$

and (III.3.9) would follow from the inequality

$$\|TT^*g\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|g\|_{L^{4/3}(\mathbb{R}\times\mathbb{R}^3)}. \tag{III.3.11}$$

We have

$$TT^*g(t,x) = \int_{\mathbb{R}} e^{i(t-s)|D|}|D|^{-1}g(s)ds.$$

Using the $L^4/L^{4/3}$ dispersion inequality (III.3.5), we obtain at fixed t,

$$\|(TT^*g)(t)\|_{L^4(\mathbb{R}^3)} \lesssim \int_{\mathbb{R}} \frac{1}{|t-s|^{1/2}}\|g(s)\|_{L^{4/3}(\mathbb{R}^3)}ds$$

By Hardy Littlewood Sobolev inequality, we deduce

$$\|TT^*g\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|g\|_{L^{4/3}(\mathbb{R}\times\mathbb{R}^3)},$$

which yields (III.3.11) and thus concludes the proof of (III.3.7).

□

III.4. Proof of the Strichartz estimate for the full wave equation

We are now ready to prove Theorem III.2.1. We can treat separately the terms

$$u_L(t) = \cos(t|D|)u_0 + \frac{\sin(t|D|)u_1}{|D|}$$

and

$$(Bf)(t) = \int_0^t \frac{\sin\big((t-s)|D|\big)}{|D|} f(s)ds. \tag{III.4.1}$$

Using that $\cos(t|D|) = \frac{1}{2}\left(e^{it|D|} + e^{-it|D|}\right)$, $\sin(t|D|) = \frac{1}{2i}\left(e^{it|D|} - e^{-it|D|}\right)$, we obtain immediately from Proposition III.3.1

$$\|u_L\|_{L^4(\mathbb{R},L^{12}(\mathbb{R}^3))} \lesssim \|u_0\|_{\dot{H}^1} + \|u_1\|_{L^2}.$$

The other term is more delicate. We first consider

$$u_a(t) = \int_0^\infty \frac{e^{i(t-s)|D|}}{|D|} f(s)ds = e^{it|D|}F, \quad F = \int_0^\infty \frac{e^{-is|D|}}{|D|} f(s)ds$$

and

$$u_b(t) = \int_0^\infty \frac{e^{-i(t-s)|D|}}{|D|} f(s)ds$$

Using that $e^{-is|D|}/|D|$ is a bounded operator from L^2 to $\dot{H}^1$, we obtain that $F \in \dot{H}^1$ with

$$\|F\|_{\dot{H}^1} \lesssim \|f\|_{L^1(\mathbb{R},L^2(\mathbb{R}^3))}.$$

By the Strichartz estimate for the half-wave equation, Proposition III.3.1, we deduce

$$\|u_a\|_{L^4(\mathbb{R},L^{12}(\mathbb{R}^3))} \lesssim \|f\|_{L^1(\mathbb{R},L^2(\mathbb{R}^3))}.$$

Similarly

$$\|u_b\|_{L^4(\mathbb{R},L^{12}(\mathbb{R}^3))} \lesssim \|f\|_{L^1(\mathbb{R},L^2(\mathbb{R}^3))}.$$

Combining, we obtain

$$\|Af\|_{L^4(\mathbb{R},L^{12}(\mathbb{R}^3))} \lesssim \|f\|_{L^1(\mathbb{R},L^2(\mathbb{R}^3))}, \tag{III.4.2}$$

where A is the operator defined by

$$Af(t) = \int_0^\infty \frac{\sin\big((t-s)|D|\big)}{|D|} f(s)ds.$$

Note that Af is analogous to Bf defined above, the only difference between the two being that the integral defining Af is on $[0,\infty)$, whereas the integral defining Bf is on $[0,t[$. An important functional analysis result, due to Michael Christ and Alexander Kiselev [**9**], shows that the boundedness of A implies the boundedness of B. We state this result in a version that was proposed by Christopher Sogge:

LEMMA III.4.1. *Let X and Y be Banach spaces. Let $1 \leq p < q \leq \infty$. Let K a continuous function from $\mathbb{R}^2$ to the space of bounded linear operators from X to Y. Let*

$$(Af)(t) = \int_{-\infty}^\infty K(t,\tau)f(\tau)d\tau,$$

and assume that A is a bounded operator from $L^p(\mathbb{R},X)$ to $L^q(\mathbb{R},Y)$, with operator norm C. Define the operator B by

$$(Bf)(t) = \int_{-\infty}^t K(t,\tau)f(\tau)d\tau.$$

Then B extends to a bounded operator from $L^p(\mathbb{R},X)$ to $L^q(\mathbb{R},Y)$, with operator norm $\leq \frac{2C\theta^2}{1-\theta}$, where $\theta = 2^{\frac{1}{q}-\frac{1}{p}}$.

Applying Christ and Kiselev Lemma to

$$K(t,\tau) = 1\!\!1_{\tau>0} \frac{\sin\big((t-\tau)|D|\big)}{|D|}\chi(\varepsilon|D|), \tag{III.4.3}$$

where $\chi \in C_0^\infty(\mathbb{R}^3)$ is equal to 1 close to 0, one obtains

$$\forall \varepsilon > 0, \quad \forall f \in L^1(\mathbb{R},L^2), \quad \|\chi(\varepsilon D)Bf\|_{L^4L^{12}} \lesssim \|f\|_{L^1L^2},$$

where Bf is as in (III.4.1). Letting $\varepsilon \to 0$ we obtain the desired result.

EXERCISE III.1. Justify this last argument.

CHAPTER IV

Cauchy theory for the non-linear equation

In this chapter we will consider the nonlinear wave equation with a power-like nonlinearity

$$\partial_t u^2 - \Delta u = \sigma u^p, \tag{NLW}$$

on $I \times \mathbb{R}^N$, where N is an interval, where the power p is an integer ≥ 2 and σ is nonzero real parameter. Considering the unknown λu instead of u for a suitable choice of $\lambda > 0$, we see that we can assume

$$\sigma \in \{\pm 1\}.$$

We will briefly consider the general case, then restrict to the quintic case $p = 5$ in space dimension 3. We will also comment on the cubic case $p = 3$, in the same space dimension.

IV.1. Scaling invariance. Critical Sobolev space

Let u be a (nonzero) C^2 solution of (NLW) on $(a, b) \times \mathbb{R}^N$, where $a < b$. Let $u_\lambda(t, x) = \lambda^\alpha u(\lambda t, \lambda x)$, where $\lambda > 0$ and $\alpha = \alpha(p, N)$ will be specified later. We have

$$\partial_t^2 u_\lambda - \Delta u_\lambda = \lambda^{\alpha+2-\alpha p} \sigma u_\lambda^p.$$

Thus, if $\alpha = \frac{2}{p-1}$, we see that u_λ is a solution of (NLW) on $\left(\frac{a}{\lambda}, \frac{b}{\lambda}\right) \times \mathbb{R}^N$. We will assume that α has this particular value in the sequel, denoting

$$u_\lambda(t, x) = \lambda^{\frac{2}{p-1}} u(\lambda t, \lambda x).$$

Let

$$\dot{\mathcal{H}}^s = \dot{H}^s(\mathbb{R}^N) \times \dot{H}^{s-1}(\mathbb{R}^N).$$

The critical Sobolev exponent is by definition the unique s such that

$$\|\vec{u}_\lambda(0)\|_{\dot{\mathcal{H}}^{s_c}} = \|\vec{u}(0)\|_{\dot{\mathcal{H}}^{s_c}}.$$

Since by explicit computation

$$\|\vec{u}_\lambda(0)\|_{\dot{\mathcal{H}}^s} = \lambda^{\frac{2}{p-1}+s-N/2} \|\vec{u}(0)\|_{\dot{\mathcal{H}}^s}. \tag{IV.1.1}$$

We see that

$$s_c = \frac{N}{2} - \frac{2}{p-1}.$$

We observe that s_c grows with p, and is always strictly smaller than $N/2$.

Consider a solution u of (NLW) defined on a finite interval $[0, T[$. The corresponding solution u_λ is defined on $[0, T/\lambda[$. Growing λ has the effect of decreasing the time of existence. If $s > s_c$, the $\dot{\mathcal{H}}^s$ norm of $\vec{u}_\lambda(0)$ becomes larger with λ. If $s < s_c$ it becomes smaller. Thus in the case where $s < s_c$, the effect of scaling is to simultaneously decrease the norm of the initial data in $\dot{\mathcal{H}}^s$, $s < s_c$ and shrinking its interval of existence. This is contrary to the intuition that for smaller solutions, the effect of the nonlinearity is weaker, and the solution should behave in a linear way (and in particular has a long time of existence). This leads to an informal conjecture that s_c is a threshold for local well-posedness. It turns out that this conjecture is true for the wave equation: the equation (NLW) is locally well-posed[1] in $\dot{\mathcal{H}}^s$ for $s \geq s_c$, and ill-posed if $\dot{\mathcal{H}}^s$ for $s < s_c$.

We will focus on the quintic case $p = 5$ in space dimension $N = 3$:

$$(\partial_t^2 - \Delta) u = \sigma u^5. \tag{W5}$$

In this case the critical Sobolev case is $\dot{\mathcal{H}}^1$, and the equation is called "energy critical". As usual, we will take initial data, say at $t = t_0$:

$$(u, \partial_t u)_{t=t_0} = (u_0, u_1). \tag{ID}$$

[1]By "well-posed in X", we mean that there is existence and uniqueness of solutions with initial data in X and a reasonable stability theory. We will not give a more rigorous definition of local well-posedness. See e.g. Definition 3.4, Remark 3.5 of T. Tao's book [**34**]

An other equation of interest, that we will sometimes study in the exercise sheets, is the cubic equation

$$(\partial_t^2 - \Delta)u = \sigma u^3, \tag{W3}$$

in dimension $1+3$, for which $s_c = 1/2$. In all the sequel, we fix $N=3$.

We will often denote $\mathcal{H} = \dot{\mathcal{H}}^1$ to lighten notations.

IV.2. Definition of solutions

As for the linear wave equation, the notion of classical (C^2) solution is too restrictive for the equation (W5), and we will define the following weaker notion of solution, based on Duhamel's formulation of the equation:

Definition IV.2.1. A *finite energy solution* of (W5), (ID) on an interval I with $t_0 \in I$ is a function $u \in L^5_{\text{loc}}(I, L^{10})$ such that $\forall t \in I$,

$$u(t) = \cos\big((t-t_0)|D|\big)u_0 + \frac{\sin\big((t-t_0)|D|\big)}{|D|}u_1 + \int_{t_0}^t \frac{\sin\big((t-s)|D|\big)}{|D|}u^5(s)ds, \tag{IV.2.1}$$

where $(u_0, u_1) \in \dot{\mathcal{H}}^1$.

In the definition, by $u \in L^5_{\text{loc}}(I, L^{10}(\mathbb{R}^3))$, we mean that $u \in L^5(J, L^{10})$ for any compact interval $J \subset I$.

Note that if u is a finite-energy solution in the above sense, one has $u^5 \in L^1_{\text{loc}}(I, L^2(\mathbb{R}^3))$, and thus by energy estimates (see Remark III.2.3),

$$\vec{u} \in C^0(I, \dot{\mathcal{H}}^1).$$

Also, by Chapter II, u satisfies the equation (W5) in the sense of distribution on $I \times \mathbb{R}^3$.

The solutions given by the Duhamel formula as in Definition IV.2.1 are called "strong" solutions in the book of Terence Tao [**34**], by opposition to the weaker notion of distributional solutions (that do not impose continuity in time) and the stronger notion of classical solutions (that are C^2 and satisfy the equation in a classical sense). Note however that this terminology is not universal. For example the solutions of Definition IV.2.1 are called ... "weak" solutions in the book [**30**] of Christopher Sogge.

We refer to Section 3.2 of [**34**] "What is a solution?", for a discussion on different types of solutions.

In the sequel, by "solution to (W5)" we will always mean (unless specified otherwise) a solution in the sense of Definition IV.2.1.

Exercise IV.1. Check that the definition of finite energy solutions above does not depend on the choice of the initial time. In other words, if u is a solution of (W5) on I and $t_1 \in I$, then for all $t \in I$,

$$u(t) = \cos\big((t-t_1)|D|\big)u(t_1) + \frac{\sin\big((t-t_1)|D|\big)}{|D|}\partial_t u(t_1) + \int_{t_1}^t \frac{\sin\big((t-s)|D|\big)}{|D|}u^5(s)ds.$$

IV.3. Existence and uniqueness

3.a. A local statement. We introduce the following notations:

$$S_L(t)\vec{u}_0 = \cos(t|D|)\,u_0 + \frac{\sin(t|D|)}{|D|}u_1, \quad \vec{S}_L(t)\vec{u}_0 = \left(S_L(t)\vec{u}_0, \partial_t S_L(t)\vec{u}_0\right),$$

where $\vec{u}_0 = (u_0, u_1)$. We start with the following local statement:

Theorem IV.3.1. *There exists $\delta_0 > 0$ with the following property. Let I be an interval with $t_0 \in I$. Let $\vec{u}_0 \in \dot{\mathcal{H}}^1$. Assume*

$$\|S_L(\cdot - t_0)\vec{u}_0\|_{L^5(I, L^{10})} = \delta \leq \delta_0. \tag{IV.3.1}$$

Then there exists a unique solution u of (W5), (ID) *on I. Furthermore*

$$\sup_{t\in I}\left\|\vec{u}(t) - \vec{S}_L(t-t_0)\vec{u}_0\right\|_{\dot{\mathcal{H}}^1} + \|u - S_L(\cdot - t_0)\vec{u}_0\|_{L^5(I,L^{10})} \lesssim \delta^5. \tag{IV.3.2}$$

In the Theorem, $S_L(\cdot - t_0)\vec{u}_0$ denotes the map $t \mapsto S_L(t-t_0)\vec{u}_0$.

Theorem IV.3.1 has two important consequences:

Local well-posedness: Note that $(5, 10)$ is a $\dot{\mathcal{H}}^1$-admissible couple in dimension 3 (it satisfies (III.2.2)). By Theorem III.2.1, if $\vec{u}_0 \in \dot{\mathcal{H}}^1$, then $S_L(\cdot)\vec{u}_0 \in L^5(\mathbb{R}, L^{10}(\mathbb{R}^3))$. Thus if $T > 0$ is small enough, then

$$\|\vec{u}_0\|_{L^5([-T,+T],L^{10})} \leq \delta_0,$$

and Theorem IV.3.1 implies that there exists a solution to (W5), (ID) on $[-T, +T]$.

Small data global well-posedness: If $\vec{u}_0 \in \dot{\mathcal{H}}^1$ and $\|u_0\|_{\dot{H}^1} \leq \delta_0/C_S$, where C_S is the constant in the Strichartz inequality (III.2.3) with $p=5$, $q=10$, then $\|S_L(\cdot)\vec{u}_0\|_{L^5(\mathbb{R},L^{10})} \leq \delta_0$, and one can use Theorem IV.3.1 with $I=\mathbb{R}$. This shows that the corresponding solution u is globally defined, and that $u \in L^5(\mathbb{R}, L^{10})$.

Proof of Theorem IV.3.1. Assume without generality that $t_0 = 0$. We use the Banach fixed point theorem, proving that the operator A, defined by

$$Av\,(t) = S_L(t)\vec{u}_0 + Bv\,(t), \quad Bv\,(t) = \sigma \int_0^t \frac{\sin\left((t-s)|D|\right)}{|D|} v^5(s)ds, \tag{IV.3.3}$$

is a contraction on X defined by

$$X = \left\{ v \in L^5(I, L^{10}),\ \|v\|_{L^5(I,L^{10})} \leq 2\delta_0 \right\}.$$

We first prove that A maps X into X. Indeed, If $v \in X$, then by Theorem III.2.1 (see Remark III.2.2),

$$\|Bv(t)\|_{L^5(I,L^{10})} \leq C_S \left\|v^5\right\|_{L^1(I,L^2)} \leq C_S \|v\|^5_{L^5(I,L^{10})} \leq C_S\delta_0^5 \leq \delta_0,$$

assuming $\delta_0 \leq C_S^{-1/4}$. Thus $Av \in X$.

We next prove that A is a contraction on X. Let $v, w \in X$. Using $w^5 - v^5 = (w-v)(w^4 + w^3 v + w^2 v^2 + wv^3 + v^4)$ and Young's inequality $ab \leq a^p/p + b^q/q$, $1/p + 1/q = 1$, one obtains

$$|v^5 - w^5| \leq \frac{5}{2} |v-w|\,(v^4 + w^4).$$

Combining with Hölder's inequality, we obtain

$$\left\|v^5 - w^5\right\|_{L^1(I,L^2)} \leq \frac{5}{2}\|v-w\|_{L^5(I,L^{10})} \left(\|v\|^4_{L^5(I,L^{10})} + \|w\|^4_{L^5(I,L^{10})}\right). \tag{IV.3.4}$$

By Strichartz estimates

$$\|Av - Aw\|_{L^5(I,L^{10})} = \|Bv - Bw\|_{L^5(I,L^{10})} \leq C_S \left\|v^5 - w^5\right\|_{L^1(I,L^2)} \leq 5C_S\|v-w\|_{L^5(I,L^{10})}\delta_0^4.$$

If δ_0 is small enough ($\delta_0 = (10C_S)^{-1/4}$ works), one has

$$\|Av - Aw\|_{L^5(I,L^{10})} \leq \frac{1}{2}\|v-w\|_{L^5(I,L^{10})}.$$

This shows that A is a contraction on X.

Let u be the only fixed point of A in X. Since $u = Au$ and $u \in L^5(I, L^{10})$ we see that u is a solution of (W5) on I.[2] Using

$$u - S_L(\cdot)\vec{u}_0 = Bu,$$

and $\|Bu\|_{L^5(I,L^{10})} \leq \delta^5$, and Strichartz inequality, we obtain (IV.3.2). It remains to prove the uniqueness statement. From the contraction argument, we see that u is the unique solution of (W5) such that $\|u\|_{L^5(I,L^{10})} \leq \delta_0$. We prove a stronger statement, Lemma IV.3.2 below, that will conclude the proof. □

Lemma IV.3.2. *Let u, v be two solutions of* (W5) *on an interval I with $t_0 \in I$. Assume $\vec{u}(t_0) = \vec{v}(t_0)$. Then $u = v$.*

Proof. Assume again $t_0 = 0$ to simplify notations. Let $\delta_0 > 0$ be as in Theorem IV.3.1. We let $K = [a,b]$ be a compact subinterval of I such that $t_0 \in K$. We will prove that $u(t) = v(t)$ for $t \in K$. Since K is compact, we have by Definition IV.2.1,

$$u \in L^5(K, L^{10}), \quad v \in L^5(K, L^{10}).$$

We can thus divide K into p subintervals $[\tau_j, \tau_{j+1}]$, $0 \leq j \leq p-1$, with $\tau_0 < \tau_1 < \ldots < \tau_p$, such that

$$\forall j \in \{0, \ldots, J-1\}, \quad \max\left(\|u\|_{L^5([\tau_j,\tau_{j+1}],L^{10})}, \|v\|_{L^5([\tau_j,\tau_{j+1}],L^{10})}\right) \leq \delta_0.$$

Let j_0 be an index such that $0 \in [\tau_{j_0}, \tau_{j_0+1}]$. By the proof of Theorem III.2.1, with $I = [\tau_{j_0}, \tau_{j_0+1}]$, noting that u and v are in X, we obtain $u(t) = v(t)$ for $t \in [\tau_{j_0}, \tau_{j_0+1}]$. This implies

$$\vec{u}(\tau_{j_0}) = \vec{v}(\tau_{j_0}) \text{ and } \vec{u}(\tau_{j_0+1}) = \vec{v}(\tau_{j_0+1}).$$

We can then iterate the preceding arguments on the intervals $[\tau_j, \tau_{j+1}]$, $j = j_0+1$, $j = j_0+2$ until $j = J-1$, and $j = j_0 - 1$, $j = j_0 - 2$ until $j = 0$ to obtain that $u(t) = v(t)$ for $t \in K$, concluding the proof. □

[2] Recall that "solution" is to be taken in the sense of Definition IV.2.1.

3.b. Maximal solution. Using the above local existence theorem, we can now glue the solutions above to construct a maximal solution of (W5).

COROLLARY IV.3.3. *Let $\vec{u}_0 \in \dot{\mathcal{H}}^1$ and $t_0 \in \mathbb{R}$. Then there is a unique maximal solution of* (W5), (ID). *Denoting by $I_{\max} = (T_-, T_+)$ its interval of existence, we have the following* blow-up criteria*:*

$$T_+ < \infty \Longrightarrow u \notin L^5([t_0, T_+[, L^{10}), \quad T_- > -\infty \Longrightarrow u \notin L^5(]T_-, t_0], L^{10}). \tag{IV.3.5}$$

The phrase "maximal solution" in the theorem means that if v is another solution of (W5), (ID) defined on an interval I with $t_0 \in I$, then $I \subset I_{\max}$ and $u(t) = v(t)$ for all $t \in I$.

PROOF. Let $\mathcal{J}$ be the set of all open intervals I such that $t_0 \in I$, and there exists a solution v of (W5), (ID) on I. Let

$$I_{\max} = \bigcup_{I \in \mathcal{J}} I.$$

By Theorem IV.3.1, $\mathcal{J}$ is nonempty. Thus $I_{\max}$ is an open interval containing t_0. If $t \in I_{\max}$, there exists an interval I and a solution v of (W5), (ID) on I. By the uniqueness Lemma IV.3.2, the value $v(t)$ does not depend on the choice of I. We denote by $u(t)$ this common value. Let K be a compact subinterval of $I_{\max}$. We next prove:

$$u \in L^5(K, L^{10}). \tag{IV.3.6}$$

Indeed, for all $t \in K$, there exist an open interval $I \in \mathcal{J}$ such that $t \in I$ and u is a solution of (W5) on I. This implies in particular that $u \in L^5([t-\varepsilon, t+\varepsilon], L^{10})$ if $\varepsilon = \varepsilon(t)$ is small enough. Using the compactness of K, we can cover K by a finite numbers of interval $]t - \varepsilon(t), t + \varepsilon(t)[$, and thus we obtain (IV.3.6).

If $t \in I_{\max}$, by the definition of $I_{\max}$ and the uniqueness Lemma IV.3.2, we have that

$$u(t) = S_L(t)\vec{u}_0 + \sigma \int_0^t \frac{\sin\big((t-s)|D|\big)}{|D|} u^5(s)ds,$$

which concludes the proof that u is a solution of (W5), (ID) on $I_{\max}$. The maximality of u is a direct consequence of the definition of $I_{\max}$ and Lemma IV.3.2.

Finally, we prove that if $T_+ < \infty$, then $u \notin L^5([t_0, T_+), L^{10}(\mathbb{R}^3))$. We argue by contradiction, assuming that there exists a solution u with maximal time of existence $T_+ < \infty$ such that $u \in L^5([t_0, T_+), L^{10})$. This implies $u^5 \in L^1([t_0, T_+), L^2)$. By energy inequality, we obtain that u can be extended to a function (that we still denote by u) such that $(u, \partial_t u) \in C^0([t_0, T_+], \dot{\mathcal{H}}^1)$. Since u is a solution on $[t_0, T_+)$, we also have, for any $t, t_1 \in [t_0, T_+)$,

$$u(t) = S_L(t - t_1)\vec{u}(t_1) + \sigma \int_{t_1}^t \frac{\sin\left((t-s)|D|\right)}{|D|} u^5(s)ds.$$

Letting $t_1 \to T_+$, we obtain

$$\forall t \in [t_0, T_+), \quad u(t) = S_L(t - T_+)\vec{u}(T_+) + \sigma \int_{T_+}^t \frac{\sin\left((t-s)|D|\right)}{|D|} u^5(s)ds,$$

i.e. that u is solution of (NLW) on $[t_0, T_+]$. Let v be the local solution of (NLW) with initial data $\vec{u}(T_+)$ at $t = T_+$, given by Theorem IV.3.1, and let J be its interval of definition. By the uniqueness statement, Lemma IV.3.2, $u(t) = v(t)$ for $t \in J \cap [t_0, T_+]$. Letting

$$w(t) = \begin{cases} u(t) & \text{if } t \in I_{\max} \\ v(t) & \text{if } t \in J \cap [T_+, +\infty), \end{cases}$$

we obtain a solution w of (NLW) that extends u on an open interval $I_{\max} \cup J$ that strictly contains $I_{\max}$. This contradicts the definition of $I_{\max}$, concluding the proof.

□

Let us mention that it is not possible to improve the blow-up criterion to

$$T_+ < \infty \Longrightarrow \limsup_{t \to \infty} \|\vec{u}(t)\|_{\dot{\mathcal{H}}^1} = +\infty.$$

Indeed, it was proved by Krieger, Schlag and Tataru [**25**] that there exist solutions of (W5) with $\sigma = 1$, with finite time of existence T_+ and such that

$$\limsup_{t \to T_+} \|\vec{u}(t)\|_{\dot{\mathcal{H}}^1} < \infty.$$

IV.4. Finite speed of Propagation

Remark IV.4.1. The proof of Theorem III.2.1 implies that if I is an interval, $t_0 \in I$, and u is a solution of (W5), (ID) on I such that $\|u\|_{L^5(I,L^{10})} \leq \delta_0/2$, then u is the limit, in $L^5(I, L^{10})$, of the sequence u^n defined by $u^0 = 0$, $u^n = Au^n$, where A is the operator defined in the proof. Indeed, by Strichartz estimates,

$$\|S_L(\cdot - t_0)\vec{u}_0\|_{L^5(I,L^{10})} \leq \|u\|_{L^5(I,L^{10})} + C_S\|u\|^5_{L^5(I,L^{10})} \leq \frac{\delta_0}{2} + C_S\delta_0^5/32 \leq \delta_0.$$

Thus $\vec{u}_0$ satisfies the assumption of Theorem III.2.1 and the conclusion follows from the fact that u is a fixed point of the contraction A.

This remark will be used at least twice in this course to obtain properties of the solution u. We will first use it to prove the finite speed of propagation property for the nonlinear equation:

Theorem IV.4.2. *Let $(t_0, x_0) \in \mathbb{R}^{1+3}$, $t_1 > t_0$, $R > 0$. We denote $\Gamma = \Big\{(t, x) \in \mathbb{R}\times\mathbb{R}^N \ : \ t_0 \leq t \leq t_1,\ |x - x_0| \leq R - |t - t_0|\Big\}$ Let u and v be two solutions of* (W5) *on $[t_0, t_1]$. We suppose $(u, \partial_t u)(t_0, x) = (v, \partial_t v)(t_0, x)$ for all $x \in B_R(x_0)$. Then $u(t, x) = v(t, x)$ for almost all $(t, x) \in \Gamma$.*

Proof. Dividing the interval $[t_0, t_1]$ into subintervals $[\tau_j, \tau_{j+1}]$, $0 \leq j \leq J - 1$, $t_0 = \tau_0 < \tau_1 < \ldots < \tau_J$, such that

$$\forall j \in \{0, \ldots , J - 1\}, \quad \max\left(\|u\|_{L^5([\tau_j,\tau_{j+1}],L^{10}(\mathbb{R}^3))}, \|v\|_{L^5([\tau_j,\tau_{j+1}],L^{10}(\mathbb{R}^3))}\right) \leq \delta_0/2,$$

we see that it is sufficient to prove the theorem with the additional assumption

$$\max\left(\|u\|_{L^5([t_0,t_1],L^{10}(\mathbb{R}^3))}, \|v\|_{L^5([t_0,t_1],L^{10}(\mathbb{R}^3))}\right) \leq \delta_0/2.$$

Thus $u = \lim_{n\to\infty} u^n$, $v = \lim_{n\to\infty} v^n$ in $L^5(I, L^{10})$, $I = [t_0, t_1]$, where u^n and v^n are defined by

$$u^0 = v^0 = 0, \quad u^{n+1} = Au_n, \quad v^{n+1} = \widetilde{A}v^n,$$

where A is as in the proof of Theorem IV.3.1 (see (IV.3.3)), and $\widetilde{A}$ is the analog of A for the initial data of v:

$$\widetilde{A}w(t) = S_L(t)\vec{v}(0) + \int_0^t \frac{\sin\left((t - s)|D|\right)}{|D|} w^5(s)ds.$$

(As usual, we assume $t_0 = 0$ to simplify notations).

We prove by induction on n that $u^n(t, x) = v^n(t, x)$ for almost every $(t, x) \in \Gamma$. This is true for $n = 0$, since $u^0 = v^0 = 0$.

Next, we assume that $u^n(t, x) = v^n(t, x)$ for almost every $(t, x) \in \Gamma$. We have

$$u^{n+1}(t) - v^{n+1}(t) = S_L(t)(\vec{u}(0) - \vec{v}(0)) + \int_0^t \frac{(\sin(t - s)|D|)}{|D|} (u^n(s) - v^n(s))\, ds.$$

By finite speed of propagation for the linear wave equation and the assumption that $\vec{u}^0(x) = \vec{v}^0(x)$ for $|x - x_0| < R$, we obtain that $S_L(t)(\vec{u}(0) - \vec{v}(0)) = 0$ for almost all $(t, x) \in \Gamma$. On the other hand, if $s \in [0, t]$, the inductive hypothesis implies that $u^n(s, x) = v^n(s, x)$ for $|x - x_0| < R - s$. Combining with finite speed of propagation, we see that

$$\frac{(\sin(t - s)|D|)}{|D|} (u^n(s) - v^n(s)) = 0$$

for almost every (t, x) with $|x - x_0| < R - s - (t - s) = R - t$, i.e. for almost every $(t, x) \in \Gamma$.

Thus $u^n = v^n$ almost everywhere on Γ. Passing to the limit, we obtain $u^n = v^n$ on Γ. ☐

IV.5. Stability

We now prove that the flow of the equation (W5) is continuous in $\dot{\mathcal{H}}^1$, i.e. that if the initial data of two solutions u and v are close in this space, then $\vec{u}(t)$ and $\vec{v}(t)$ are close for all times t in their domain of existence. In the statement, we must take into account the fact that the solutions u and v might have different maximal interval of existence.

Theorem IV.5.1. *Let $t_0 \in \mathbb{R}$, $\vec{u}_0 = (u_0, u_1) \in \dot{\mathcal{H}}^1$. Let u be the solution of* (W5), (ID)*. Let I be a compact interval such that $t_0 \in I \subset I_{\max}(\vec{u}_0)$. Let $(\vec{u}_0^k)_k$ be a sequence in $\dot{\mathcal{H}}^1$ such that $\lim_n \vec{u}_0^k = \vec{u}_0$ in $\dot{\mathcal{H}}^1$. Let u^k be the corresponding solutions. Then for large k, we have $I \subset I_{\max}(\vec{u}_0^k)$. Moreover*

$$\lim_{k\to\infty} \left(\sup_{t\in I} \left\|\vec{u}^k(t) - \vec{v}^k(t)\right\|_{\dot{\mathcal{H}}^1} + \|u^k - v^k\|_{L^5(I,L^{10})}\right) = 0.$$

Proof. We will consider $T>0$ such that

$$\|u\|_{L^5([0,T],L^{10})}\leq \delta_0 \tag{IV.5.1}$$

(where δ_0 is a small parameter), and prove that $T^+(u^k)>T$ for large k and

$$\|u-u^k\|_{L^5([0,T],L^{10})}+\sup_{0\leq t\leq T}\left\|\vec{u}(t)-\vec{u}^k(t)\right\|_{\dot{\mathcal{H}}^1}\underset{k\to\infty}{\longrightarrow}0. \tag{IV.5.2}$$

The conclusion of the theorem will then follow by iteration, dividing as above the interval I into subintervals where the L^5L^{10} norm of u is small.

We have

$$u(t)-u^k(t)=S_L(t)(\vec{u}_0-\vec{u}_0^k)+\int_0^t\frac{\sin\left((t-s)|D|\right)}{|D|}\left(u^5(s)-(u^k)^5(s)\right)ds. \tag{IV.5.3}$$

As in (IV.3.4), we have

$$\left\|u^5-(u^k)^5\right\|_{L^1([0,t],L^2)}\leq\frac{5}{2}\left\|u-u^k\right\|_{L^5([0,t],L^{10})}\left(\|u\|^4_{L^5([0,t],L^{10})}+\left\|u^k\right\|^4_{L^5([0,t],L^{10})}\right).$$

Using the triangle inequality, and (IV.5.1), we deduce

$$\left\|u^5-(u^k)^5\right\|_{L^1([0,t],L^2)}\leq\frac{5}{2}\left\|u-u^k\right\|_{L^5([0,t],L^{10})}\left(2\delta_0^4+\left\|u-u^k\right\|^4_{L^5([0,t],L^{10})}\right).$$

Thus, by (IV.5.3) and Strichartz estimate, we have that for all $t\in[0,T]$

$$a_k(t)\leq C\left(\varepsilon_k+\delta_0^4a_k(t)+a_k(t)^5\right),$$

where $a_k(t)=\left\|u-u^k\right\|_{L^5([0,t],L^{10})}$, $\varepsilon_k=\left\|\vec{u}_0-\vec{u}_0^k\right\|_{\dot{\mathcal{H}}^1}\underset{k\to\infty}{\longrightarrow}0$, and C is a constant. Taking δ_0 small (so that $C\delta_0^4\leq 1/2$), we deduce

$$a_k(t)\leq 2C\varepsilon_k+2Ca_k(t)^5. \tag{IV.5.4}$$

We temporarily fix k, large enough so that $2C(4C\varepsilon_k)^5\leq C\varepsilon_k$, and prove

$$\forall t\in[0,T],\quad a_k(t)\leq 3C\varepsilon_k. \tag{IV.5.5}$$

Indeed, (IV.5.5) is true for small $t>0$, since a is continuous and $a(0)=0$. If (IV.5.5) does not hold, using again the continuity of a, we see that there exists a $t\in[0,T]$ such that $3C\varepsilon_k<a_k(t)\leq 4C\varepsilon_k$. By (IV.5.4), and the smallness of ε_k we see that $a_k(t)\leq 3C\varepsilon_k$. This is a contradiction, concluding the proof of (IV.5.5). This type of reasoning is called a *bootstrap argument*. By (IV.5.5),

$$\lim_{k\to\infty}a_k(T)=0$$

Using (IV.5.3) and Strichartz estimate again, we deduce

$$\sup_{t\in[0,T]}\left\|\vec{u}(t)-u^k(t)\right\|_{\dot{\mathcal{H}}^1}\underset{k\to\infty}{\longrightarrow}0,$$

which concludes the proof. □

IV.6. Persistence of regularity, conservation of the energy

The energy of a solution u of (W5) is defined as

$$E(\vec{u}(t))=\frac{1}{2}\int(\partial_tu(t,x))^2dx+\frac{1}{2}\int|\nabla u(t,x)|^2dx-\frac{\sigma}{6}\int(u(t,x))^6dx, \tag{IV.6.1}$$

where all integrals are taken over $\mathbb{R}^3$. Multiplying the equation (W5) by $\partial_tu(t,x)$, integrating on $\mathbb{R}^3$ and integrating by part, we would obtain that the derivative of the energy is 0, and thus that it is independent of time. However this computation is purely formal. To make it rigorous, we need to work on more regular solutions. The key ingredients for this are the stability theorem from the previous section and a *persistence of regularity* property:

Theorem IV.6.1. *Let* $\vec{u}_0=(u_0,u_1)\in\dot{\mathcal{H}}^1$, *$u$ be the solution of* (W5), (ID) *given by Corollary IV.3.3, and* $I_{\max}$ *its maximal interval of existence. Let $\ell\geq 2$ be an integer. Assume* $\vec{u}_0\in\dot{\mathcal{H}}^\ell$. *Then*

$$\vec{u}\in C^0\left(I_{\max},\dot{\mathcal{H}}^\ell\cap\dot{\mathcal{H}}^1\right),\quad\forall j\in\{1,\dots,\ell\},\quad\partial_t^ju\in C^0\left(I_{\max},\dot{H}^{\ell-j}\cap L^2\right) \tag{IV.6.2}$$

$$\forall\alpha,j\text{ with }j+|\alpha|\leq\ell-1,\quad\partial_x^\alpha\partial_t^ju\in L^5_{\mathrm{loc}}(I_{\max},L^{10}). \tag{IV.6.3}$$

In particular, if $\ell\geq 4$, $u\in C^2\left(I_{\max}\times\mathbb{R}^3\right)$

In the theorem, the derivative ∂_t can be interpreted as a derivative for a function of the time variable, with value in $L^2(\mathbb{R}^3)$.

REMARK IV.6.2. Recall that $\dot{H}^s$ is not complete for $s \geq \frac{N}{2}$. However, if $s_1 < \frac{N}{2}$ and $s_2 \in \mathbb{R}$, the space $\dot{H}^{s_1} \cap \dot{H}^{s_2}$, endowed with the norm defined by

$$\|f\|^2_{\dot{H}^{s_1} \cap \dot{H}^{s_2}} = \|f\|^2_{\dot{H}^{s_1}} + \|f\|^2_{\dot{H}^{s_2}}$$

is a Hilbert space. This encompasses the spaces $\dot{H}^\ell \cap \dot{H}^1$ and $\dot{H}^{\ell-1} \cap L^2$ that appear in the statement of the preceding theorem. Also, we have

$$\dot{H}^1(\mathbb{R}^3) \cap \dot{H}^2(\mathbb{R}^3) \subset (L^\infty \cap C^0)(\mathbb{R}^3) \tag{IV.6.4}$$

and

$$C_0^\infty(\mathbb{R}^3) \subset \bigcap_{s \geq 1} \dot{H}^s \subset C^\infty, \tag{IV.6.5}$$

with continuous embeddings. We leave the proof of these facts as an exercise to the reader. *Hint*: to prove (IV.6.4), use the Fourier representation of u:

$$u(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{ix\cdot\xi} \hat{u}(\xi) d\xi.$$

PROOF OF THEOREM IV.6.1. We prove the result for $\ell = 2$. The proof for $\ell \geq 3$ is very close and left to the reader. As usual, we assume $t_0 = 0$.

Step 1. Reduction to an interval with small Strichartz norm.

Let K be a compact sub-interval of $I_{\max}$. By time reversibility, we only consider positive times. We denote $[0, T] = K \cap [0, \infty)$. We let $T_0 = 0 < T_1 < \ldots < T_{n-1} < T_n = T$ such that

$$\|u\|_{L^5((T_j, T_{j+1}), L^{10})} \leq \delta/2, \tag{IV.6.6}$$

where $\delta > 0$ is the small constant given by local well-posedness theory. We will prove by induction on j that

$$\partial_{x_k} u \in L^5((T_j, T_{j+1}), L^{10}), \quad k = 1, 2, 3, \quad \partial_t u \in L^5((T_j, T_{j+1}), L^{10})$$

and

$$u \in C^0([T_j, T_{j+1}], \dot{H}^2), \quad \partial_t u \in C^0([T_j, T_{j+1}], \dot{H}^1).$$

By time translation, it is sufficient to consider the case $j = 0$. To simplify notation, we will denote $T = T_1$. We are thus reduced to prove the conclusion of the theorem with $\ell = 2$, assuming

$$\|u\|_{L^5((0,T), L^{10})} \leq \delta/2, \tag{IV.6.7}$$

Step 2.

By the proof of Theorem IV.3.1, the restriction of u to $[0, T]$ is the limit, in $L^5([0, T), L^{10})$, of the sequence u^n defined as above by $u^0 = 0$, $u^{n+1} = Au^n$, where A is defined by (IV.3.3). Let $j \in \{1, 2, 3\}$. We have

$$\partial_{x_j}(u^{n+1}) = S_L(t)\partial_{x_j}\vec{u}_0 + 5\sigma \int_0^t \frac{\sin\big((t-s)|D|\big)}{|D|} (u^n(s))^4 \, \partial_{x_j} u^n(s) ds, \tag{IV.6.8}$$

where ∂_{x_j} is the distributional derivative with respect to x_j, and we have used the formula $\partial_{x_j}(v^5) = 5v^4 \partial_{x_j} v$, which is valid for $v \in \dot{H}^1 \cap \dot{H}^2$ (this can be checked easily using that the functions in $\dot{H}^1 \cap \dot{H}^2$ are continuous).

Fixing $j \in \{1, 2, 3\}$, we prove by induction on n that

$$\|\partial_{x_j} u^n\|_{L^5([0,T], L^{10})} \leq M_j, \tag{IV.6.9}$$

where

$$M_j = 2 \left\| S_L(t)(\partial_{x_j} u_0, \partial_{x_j} u_1) \right\|_{L^5([0,T], L^{10})},$$

is finite since $(u_0, u_1) \in \dot{H}^2 \times \dot{H}^1$. The case $n = 0$ is trivial since $u^0 = 0$.

Next we assume that $\partial_{x_j} u^n \in L^5([0, T], L^{10})$ and satisfies (IV.6.9). Then by Strichartz estimates, the definition of u^{n+1}, the inductive hypothesis, Hölder inequality and the smallness of δ:

$$\|\partial_{x_j} u^n\|_{L^5([0,+T], L^{10})} \leq M_j/2 + C_S M_j \delta^4 \leq M_j. \tag{IV.6.10}$$

This shows that (IV.6.9) holds for all n.

By (IV.6.9) and Banach-Alaoglu, we obtain that there exists a subsequence $(u^{n_k})_k$ of $(u^n)_n$ and $v \in L^5((0, T), L^{10})$ such that

$$\partial_{x_j} u^{n_k} \xrightarrow[k \to \infty]{} v_j \text{ in } L^5((0, T), L^{10}).$$

By uniqueness of limits in the sense of distribution on $\mathbb{R}\times\mathbb{R}^3$, we obtain

$$\partial_{x_j} u = v_j \in L^5([0,T],L^{10}).$$

The same argument, using the equality (between $C^0([0,T],L^2)$ functions):

$$\partial_t(u^{n+1}) = S_L(t)(u_1,\Delta u_0+\mu u_0^5)+5\sigma\int_0^t \frac{\sin\big((t-s)|D|\big)}{|D|}\,(u^n(s))^4\,\partial_t u^n(s)ds, \tag{IV.6.11}$$

yields that $(\partial_t u^n)_n$ is bounded in $L^5([0,T],L^{10})$ and that

$$\partial_t u \in L^5([0,T],L^{10}).$$

To prove, (IV.6.11), we use that for all $t\in(0,T)$

$$\lim_{n\to\infty}\frac{(u^n)^5(t+h)-(u^n)^5(t)}{h} = 5\partial_t u^n(t)u^n(t)^4 \text{ in } L^2(\mathbb{R}^3)$$

Indeed, fixing t, we have $u^n(t+h)=u^n(t)+h\partial_t u^n(t)+h\varepsilon(h)$, where ε (depending also on t and n) satisfies $\lim_{h\to 0}\|\varepsilon(h)\|_{L^2}=0$. Thus

$$\begin{aligned}(u^n)^5(t+h) = (u^n)^5(t)+h(\partial_t u^n(t)+\varepsilon_n(t))(u^n)^4(t)+\sum_{j=2}^5\binom{j}{5}u^n(t)^{5-j}h^j(\partial_t u^n(t)+\varepsilon_n(t))^j\\ = (u^n)^5(t)+h\partial_t u^n(t)(u^n)^4(t)+o_n(h) \text{ in } L^2,\end{aligned}$$

where we used that $u^n\in(L^\infty\cap C^0)([0,T],\dot{H}^1\cap\dot{H}^2)$ (which can be proved by induction). Thus $u^n\in(L^\infty\cap C^0)([0,T]\times\mathbb{R}^3)$, and as a consequence $h(\partial_t u^n(t)+\varepsilon_n(t))$ is uniformly bounded.

Step 3. The formulas

$$\partial_{x_j}(u) = S_L(t)\partial_{x_j}\vec{u}_0+5\sigma\int_0^t\frac{\sin\big((t-s)|D|\big)}{|D|}\,(u(s))^4\,\partial_{x_j}u(s)ds,$$
$$\partial_t(u^n) = S_L(t)(u_1,\Delta u_0+\mu u_0^5)+5\sigma\int_0^t\frac{\sin\big((t-s)|D|\big)}{|D|}\,(u(s))^4\,\partial_t u(s)ds,$$

follow from step 2, letting $n\to\infty$ in (IV.6.8), (IV.6.11). Using these formulas and energy inequalities, we deduce from the fact that $\partial_t u$, $(\partial_{x_j}u)_{j=1,2,3}$ are in $L^5([0,T],L^{10})$ that

$$\partial_t\vec{u}\in C^0([0,T],\dot{\mathcal{H}}^1),\quad \partial_{x_j}u\in C^0([0,T],\dot{\mathcal{H}}^1).$$

This concludes the proof for $\ell=2$. The proof for $\ell\geq 3$ is by induction. More precisely assuming that the result holds true for some $\ell\geq 2$, a similar proof as before yields the result for $\ell+1$, dividing the interval $[0,T]$ in a finite number of subintervals (a_j,a_{j+1}) such that

$$\sup_{0\leq|\alpha|+j\leq\ell-1}\left\|\partial_t^j\partial_x^\alpha u\right\|_{L^5((a_j,a_{j+1}),L^{10})}\leq\delta/2.$$

□

Remark IV.6.3. If $T_+<\infty$, then

$$\lim_{t\to T_+}\|\vec{u}(t)\|_{\dot{\mathcal{H}}^1\cap\dot{\mathcal{H}}^2}=+\infty.$$

This is an immediate consequence of the embedding $\dot{H}^2\cap\dot{H}^1$ and the blow-up criterion in L^5L^{10}.

Corollary IV.6.4. *Let u be a solution with initial data $(u_0,u_1)\in(C_0^\infty(\mathbb{R}^3))^2$. Then the corresponding solution u of* (W5), (ID) *is in $C^\infty(I_{\max}\times\mathbb{R}^3)$, where $I_{\max}=I_{\max}(\vec{u}_0)$ is the maximal interval of existence of u.*

Proof. The corollary follows immediately from Theorem IV.6.1, using Sobolev embeddings (see (IV.6.5)).
□

We are now in position to prove rigorously the conservation of the energy:

Theorem IV.6.5. *Let the energy E be defined by* (IV.6.1). *Let $\vec{u}$ be a solution of* (W5). *Then $E(\vec{u}(t))$ is independent of $t\in I_{\max}(u)$.*

Proof. Let $t_0, t_1 \in I_{\max}(u)$. Let $\vec{u}_0^n = (u_0^n, u_1^n) \in \left(C_0^\infty(\mathbb{R}^3)\right)^2$ such that

$$\lim_{n\to\infty} \|\vec{u}_0^n - \vec{u}(t_0)\|_{\dot{H}^1} = 0. \tag{IV.6.12}$$

Let u^n be the solution of (W5) with initial data $u^n(0) = u_0^n$, $\partial_t u^n(0) = u_1^n$. By the stability theorem IV.5.1, $[t_0, t_1]$ is included in the maximal interval of existence of u^n for large n.

By Corollary IV.6.4, $u^n \in C^\infty([t_0, t_1] \times \mathbb{R}^3)$. Since it satisfies (W5) in the sense of distribution, it must also satisfy this equation in the classical sense. By finite speed of propagation $u^n(t)$ is a compactly supported function (in space) for all $t \in [t_0, t_1]$. We have

$$\int \partial_t^2 u^n \partial_t u^n - \int \Delta u^n \partial_t u^n - \sigma \int (u^n)^5 \partial_t u^n = 0$$

Since $\int \Delta u^n \partial_t u^n = \int \sum_{j=1,2,3} \partial_{x_j} u^n \partial_t \partial_{x_j} u^n$, we deduce

$$\frac{d}{dt} E(\vec{u}^n(t)) = 0, \quad t_0 \leq t \leq t_1.$$

Thus $E(\vec{u}^n(t_0)) = E(\vec{u}^n(t_1))$. Passing to the limit $n \to \infty$ and using Theorem IV.5.1, we deduce

$$E(\vec{u}(t_0)) = E(\vec{u}(t_1)),$$

concluding the proof. We have used that by the Sobolev embedding $\dot{H}^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$, the convergence in $\dot{H}^1$ implies the convergence in L^6. □

In the case $\sigma = -1$, all the terms in the definitions of the energy are positive, and we have

$$\vec{u}(t) \leq 2E(\vec{u}(t)).$$

This implies that the $\dot{H}^1$ norm of any solution u of (W5) is bounded on its maximal interval of existence. This is not sufficient to ensure global existence. We will see however that in this case, all solutions are indeed global.

Definition IV.6.6. The equation (W5) or the corresponding nonlinearity is called *defocusing* (or *repulsive*) when $\sigma = -1$ and *focusing* (or *attractive*) when $\sigma = 1$.

Let us mention that we can also construct classical solutions of (W5) (or of any equation of the form (NLW) with $p \in \mathbb{N}$, $p \geq 2$, in space dimension 3), without Strichartz estimates, using the representation formulas of Chapter 1 and a fixed point argument. These solutions coincide with the finite energy solutions of Definition IV.2.1 when $\vec{u}_0 \in C_0^3(\mathbb{R}^3) \times C_0^2(\mathbb{R}^3)$ for example. This is an alternative approach to obtain Corollary IV.6.4. We refer to [**30**, Section I.5] for the details.

CHAPTER V

Examples of dynamics

In this chapter, we consider again the quintic wave equation in space dimension 5:

$$(\partial_t^2 - \Delta)u = \sigma u^5. \tag{W5}$$

We give 3 examples of dynamics of (W5). Section V.1 concerns solutions that blow up in finite times (in the case $\sigma = 0$). Section V.2 deals with global solutions which behave asymptotically as solutions of the linear wave equation. In Section V.3, we will consider stationary solutions and traveling waves.

V.1. Blow-up in finite time

In the focusing case $\sigma = 1$, there exists solutions blowing-up in finite time:

Theorem V.1.1. *Let $T > 0$. There exists a solution u of* (W5)*, with C^∞, compactly supported initial data $\vec{u}_0$ at $t = 0$, such that $T_+(\vec{u}_0) = T$.*

Proof. By scaling invariance, it is sufficient to construct one solution of (W5) blowing-up in finite time, with compactly supported, smooth initial data.

Let Y be a solution of the ODE $Y'' = Y^5$ defined on $[0, 1[$, and blowing-up at $t = 1$. For example $Y(t) = c(1-t)^{-1/2}$, where $\frac{3}{4} = c^4$. Note that Y is a solution of (W5) (in the classical sense).

Let $\varphi \in C_0^\infty(\mathbb{R}^3)$ such that $\varphi(x) = 1$ for $|x| \leq 2$. Let u be the solution of (LW) with initial data $(\varphi Y(0), \varphi Y'(0))$. Let T_+ be the maximal time of existence of u. By finite speed of propagation,

$$u(t,x) = Y(t), \quad |x| \leq 2 - t, \quad t \in [0, T_+[.$$

If $T_+ > 1$, we have

$$\int_0^1 \left(\int_{|x|\leq 1} u^{10}(t,x)dx \right)^{1/2} dt = c^5 \int_0^1 \frac{1}{(1-t)^{5/2}} dt = +\infty,$$

a contradiction with the fact that u must be in $L^5([0,1], L^{10})$. Thus $T_+ \leq 1$, concluding the proof. ☐

The preceding proof is not completely rigorous: we have used finite speed of propagation for the equation (W5) outside of the framework of Theorem IV.4.2. However Y is not a solution of (W5) in the sense of Definition IV.2.1. We thus need the analog of V.1.2 for classical solutions:

Theorem V.1.2. *Let $(t_0, x_0) \in \mathbb{R}^{1+3}$, $t_1 > t_0$, $R > 0$. We denote by $\Gamma = \Big\{(t,x) \in \mathbb{R} \times \mathbb{R}^N \ : \ t_0 \leq t \leq t_1,\ |x - x_0| \leq R - |t - t_0|\Big\}$ Let $u, v \in C^2(\Gamma)$ be two classical solutions of* (W5) *on Γ. We suppose $(u, \partial_t u)(t_0, x) = (v, \partial_t v)(t_0, x)$ for all $x \in B_R(x_0)$. Then $u(t,x) = v(t,x)$ all $(t,x) \in \Gamma$.*

We leave the proof of Theorem V.1.2 as an exercise to the reader:

Exercise V.1. Let u and v be as in Theorem V.1.2. Assume $t_0 = 0$, $x_0 = 0$. Let

$$V(t) = \frac{1}{2}\int_{|x|<R-t} (u(t,x) - v(t,x))^2 dx + \frac{1}{2}\int_{|x|<R-t} (\partial_t u(t,x) - \partial_t v(t,x))^2 dx + \frac{1}{2}\sum_{j=1}^{3} \int_{|x|<R-t} (\partial_{x_j} u(t,x) - \partial_{x_j} v(t,x))^2 dx.$$

1. Prove that $V'(t) \leq CV(t)$ for $t \in [0, t_1]$.
2. Prove that $V(t) = 0$ for all $t \in [0, t_1]$.

context.

Remark V.1.3. There are at least two other ways to prove the existence of solutions of the focusing (W5) that blow up in finite time:

- A monotonicity/convexity argument based on the computation of the first and second derivatives in time of $\int |u(t,x)|^2 dx$. This argument goes back to an article of Howard Levine [**26**], and implies that any solution with negative energy blows up in finite time. It is detailed in [**23**] (see the proof of Theorem 3.7 there).
- A proof based on differential inequalities and the fact that $\frac{\sin(t|D|)}{|D|}$ is a positive functional in space dimension 3, as seen in Exercise 6 of the worksheet of Chapter 1 (see [**21**]).

V.2. Scattering

2.a. Definition and characterization.

DEFINITION V.2.1. The solution u of (W5) is said to *scatter* in the future to a linear solution if $T_+(u) = +\infty$ and there exists $\vec{v}_0 \in \dot{\mathcal{H}}^1(\mathbb{R}^3)$ such that

$$\lim_{t\to\infty} \left\| \vec{S}_L(t)\vec{v}_0 - \vec{u}(t) \right\|_{\dot{\mathcal{H}}^1} = 0. \tag{V.2.1}$$

In the remainder of this section, we will simply say that a solution as in Definition V.2.1 scatters or is a scattering solution. We next give a characterization of scattering solutions:

PROPOSITION V.2.2. *The solution u of* (W5), (ID) *scatters if and only if $u \in L^5([0,T_+), L^{10})$, where T_+ is the maximal time of existence of u.*

PROOF. Let u be a solution such that $u \in L^5([0,T_+), L^{10})$. By the blow-up criterion, we already know that $T_+(u) = +\infty$. Let $\vec{v}_0 \in \dot{\mathcal{H}}^1$. Since $\vec{S}_L(t)$ conserves the $\dot{\mathcal{H}}^1$ norm, we have

$$\text{(V.2.1)} \iff \lim_{t\to\infty} \left\| \vec{v}_0 - \vec{S}_L(-t)\vec{u}(t) \right\|_{\dot{\mathcal{H}}^1} = 0.$$

We are thus reduced to prove that $\vec{S}_L(-t)\vec{u}_0(t)$ has a limit in $\dot{\mathcal{H}}^1$. Since u is a solution in the sense of Definition IV.2.1, we have

$$\vec{S}_L(-t)\vec{u}(t) = \vec{u}_0 + \int_0^t \vec{S}_L(-s)(0, u^5(s))ds.$$

Using $u \in L^5([0,+\infty, L^{10})$ and

$$\left\| \vec{S}_L(-s)(0, u^5(s)) \right\|_{\dot{\mathcal{H}}^1} = \left\| u^5(s) \right\|_{L^2} = \|u(s)\|_{L^{10}}^5,$$

we see that

$$\int_0^\infty \left\| \vec{S}_L(-s)(0, u^5(s)) \right\|_{\dot{\mathcal{H}}^1} ds = \|u\|^5_{L^5([0,\infty),L^{10})} < \infty.$$

Thus $\int_0^t \vec{S}_L(-s)(0,u^5(s))ds$ converges in $\dot{\mathcal{H}}^1$ as t goes to ∞, which shows that u scatters to a linear solution.

Next, we consider a solution u of (W5) that scatters to a linear solution. Thus $T_+(u) = \infty$, and there exists $\vec{v}_0 \in \dot{\mathcal{H}}^1$ such that

$$\lim_{t\to\infty} \left\| \vec{u}(t) - \vec{S}_L(t)\vec{v}_0 \right\|_{\dot{\mathcal{H}}^1} = 0.$$

Fix $T_0 \geq 0$ such that

$$\|S_L(\cdot)\vec{v}_0\|_{L^5([T_0,\infty[,L^{10})} \leq \delta_0/2,$$

where δ_0 is given by the local well-posedness theory (Theorem IV.3.1). Let $T \geq T_0$. Then, by Strichartz estimates

$$\|S_L(\cdot)\vec{u}(T)\|_{L^5([0,\infty[,L^{10})} \leq \|S_L(\cdot)\vec{v}_0\|_{L^5([T,\infty[,L^{10})} + C_S \left\| \vec{u}(T) - \vec{S}_L(T)\vec{v}_0 \right\|_{\dot{\mathcal{H}}^1} \leq \delta_0$$

for large T. By Theorem IV.3.1 and the uniqueness Lemma IV.3.2,

$$u \in L^5([T,+\infty), L^{10})$$

which concludes the proof. □

Combining Theorem IV.3.1, Strichartz estimates and Proposition V.2.2, we obtain:

COROLLARY V.2.3 (Small data scattering). *There exists a constant $\varepsilon > 0$ such that for all $\vec{u}_0 \in \dot{\mathcal{H}}^1$ with $\|\vec{u}_0\|_{\dot{\mathcal{H}}^1} \leq \varepsilon$, the solution of* (W5), (ID) *scatter in both time directions.*

Two natural questions arise:

Existence of wave operators: Given $\vec{v}_0 \in \dot{\mathcal{H}}^1$, does there exist a solution u of (W5) with $T_+(u) = +\infty$ and

$$\lim_{t\to\infty} \left\| \vec{u}(t) - \vec{S}_L(t)\vec{v}_0 \right\|_{\dot{\mathcal{H}}^1} = 0? \tag{V.2.2}$$

Asymptotic completeness: Do all solutions of (W5) scatter?

It turns out that the answer to the first question is always positive, independently of the sign σ in (W5). The asymptotic completeness is a much more delicate issue. We already know that it is not true in the focusing case $\sigma = 1$, since there exist solutions blowing-up in finite time (see Section V.1). On the other hand, the asymptotic completeness holds in the defocusing case $\sigma = -1$ (see [**4**]). We will prove this fact for radial solutions. The general proof is more complicated but relies on the same type of arguments.

2.b. Existence of wave operators.

THEOREM V.2.4. *Let* $\vec{v}_0 \in \dot{\mathcal{H}}^1$. *Then there exists a solution* u *of* (W5) *with* $T_+(u) = +\infty$ *and such that* (V.2.2) *holds.*

PROOF. Let $\vec{v}_0 \in \dot{\mathcal{H}}^1$. Let u be a scattering solution of (W5), defined on $[t_0, \infty) \times \mathbb{R}^3$, such that (V.2.2) holds. Letting $t \to \infty$ in the equality

$$\vec{S}_L(-t)\vec{u}(t) = \vec{S}_L(-t_0)\vec{u}_0 + \sigma \int_{t_0}^{t} \vec{S}_L(-s)\left(0, u^5(s)\right) ds,$$

we obtain

$$\vec{v}_0 = S_L(-t_0)\vec{u}_0 + \sigma \int_{t_0}^{\infty} \vec{S}_L(-s)\left(0, u^5(s)\right) ds. \tag{V.2.3}$$

Note that the integral is convergent in $\dot{\mathcal{H}}^1$ by conservation of the energy for the linear wave equation and since $u \in L^5([t_0, \infty[, L^{10})$. In view of (V.2.3), we can rewrite Duhamel's formula as

$$u(t) = S_L(t)\vec{v}_0 - \sigma \int_{t}^{\infty} S_L(t-s)\left(0, u^5(s)\right) ds. \tag{V.2.4}$$

Conversely, if $u \in L^5([t_0, \infty), L^{10})$ satisfies (V.2.4) for $t \geq t_0$, then it is a solution of (W5) such that $T_+(u) = \infty$ and (V.2.2) holds. This shows that the problem of existence of wave operator can be interpreted as a Cauchy problem with initial data at time infinity. To solve this problem, we fix T large such that

$$\|S_L(\cdot)\vec{v}_0\|_{L^5([T,\infty),L^{10})} \leq \delta_0,$$

for some small $\delta_0 > 0$ and we prove that the operator A defined by

$$Av(t) = S_L(t)\vec{v}_0 - \sigma \int_{t}^{\infty} S_L(t-s)\left(0, v^5(s)\right) ds$$

is a contraction of the metric space X defined by

$$X = \left\{ v \in L^5([T, \infty), L^{10}),\ \|v\|_{L^5([T,\infty),L^{10})} \leq 2\delta_0 \right\}$$

The details are very close to the ones of the proof of Theorem IV.3.1 and are left to the reader. □

2.c. Asymptotic completeness. The existence of solutions blowing up in finite time excludes asympotic completeness for the focusing nonlinearity $\sigma = 1$. In the defocusing case, we have:

THEOREM V.2.5. *Let* u *be a solution of* (W5) *with* $\sigma = -1$. *Then* u *scatters.*

This is due to several authors. See e.g. J. Ginibre, A. Soffer, G. Velo (see [**17**]) for the radial case, and H. Bahouri and J. Shatah [**4**]. We will give elements of proofs in the end of this course.

V.3. Stationary solutions and travelling waves

3.a. Stationary solutions. We are interested by stationary solutions of the equation (W5), i.e. nonzero, $\dot{H}^1$ solutions of the elliptic equation $-\Delta Q = \sigma Q^5$. In the defocusing case $\sigma = -1$, the equation is

$$-\Delta Q + Q^5 = 0,$$

to be interpreted in the sense of distribution on $\mathbb{R}^3$. This means

$$\forall \varphi \in C_0^\infty(\mathbb{R}^3), \quad \int \nabla Q \cdot \nabla \varphi + \int Q^5 \varphi = 0.$$

Approximation Q by smooth, compactly supported functions, we obtain

$$\int |\nabla Q|^2 + \int Q^6 = 0,$$

which implies $Q = 0$ a.e. Thus in the defocusing case, the only nonstationary solution is the constant null solution. This was already known, since in this case, all solutions scatter and a scattering solution cannot be stationary since it is in $L^5(\mathbb{R}, L^{10})$.

We next consider the focusing case $\sigma = 1$. The equation is:

$$-\Delta Q = Q^5, \quad Q \in \dot{H}^1(\mathbb{R}^3). \tag{Ell}$$

Since Q must be a solution of (W5) in the sense of definition IV.2.1, we will also assume $Q \in L^{10}$.

PROPOSITION V.3.1. *Let Q be a solution of* (Ell) *with $Q \in L^{10}$. Then*

$$Q \in \bigcap_{s \geq 1} \dot{H}^s.$$

In particular, Q is bounded as well as all of its derivative.

PROOF. Since $Q \in L^{10}(\mathbb{R}^3)$, we have $Q^5 \in L^2(\mathbb{R}^3)$. Thus $\Delta Q \in L^2$. This implies $|\xi|^2 \hat{Q}(\xi) \in L^2$, and thus $Q \in \dot{H}^1 \cap \dot{H}^2$.

Using that $\dot{H}^1 \cap \dot{H}^s$ is an algebra for $s \geq 2$ one obtains by induction on s, using the equation $-\Delta Q = Q^5$, that $Q \in \dot{H}^1 \cap \dot{H}^s$ for all $s \geq 2$.

Recalling that $\dot{H}^2 \cap \dot{H}^1(\mathbb{R}^3)$ is continuously embedded into $C_b^0(\mathbb{R}^3)$ (the space of bounded, continuous functions on $\mathbb{R}^3$), we obtain that Q is C^∞ and that all its derivatives are bounded. □

EXERCISE V.2. Let $s \geq 2$ be an integer, prove that $\dot{H}^s \cap \dot{H}^1$ is an algebra.

Let us mention that the assumption $Q \in L^{10}$ in Proposition V.3.1 is not necessary, and that it is possible (but not trivial), to prove that Q satisfying (Ell) must be in $C^\infty \cap L^\infty$ (see [**36**]). Note that in this case, a simple elliptic regularity argument based on Sobolev inequalities does not work. Indeed, we have:

$$Q \in \dot{H}^1 \Longrightarrow Q \in L^6 \Longrightarrow \Delta Q = -Q^5 \in L^{6/5} \Longrightarrow \nabla Q \in L^2, \tag{V.3.1}$$

where we used the Sobolev embeddings $\dot{H}^1 \subset L^6$ and $\dot{W}^{2,6/5} \subset \dot{H}^1$. Of course, the circular implications (V.3.1) do not give any improvement on the regularity of Q.

The equation (Ell) has nonzero solutions. In the radial case, the solutions are completely classified.

THEOREM V.3.2. *Let*

$$W(x) = \frac{1}{\left(1 + \frac{|x|^2}{3}\right)^{1/2}}, \quad x \in \mathbb{R}^3 \tag{V.3.2}$$

Then W is a solution of (Ell)*. Furthermore the set of radial solutions of* (Ell) *is given by*

$$\Sigma = \{0\} \cup \left\{ \frac{\iota}{\lambda^{1/2}} W\left(\frac{\cdot}{\lambda}\right),\ \lambda > 0,\ \iota \in \{\pm 1\} \right\}.$$

PROOF. It can be checked by explicit computations that W is a solution of (Ell) . Since the equation is invariant by scaling and sign change, we obtain also that $W_{\iota,\lambda} = \frac{\iota}{\lambda^{1/2}} W\left(\frac{\cdot}{\lambda}\right)$ is a solution for any $\lambda > 0$, $\iota = 1$ or -1. We have $W_{\iota,\lambda}(0) = \frac{\iota}{\lambda^{1/2}}$. This can take any nonzero real value a.

Since any L^{10} solution of (Ell) is smooth, and smooth radial functions satisfy $Q'(0) = 0$, we are reduced to prove that for all $a \in \mathbb{R}$, there is at most one smooth solution of

$$-y'' - \frac{2}{r} y' = y^5, \quad y(0) = a,\ y'(0) = 0. \tag{V.3.3}$$

We can write the differential equation satisfied by y as $\frac{d}{dr}\left(r^2 y'\right) = -r^2 y^5$. If y is a solution of (V.3.3) we obtain, integrating twice between 0 and r,

$$y(r) = a - \int_0^r \frac{1}{s^2} \int_0^s \rho^2 y^5(\rho) d\rho\, ds.$$

Let y and z be two solutions of (V.3.3). For $0 < R \leq 1$, we let

$$M(R) = \sup_{0 < r \leq R} |y(r) - z(r)|, \quad K = \sup_{0 < r \leq 1} \left(|y(r)| + |z(r)|\right).$$

Then, using the inequality $|y^5 - z^5| \leq \frac{5}{2}|y-z|(y^4+z^4)$, we obtain, for $0 < r \leq R \leq 1$,

$$|y(r) - z(r)| \leq \frac{5K^4}{2} M(R) \int_0^r \frac{1}{s^2} \int_0^s \rho^2 d\rho ds = \frac{5}{12} K^4 M(R) r^2.$$

Taking the supremum over $r \in (0, R]$, we deduce:

$$M(R) \leq \frac{5}{12} K^4 R^2 M(R).$$

Thus $M(R) = 0$ for $\frac{5}{12} K^4 R^2 < 1$. This shows that $y(r) = z(r)$ for $r > 0$ close to 0. By uniqueness in the Cauchy-Lipschitz theorem, we deduce $y = z$ everywhere. This concludes the proof. □

The equation (Ell) without symmetry assumption admits other solutions. Multiplying (Ell) by Q and integrating by parts, we obtain

$$\int |\nabla Q|^2 = \int Q^6 = 3E(Q, 0).$$

In particular, the energy of a nonzero solution Q of (Ell) (considered as a solution of (W5)) is positive. Combining with (V.3.4), one obtains that the energy of any nonzero solution of (Ell) is greater or equal to $E(W, 0)$. The least-energy nonzero solution W of (Ell) is sometimes called the *ground state* of (W5). It was proved by Ding in 1986 (see [**11**]) that one can also construct solutions of (Ell) with arbitrarily large energy. However there is no general classification of these solutions.

We next state an important property of W, that we will not prove

Theorem V.3.3. *The ground state W is the maximizer for the Sobolev inequality on $\mathbb{R}^3$: $\|f\|_{L^6} \lesssim \|\nabla f\|_{L^2}$. That is, if $f \in \dot{H}^1$, one has*

$$\int f^6 \leq C_s \left(\int |\nabla f|^2 \right)^3, \quad C_s = \int W^6 \times \left(\int |\nabla W|^2 \right)^{-3} = \left(\int |\nabla W|^2 \right)^{-2}, \tag{V.3.4}$$

with equality if and only if

$$\exists \mu \in \mathbb{R},\ \lambda > 0,\ X \in \mathbb{R}^3 \ s.t.\ f(x) = \mu W(\lambda(x - X)).$$

This was proved independently by Aubin [**1**] and Talenti [**33**] in the mid 70's. We will discuss the proof of Theorem V.3.3 in the next chapter.

3.b. Travelling waves. Travelling wave solutions of (W5) are by definitions solutions of the form $\varphi(x - \mathbf{c}t)$, where the speed $\mathbf{c} \in \mathbb{R}^3$ is fixed, and $\varphi \in \dot{H}^1$. Using the invariance of (W5) by rotation, we can assume $\mathbf{c} = (c, 0, 0)$, where $c \in \mathbb{R}$. We are thus lead to study solutions of (W5) of the form

$$u(t, x) = \varphi(x_1 - tc, x_2, x_3), \quad c \in \mathbb{R}, \quad \varphi \in \dot{H}^1. \tag{V.3.5}$$

These solutions can be deduced from solutions of the elliptic equation (Ell).

Theorem V.3.4. *Let u be a nonzero solution of* (W5) *of the form* (V.3.5)*. Then $\sigma = 1$, $|c| < 1$, and Q defined by*

$$Q(x_1, x_2, x_3) = \varphi\left(x_1\sqrt{1 - c^2}, x_2, x_3\right) \tag{V.3.6}$$

is a solution of (Ell).

Remark V.3.5. Recall from Exercise I.8 the definition of the Lorentz boost of a function $u : \mathbb{R}^4 \to \mathbb{R}$. One can check that the Lorentz boost of a C^2, global solution u of (W5) is also a solution of (W5). The travelling waves are exactly given by applying Lorentz boosts to solutions of (Ell).

Proof of the Theorem. Let u be a nonzero travelling wave solution.

The fact that $|c| < 1$ follows from finite speed of propagation. Indeed, arguing by contradiction, we consider a solution u of (W5) of the form (V.3.5), with $c \geq 1$ (where we have assumed c positive to fix ideas, the case $c \leq -1$ can be deduced by the transformation $x_1 \mapsto -x_1$).

We fix $L > 0$ such that

$$\int_{x_1 > L} |\nabla u_0|^2 + u_1^2 = \varepsilon^2, \tag{V.3.7}$$

where $\varepsilon > 0$ is small. Let $(v_0, v_1) \in \dot{\mathcal{H}}^1(\mathbb{R}^3)$ such that

$$x_1 \geq L \Longrightarrow (v_0, v_1)(x) = (u_0, u_1)(x), \quad \int |\nabla v_0|^2 + v_1^2 dx \leq 2\varepsilon^2. \tag{V.3.8}$$

(Defining v_j for $j=0,1$ by $v_j(x_1,x_2,x_3)=u_j(2L-x_1,x_2,x_3)$ for $x_1\leq L$ would work for example). Let v be the solution of (W5) with initial data (v_0,v_1) at $t=0$. By the small data theory (Theorem IV.3.1), $v\in L^5(\mathbb{R},L^{10})$. By (V.3.8) and finite speed of propagation,

$$\forall t\geq 0,\quad \forall x\in\mathbb{R}^3,\quad x_1\geq L+t\Longrightarrow v(t,x)=u(t,x)=\varphi(x_1-ct,x_2,x_3).$$

Thus

$$\int_{\mathbb{R}^3}|v(t,x)|^{10}dx\geq\int_{x_1\geq L+t}|\varphi(x_1-ct,x_2,x_3)|^{10}dx\geq\int_{x_1\geq L+(1-c)t}|\varphi(x)|^{10}dx\geq a,$$

where $a=\int_{x_1\geq L}|\varphi|^{10}dx>0$ by (V.3.7). This concludes the proof.

We thus have $c<1$. In this case, it is easy to check, using (W5), that Q defined by (V.3.6) satisfies $-\Delta Q=\sigma Q^5$. This implies since Q is not identically 0, that $\sigma=1$ and that Q is solution to (Ell), which concludes the proof. □

3.c. Energy trapping. In this subsection, we state and prove a property of solutions of (W5) with $\sigma=1$, that is:

$$\partial_t^2u-\Delta u=u^5. \tag{W5f}$$

with energy below $E(W,0)$. This property will be proved by purely variational arguments, using only the the conservation of the energy and the continuity of the flow of (W5f) in $\dot{\mathcal{H}}^1$.

Proposition V.3.6. *Let $E<E(W,0)$. Then there exist constants $0<C_1(E)<\int|\nabla W|^2<C_2(E)$ such that for any $(u_0,u_1)\in\dot{\mathcal{H}}^1(\mathbb{R}^3)$ with $E(u_0,u_1)=E$, one has*

$$\int|\nabla u_0|^2+u_1^2\leq C_1(E)\ \text{or}\ \int|\nabla u_0|^2+u_1^2\geq C_2(E).$$

Corollary V.3.7 (Energy trapping). *Let u be a solution of* (NLW) *with $\sigma=1$, with initial data $(u_0,u_1)\in\dot{\mathcal{H}}^1$ and energy $E<E(W,0)$. Then one of the following holds:*

(1) $\int|\nabla u_0|^2+\int u_1^2<\int|\nabla W|^2$. *In this case*

$$\forall t\in I_{\max}(u),\quad\int|\nabla u(t)|^2+\int(\partial_tu(t))^2\leq C_1(E).$$

(2) $\int|\nabla u_0|^2+\int u_1^2>\int|\nabla W|^2$. *In this case*

$$\forall t\in I_{\max}(u),\quad\int|\nabla u(t)|^2+\int(\partial_tu(t))^2\geq C_2(E).$$

The corollary is an immediate consequence of Proposition V.3.6, the intermedaite value theorem, and the continuity of the map $t\mapsto\int|\nabla u(t)|^2+\int(\partial_tu(t))^2$. We next prove Proposition V.3.6.

Proof. By the Sobolev ineqality with the optimal constant, we have

$$\int u_0^6\leq\frac{1}{\left(\int|\nabla W|^2\right)^2}\left(\int|\nabla u_0|^2\right)^3.$$

Thus

$$E=\frac{1}{2}\int|\nabla u_0|^2+\frac{1}{2}\int u_1^2-\frac{1}{6}\int u_0^6\geq\frac{1}{2}\int|\nabla u_0|^2+\frac{1}{2}\int u_1^2-\frac{1}{6\left(\int|\nabla W|^2\right)^2}\left(\int|\nabla u_0|^2\right)^3.$$

Let F the function defined by $F(a)=\frac{1}{2}a-\frac{a^3}{6(\int|\nabla W|^2)^2}$. The preceding inequality implies

$$E\geq F(A),$$

where $A=\int u_1^2+\int|\nabla u_0|^2$. We have $F'(a)=\frac{1}{2}\left(1-a^2\left(\int|\nabla W|^2\right)^{-2}\right)$. Thus F is increasing on $\left(0,\int|\nabla W|^2\right)$ and decreasing on $\left(\int|\nabla W|^2,\infty\right)$, its maximum value is $E(W,0)$, attained at $a=\int|\nabla W|^2$. Since $E<E(W,0)$, we obtain that there exist two positive numbers $0<C_1(E)<\int|\nabla W|^2<C_2(E)$ such that $F(C_1(E))=F(C_2(E))=E$, and

$$F(a)>E\iff C_1(E)<a<C_2(E).$$

The conclusion of the proposition follows. □

Proposition V.3.6 and Corollary V.3.7 are due to F. Merle and C. Kenig: see [**23**]. The proof of this result is quite robust and can be applied to several dispersive equations with a similar structure. It goes back to the work of L. Payne and D. Sattinger on Klein-Gordon's equation, see [**28**].

In [**23**] a stronger statement than Corollary V.3.7 is proved:

Theorem V.3.8 (Dichotomy below the ground state). *Let u be as in Corollary V.3.7. Then in case* (1)*, u scatters to a linear solution, and in case* (2)*, u blows up in finite time.*

The proof is much more involved than the proof of Corollary V.3.7. We will give some elements of the scattering of solutions in case (1) in the next chapters.

V.4. Resolution into stationary solutions

We have identified 3 types of solutions to (W5f).

(1) Solutions of the ordinary differential equation $y'' = y^5$, such as $\left(\frac{3}{4t^2}\right)^{\frac{1}{4}}$, that can be truncated to obtain finite time blow-up solutions with finite energy.
(2) Scattering solutions, that are global and asymptotically close to solutions of the linear wave equation, that move with the speed of light (1 in our normalization).
(3) Stationnary solutions, and travelling wave solutions, with velocity < 1. We have also mentioned a fourth kind of solutions, constructed in [**25**]:
(4) Type II blow-up solutions, that are solutions such that $T_+(u) < \infty$ and

$$\limsup_{t\to T_+(u)} \|\vec{u}\|_{\dot{H}^1\times L^2} < \infty.$$

If we believe that the ODE solution will not play any role for the asymptotics of global solutions[1], we are lead to conjecture that this asymptotics will only be influenced by the travelling wave and linear solutions. Moreover, the different speeds of propagation would decouple asymptotically the linear and travelling wave dynamics. A similar conjecture holds for Type II Blow-up solutions (this is coherent with the solutions constructed in [**25**]).

In the radial case in dimension 3, these conjectures are true. Note that in this case, there is no travelling wave with speed $\mathbf{c} \neq 0$, and the only nonzero solutions of (Ell) are given by the transforms of W. Denote $W_\lambda(x) = \lambda^{-1/2}W(\lambda^{-1}x)$. The following result is proved in [**13**]:

Theorem V.4.1. *Let u be a radial solution of* (W5f) *such that one of the following property holds:*

(i) $$T+(u) = +\infty \text{ or}$$

(ii) $$T_+(u) < \infty \text{ and } \liminf_{t\to T_+(u)} \|\vec{u}(t)\|_{\dot{\mathcal{H}}^1} < \infty.$$

Then there exist $(v_0, v_1) \in \dot{\mathcal{H}}^1$, $J \geq 0$, and, for $j \in \{1, \dots, J\}$, a sign $\iota_j \in \{\pm 1\}$ and a scaling parameter $\lambda_j(t) > 0$ (defined for $t < T_+(u)$, close to $T_+(u)$) such that

$$\forall j \in \{1, \dots, J\}, \quad \lim_{t\to T_+(u)} \frac{\lambda_j(t)}{\lambda_{j-1}(t)} = 0, \tag{V.4.1}$$

and

- *In case* (i)*, $\lambda_0(t) = t$ and*

$$\vec{u}(t) = \vec{S}_L(t)(v_0, v_1) + \sum_{j=1}^{J} (\iota_j W_{\lambda_j(t)}, 0) + o(1) \text{ in } \dot{\mathcal{H}}^1, \quad t \to \infty \tag{V.4.2}$$

- *in case* (i)*, $J \geq 1$, $\lambda_0(t) = T_+ - t$ and*

$$\vec{u}(t) = (v_0, v_1) + \sum_{j=1}^{J} (\iota_j W_{\lambda_j(t)}, 0) + o(1) \text{ in } \dot{\mathcal{H}}^1, \quad t \to \infty \tag{V.4.3}$$

Theorem V.4.1 can be compared to Theorem V.2.5 on the defocusing equation. For this equation ((W5) with $\sigma = -1$), there is only one type of dynamics: scattering to a linear solution. The dynamics of the focusing wave equation is much richer. Let us note however that there is no known examples of solutions as in (V.4.2) or (V.4.3) with $J \geq 2$.

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[1]This belief is false in general for semi-linear wave equation, but turns out to be true in the energy-critical case. See [**14**] for an example of a global solution of the cubic wave equation which is asymptotically close to a solution of the corresponding ODE.

We will not give complete proofs of Theorems V.2.5, V.3.8 and V.4.1 in this course. We will however give elements of proofs. In the proofs of these three theorems, it is important to understand the behaviour of sequences of solutions, when the corresponding sequence of initial data is bounded in the energy space. This is useful to extract the profiles in the expansions (V.4.2) and (V.4.3) in Theorem V.4.1, and to extract a particular critical element in a contradiction argument in Theorems V.2.5 and V.3.8.

For this, we will introduce, in the next chapter, an important concentration/compactness tool, the profile decomposition, following a work of Hajer Bahouri and Patrick Gérard ([**3**]).

CHAPTER VI

Profile decomposition

This chapter concerns both the linear wave equation (LW) and the quintic nonlinear equation (W5) in space dimension 3. We will present one important tool, the profile decomposition, due to Hajer Bahouri and Patrick Gérard in this context. It is one of the key ingredients to prove the two results on the focusing equation presented above, Theorems V.3.8 and V.4.1. It can also be used to prove Theorem V.2.5 on the defocusing equation. This decomposition is related to the full understanding of the defect of compactness of the Strichartz estimates of Theorem III.2.1. We will first work on this defect of compactness in Section VI.2, then state and prove the profile decomposition in Section VI.3. We start this chapter with a few preliminaries.

VI.1. Preliminaries

1.a. Compactness of operators. Let A and B be 2 Banach spaces. We recall that a bounded linear operator $L : A \to B$ is *compact* when for any bounded sequence $(a_n)_{n\in\mathbb{N}}$ of A, one can extract from the sequence $(La_n)_{n\in\mathbb{N}}$ a subsequence that converges in B. If A is reflexive, an equivalent formulation is that for any sequence $(a_n)_n$ in A that converges weakly to $a \in A$, the sequence La_n converges strongly (to La since a bounded operator is continuous for the weak topologies).

We will be interested by the following example of noncompact operator. Let (p, q) such that $p > 2$ and $\frac{1}{p} + \frac{3}{q} = \frac{1}{2}$. By Strichartz estimates (Theorem III.2.1), the map $\vec{u}_0 \mapsto S_L(\cdot)\vec{u}_0$ is a bounded operator from $\dot{\mathcal{H}}^1$ to $L^p(\mathbb{R}, L^q)$. This map is not compact. Indeed, let $\varphi(t, x)$ be a fixed nonzero solution of the linear wave equation (LW) and $(\lambda_n)_{n\in\mathbb{N}} \in]0, \infty[^{\mathbb{N}}$ be such that $\lim_n |\log(\lambda_n)| = \infty$. Consider the sequence $(u_{Ln})_n$ of solutions of (LW) defined by

$$u_{Ln}(t, x) = \lambda_n^{1/2}\varphi(\lambda_n t, \lambda_n x),$$

with initial data

$$\vec{u}_{0n}(x) = \left(\lambda_n^{1/2}\varphi(0, \lambda_n x), \lambda_n^{3/2}\partial_t\varphi(0, \lambda_n x)\right).$$

Then the sequence $(\vec{u}_{0n})_n$ converges weakly to 0 in $\dot{\mathcal{H}}^1$. However the sequence $(u_{Ln})_n = \left(S_L(\cdot)\vec{u}_{0n}\right)_n$ does not converge strongly to 0 in L^pL^q (the norm of u_{Ln} in this space is exactly the norm of φ).

Similarly, let $(t_n, x_n)_n \in \left(\mathbb{R}^{1+3}\right)^{\mathbb{N}}$ such that

$$\lim_{n\to\infty} |t_n| + |x_n| = +\infty,$$

and

$$u_{L,n}(t, x) = \varphi(t + t_n, x + x_n),$$

then $(\vec{u}_{L,n}(0))_n$ converges to 0 weakly in $\dot{\mathcal{H}}^1$, and $(u_{L,n})_n$ does not converge strongly in $L^p(\mathbb{R}, L^q(\mathbb{R}^3))$.

The statement below, Theorem VI.2.1, says that these scaling and translations are the only causes of the defect of compactness of the map $S_L(\cdot) : \dot{\mathcal{H}}^1 \to L^pL^q$.

1.b. Notations. We introduce a few notations:

- $\mathcal{LW}$ is the set of solutions of (LW) with initial data in $\dot{\mathcal{H}}^1$.
- We denote by $\mathcal{G} =]0, \infty[\times\mathbb{R}\times\mathbb{R}^3$.
- We will use the following convention for limits, sequences and subsequences. A *sequence of elements of* X or *sequence in* X, is a family $(x_n)_{n\in\mathbf{I}}$ of elements of X, indexed by an infinite countable set I. The real sequence $(x_n)_{n\in\mathbf{I}}$ *converges* to ℓ if for all $\varepsilon > 0$, there exists a finite subset F of I such that $\forall n \in \mathbf{I} \setminus F$, $|\ell - x_n| \le \varepsilon$. In this case, we will write $\lim_{n\in\mathbf{I}} x_n = \ell$. We can deduce from this definition the definition of strong and weak convergence of sequences in a Banach space. These notions of convergence are of course identical that the one which we would obtain by identifying the countable set I to $\mathbb{N}$ by any bijective application.

  If $\mathbf{I}$ is an infinite countable set, we will write $\mathbf{I}' \sqsubset \mathbf{I}$ when $\mathbf{I}'$ is infinite and $\mathbf{I}' \subset \mathbf{I}$. A subsequence of $(x_n)_{n\in\mathbf{I}}$ is thus any sequence of the form $(x_n)_{n\in\mathbf{I}'}$ with $\mathbf{I}' \sqsubset \mathbf{I}$. To lighten notation, we will often omit the set of indices, writing a sequence $(v_n)_n$, when it is not relevant.

VI.2. Defect of compactness

2.a. Defect of compactness for the Strichartz estimate.

THEOREM VI.2.1. *Let (p,q) with $p>2$, $1/p+3/q=1/2$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{LW}$ such that $\overrightarrow{u_n}(0)$ is bounded in $\dot{\mathcal{H}}^1$. Assume that for all $(g_n)_{n\in\mathbb{N}}\in\mathcal{G}^{\mathbb{N}}$ one has*

$$\left(\lambda_n^{1/2}u_n(t_n,\lambda_n\cdot+x_n),\lambda_n^{3/2}\partial_t u_n(t_n,\lambda_n\cdot+x_n)\right)\xrightharpoonup[n\to\infty]{}(0,0). \tag{VI.2.1}$$

Then

$$\lim_{n\to\infty}\|u_n\|_{L^p(\mathbb{R},L^q)}=0. \tag{VI.2.2}$$

PROOF. We first observe that it is sufficient to prove the result for $p=\infty$, $q=6$. Indeed, assume that (VI.2.2) holds $p=\infty$ and $q=6$, and let $2<p<\infty$, q with $3/p+1/q=1/2$. By the assumptions that $\overrightarrow{u_n}(0)$ is bounded and Strichartz estimates, $(u_n)_n$ is bounded in L^aL^b for all $a>2$ and b with $1/a+3/b=1/2$. Using this fact for a pair (a,b) with $a<p<\infty$ (and thus $6<q<b$), we obtain, by Hölder's inequality, that (VI.2.2) holds for this (p,q) also.

We prove the theorem for $p=\infty$, $q=6$ by contradiction, assuming that (VI.2.1) holds and that

$$\limsup_{n\to\infty}\|u_n\|_{L^\infty(\mathbb{R},L^6)}>0. \tag{VI.2.3}$$

Thus there exists $\varepsilon>0$ and $\mathbf{I}\sqsubset\mathbb{N}$ such that

$$\forall n\in\mathbf{I},\quad \|u_n\|_{L^\infty(\mathbb{R},L^6)}\geq\varepsilon. \tag{VI.2.4}$$

As a consequence, for all $n\in\mathbf{I}$, we can choose $\tau_n\in\mathbb{R}$ such that

$$\|u_n(\tau_n)\|_{L^6}\geq\varepsilon/2. \tag{VI.2.5}$$

Also, we observe that (VI.2.1) implies that for all sequence $(\lambda_n,x_n)_{n\in\mathbf{I}}\in\left(]0,\infty[\times\mathbb{R}^3\right)^I$, we have

$$\lambda_n^{1/2}u_n(\tau_n,\lambda_n\cdot+x_n)\xrightharpoonup[n\to\infty]{}0 \text{ weakly in } \dot{H}^1, \tag{VI.2.6}$$

We are reduced to prove the following proposition (due to Patrick Gérard [**16**]), which is the analog to Theorem VI.2.1 for the defect of compactness of the Sobolev embedding of $\dot{H}^1(\mathbb{R}^3)$ into $L^6(\mathbb{R}^3)$.

PROPOSITION VI.2.2. *Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\dot{H}^1(\mathbb{R}^3)$. Assume that for all sequence $(\lambda_n,x_n)\in\left(]0,\infty[,\mathbb{R}^3\right)_n$,*

$$\lambda_n^{1/2}f_n(\lambda_n\cdot+x_n)\xrightharpoonup[n\to\infty]{}0. \tag{VI.2.7}$$

Then

$$\lim_{n\to\infty}\|f_n\|_{L^6}=0 \tag{VI.2.8}$$

Note that Proposition VI.2.2, together with (VI.2.5) and (VI.2.6) yield a contradiction. We next prove Proposition VI.2.2. □

2.b. Defect of compactness for the Sobolev inequality.

In this subsection we prove Proposition VI.2.2. This is a consequence of the improved Sobolev inequality (Theorem II.2.3). We recall the definition of the norm

$$\|f\|_{\dot{B}^1}^2=\sup_{k\in\mathbb{Z}}\frac{1}{(2\pi)^3}\int_{2^k\leq|\xi|\leq 2^{k+1}}|\xi|^2|\widehat{f}(\xi)|^2d\xi,$$

and note that

$$\|f\|_{\dot{B}^1}=\sup_{k\in\mathbb{Z}}\|\Delta_k f\|_{\dot{H}^1}, \tag{VI.2.9}$$

where the Δ_k are defined by[1]

$$\Delta_k f=\mathbb{1}_{\{2^k\leq|D|<2^{k+1}\}}f, \tag{VI.2.10}$$

that is $\widehat{\Delta_k f}(\xi)=\widehat{f}(\xi)$ if $2^k\leq|\xi|<2^{k+1}$ and $\widehat{\Delta_k f}(\xi)=0$ for all other ξ. We recall the improved Sobolev inequality

$$\|f\|_{L^6}^3\lesssim\|f\|_{\dot{B}^1}^2\|f\|_{\dot{H}^1} \tag{VI.2.11}$$

[1]The reader should be aware that this is not exactly the notation of Appendix B, where we use smooth bump functions instead of characteristic functions to localize in frequency.

We first prove the following lemma:

LEMMA VI.2.3. *Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\dot{H}^1$ and such that*

$$\limsup_{n\to\infty} \|f_n\|_{L^6} > 0. \tag{VI.2.12}$$

Then there exists a subsequence $(f_n)_{n\in\mathbf{I}}$ of $(f_n)_{n\in\mathbb{N}}$ and a sequence $(j_n)_{n\in\mathbf{I}} \in \mathbb{Z}^I$ such that

$$\lim_{n\in\mathbf{I}} \|\Delta_{j_n} f_n\|_{L^6} > 0.$$

PROOF. We argue by contradiction, assuming that for all sequence $(j_n)_{n\in\mathbb{N}} \in \mathbb{Z}^{\mathbb{N}}$,

$$\lim_{n\to\infty} \|\Delta_{j_n} f_n\|_{L^6} = 0. \tag{VI.2.13}$$

Fixing $\varepsilon > 0$, we will decompose $(f_n)_{n\in\mathbb{N}}$ as $f_n = v_n + w_n$, where

$$\lim_{n\to\infty} \|v_n\|_{L^6} = 0, \quad \forall n,\ \|w_n\|_{\dot{B}^1} \leq \varepsilon, \tag{VI.2.14}$$

$$\|f_n\|^2_{\dot{H}^1} = \|v_n\|^2_{\dot{H}^1} + \|w_n\|^2_{\dot{H}^1}. \tag{VI.2.15}$$

Assuming (VI.2.14) and (VI.2.15), and applying the Sobolev inequality (VI.2.11) on w_n, we obtain

$$\limsup_{n\to\infty} \|f_n\|_{L^6} = \limsup_{n\to\infty} \|w_n\|_{L^6} \leq C\varepsilon^2$$

for some constant $C > 0$. Since $\varepsilon > 0$ is arbitrary, this contradicts (VI.2.12).

To construct v_n and w_n we let, for any $n \in \mathbb{N}$,

$$J_n = \{j \in \mathbb{Z} \;:\; \|\Delta_j f_n\|_{\dot{H}^1} \geq \varepsilon\}$$

$$v_n = \sum_{j\in J_n} \Delta_j f_n, \quad w_n = \sum_{j\in\mathbb{Z}\setminus J_n} \Delta_j f_n = f_n - v_n.$$

We have

$$(2\pi)^3\|f_n\|^2_{\dot{H}^1} = \int |\xi|^2 \left|\widehat{f}_n(\xi)\right|^2 = \sum_{j\in J_n} \int |\xi|^2 \left|\widehat{f}_n(\xi)\right|^2 d\xi + \sum_{j\in\mathbb{Z}\setminus J_n} \int |\xi|^2 \left|\widehat{f}_n(\xi)\right|^2 d\xi$$
$$= (2\pi)^3 \left(\|v_n\|^2_{\dot{H}^1} + \|w_n\|^2_{\dot{H}^1}\right).$$

Hence (VI.2.15).

Next, we see that

$$\|v_n\|^2_{\dot{H}^1} = \sum_{j\in J_n} \|\Delta_j f_n\|^2_{\dot{H}^1} \geq |J_n|\varepsilon^2$$

(where $|J_n|$ is the number of elements of J_n). Thus

$$|J_n| \leq C\varepsilon^{-2},$$

where $C = \sup_n \|f_n\|^2_{\dot{H}^1}$. This proves, together with (VI.2.13), $\lim_n \|v_n\|_{L^6} = 0$.

We next prove the statement about w_n in (VI.2.14). We have

$$\|w_n\|_{\dot{B}^1} = \sup_{k\in\mathbb{Z}} \|\Delta_k w_n\|_{\dot{H}^1}, \tag{VI.2.16}$$

and

$$\Delta_k w_n = \begin{cases} \Delta_k f_n & \text{if } k \notin J_n \\ 0 & \text{if } k \in J_n. \end{cases}$$

and it follows from the defition of J_n that $\|\Delta_k w_n\|_{\dot{H}^1} \leq \varepsilon$. This concludes the proof. □

PROOF OF PROPOSITION VI.2.2. Let $(f_n)_{n\in\mathbb{N}}$ be as in the proposition. We argue by contradiction, assuming that (VI.2.8) does not hold, i.e. that there exists $\mathbf{I} \sqsubset \mathbb{N}$ such that

$$\lim_{n\in\mathbf{I}} \|f_n\|_{L^6} > 0.$$

By Lemma VI.2.3, there exists $\mathbf{I}' \sqsubset \mathbf{I}$ and a sequence $(j_n)_{n\in\mathbf{I}'}$ such that

$$\lim_{n\in\mathbf{I}'} \|\Delta_{j_n} f_n\|_{L^6} > 0.$$

Rescaling f_n, we can assume $j_n = 0$ for all n, i.e.

$$\lim_{n\in\mathbf{I}'} \|\Delta_0 f_n\|_{L^6} > 0. \tag{VI.2.17}$$

Since $(f_n)_n$ is bounded in $\dot{H}^1$, we have

$$\limsup_{n\in\mathbf{I}'} \|\Delta_0 f_n\|_{L^2} < \infty. \tag{VI.2.18}$$

Next we prove

$$\lim_{n\in\mathbf{I}'} \|\Delta_0 f_n\|_{L^\infty} = 0. \tag{VI.2.19}$$

Indeed,

$$\Delta_0 f_n(x) = \frac{1}{(2\pi)^3}\int_{1\leq|\xi|\leq 2} e^{ix\cdot\xi}\widehat{f}_n(\xi)d\xi,$$

which proves (since $\widehat{f}_n \in L^1(\{1\leq|\xi|\leq 2\})$) that $\Delta_0 f_n$ is a continuous bounded function. Let $x_n\in\mathbb{R}^3$ such that $|\Delta_0 f_n(x_n)| \geq \frac{1}{2}\|\Delta_0 f_n\|_{L^\infty}$. Then

$$\Delta_0 f_n(x_n) = \frac{1}{(2\pi)^3}\int\left(\int_{1\leq|\xi|\leq 2} e^{-iy\cdot\xi}d\xi\right)f_n(y+x_n)dy.$$

Note that $y\mapsto\int_{1\leq|\xi|\leq 2} e^{-iy\cdot\xi}d\xi$ is the Fourier transform of $1\!\!1_{\{1\leq|\xi\leq 2\}}$. As a consequence, it is in $\bigcap_{s\in\mathbb{R}}\dot{H}^s$.

Using that $f_n(x_n+\cdot)$ converges weakly to 0 in $\dot{H}^1$, we obtain $\lim_n \Delta_0 f_n(x_n) = 0$ and thus (VI.2.19). Combining (VI.2.18) and (VI.2.19), we deduce that $\Delta_0 f_n$ goes to 0 in L^6, which contradicts (VI.2.17) and concludes the proof. □

VI.3. Profile decomposition

3.a. Preliminaries on the group of transformations. We start with some preliminaries on the group $\mathcal{G}$ of transformation (scaling and translation) of solutions of the wave equation. Recall that for $g=(\lambda,T,X)\in\mathcal{G}=(0,\infty)\times\mathbb{R}\times\mathbb{R}^3$ and u a function defined on $\mathbb{R}\times\mathbb{R}^3$, we write:

$$u^g(t,x) = \frac{1}{\lambda^{1/2}}u\left(\frac{t-T}{\lambda},\frac{x-X}{\lambda}\right). \tag{VI.3.1}$$

The set $\mathcal{G}$ is equipped with a group structure with the operation $\circ$:

$$g\circ h = (\lambda\mu, T+\lambda S, X+\lambda Y),$$

where $g=(\lambda,T,X)$, $h=(\mu,S,Y)$, and the operation is chosen so that that $(u^h)^g = u^{g\circ h}$. Denoting by e the unit element and g^{-1} the inverse of g, we have

$$e=(1,0,0),\quad g^{-1}=\left(\frac{1}{\lambda},-\frac{T}{\lambda},-\frac{X}{\lambda}\right)$$

and

$$g^{-1}\circ h = \left(\frac{\mu}{\lambda},\frac{S-T}{\lambda},\frac{Y-X}{\lambda}\right).$$

We identify $\mathcal{LW}$ (the set of finite energy solution of the linear wave equation) with $\mathcal{H}=\dot{H}^1\times L^2$ using the vector space isomorphism:

$$u\mapsto\vec{u}(0).$$

The scalar product and norm on $\mathcal{LW}$ given by this identification will be denoted by $\langle\cdot|\cdot\rangle$ and $\|\cdot\|$. Thus:

$$\langle u|v\rangle = (\vec{u}(0),\vec{v}(0))_{\mathcal{H}},\quad \|u\| = \|\vec{u}(0)\|_{\mathcal{H}}.$$

For all $g\in\mathcal{G}$, the linear operator $u\mapsto u^g$ is an isometry of $\mathcal{LW}$:

Claim VI.3.1. *Let $u,v\in\mathcal{LW}$ and $g\in\mathcal{G}$. Then*

$$\langle u^g|v^g\rangle = \langle u|v\rangle.$$

Proof. The claim is equivalent to the fact that

$$\int\frac{1}{\lambda^3}\nabla u\left(\frac{-T}{\lambda},\frac{x-X}{\lambda}\right)\cdot\nabla v\left(\frac{-T}{\lambda},\frac{x-X}{\lambda}\right)dx + \int\frac{1}{\lambda^3}\partial_t u\left(\frac{-T}{\lambda},\frac{x-X}{\lambda}\right)\partial_t v\left(\frac{-T}{\lambda},\frac{x-X}{\lambda}\right)dx$$

is independent of $T\in\mathbb{R}$, $X\in\mathbb{R}^3$ and $\lambda>0$. The fact that it does not depend on T follows from conservation of the energy. The other statements follow by the change of variable $y=\frac{x-X}{\lambda}$. □

The group $\mathcal{G}$ endowed with the usual topology of $(0,\infty)\times\mathbb{R}\times\mathbb{R}^3$, is a topological group. This topology is compatible with the weak topology on $\mathcal{LW}$ (which is by definition given by the weak topology on $\mathcal{H}$ through the identification $u\mapsto\vec{u}(0)$):

Lemma VI.3.2. *Let $(u_n)_n$ be a bounded sequence in $\mathcal{LW}$ and $(g_n)_n$ a sequence in $\mathcal{G}$. Assume*

$$u_n \underset{n}{\rightharpoonup} u \text{ in } \mathcal{LW}, \quad g_n \underset{n}{\to} g \text{ in } \mathcal{G}.$$

Then

$$u_n^{g_n} \underset{n}{\rightharpoonup} u^g \text{ in } \mathcal{LW}.$$

Proof. Since $(u_n^{g_n})_n$ is bounded in $\mathcal{LW}$, is it sufficient to check, by density, that for all $\varphi \in \mathcal{LW}$ such that $\vec{\varphi}(0) \in (C_c^\infty(\mathbb{R}^3))^2$, one has

$$\lim_n \langle u_n^{g_n} | \varphi\rangle = \langle u^g | \varphi\rangle.$$

Then

$$\langle u_n^{g_n} | \varphi\rangle = \langle u_n, \varphi^{g_n^{-1}}\rangle = \langle u_n | \varphi^{g^{-1}}\rangle + \langle u_n | \varphi^{g_n^{-1}} - \varphi^{g^{-1}}\rangle.$$

We have, by the weak convergence of u_n,

$$\lim_n \langle u_n | \varphi^{g^{-1}}\rangle = \langle u | \varphi^{g^{-1}}\rangle = \langle u^g | \varphi\rangle.$$

Furthermore, it can be easily checked that

$$\lim_n \|\varphi^{g_n^{-1}} - \varphi^{g^{-1}}\| = 0,$$

using that $\vec{\varphi}(0)$ is smooth and compactly supported. This concludes the proof. □

Definition VI.3.3. Let $(g_n)_{n\in\mathbf{I}}$ be a sequence in $\mathcal{G}$. We will write:

$$\lim_{n\in\mathbf{I}} g_n = \infty,$$

if g_n goes out of every compact of $\mathcal{G}$, i.e.

$$\lim_{n\in\mathbf{I}} |x_n| + |t_n| + \lambda_n^{-1} + \lambda_n = +\infty.$$

We say that two sequences $(g_n)_{n\in\mathbf{I}}$, $(h_n)_{n\in\mathbf{I}}$ in $\mathcal{G}$ are *orthogonal*, and we note $(g_n)_n \perp (h_n)_n$ when

$$\lim_n g_n^{-1} \circ h_n = \infty$$

or equivalently

$$\lim_{n\in\mathbf{I}} \frac{|x_n - y_n| + |t_n - s_n|}{\lambda_n} + \frac{\mu_n}{\lambda_n} + \frac{\lambda_n}{\mu_n} = +\infty,$$

where $g_n = (\lambda_n, t_n, x_n)$, $h_n = (\mu_n, s_n, y_n)$.

Remark VI.3.4. The reader should be careful with the notation $\lim_{n\in\mathbf{I}} g_n = \infty$. Note for example that $\lim_n \lambda_n = 0 \Rightarrow \lim_n g_n = \infty$, and that $\lim_n g_n = \infty \iff \lim_n g_n^{-1} = \infty$.

Remark VI.3.5. If the two sequences $(g_n)_n$ and $(h_n)_n$ are not orthogonal, there exist $g \in \mathcal{G}$ and $\mathbf{I}' \sqsubset \mathbf{I}$ and

$$\lim_{n\in\mathbf{I}'} g_n^{-1} \circ h_n = g.$$

Recall the notation u^g from (VI.3.1).

Claim VI.3.6. *Let $(g_n)_n$ and $(h_n)_n$ be two sequences in $\mathcal{G}$ that are orthogonal. Let u and v be two solutions of the linear wave equation. Then*

$$\lim_n \langle u^{g_n} | v^{h_n}\rangle = 0.$$

Equivalently, if $g_n \to \infty$, and $u \in \mathcal{LW}$, then

$$u^{g_n} \underset{n}{\rightharpoonup} 0 \text{ in } \mathcal{LW}.$$

Proof. By the preceding claim, we have

$$\langle u^{g_n} | v^{h_n}\rangle = \langle u^{h_n^{-1}\circ g_n} | v\rangle.$$

It is thus sufficient to prove the result with $h_n = e$ and $g_n \to \infty$ (which is also the second statement of the claim). We let $g_n = (\lambda_n, t_n, x_n)$. We are left with proving

$$\lim_{n\to\infty} \left(\overrightarrow{u^{g_n}}(0), \vec{v}(0)\right)_{\dot{\mathcal{H}}^1} = 0, \tag{VI.3.2}$$

where

$$\overrightarrow{u^{g_n}}(0,x) = \left(\frac{1}{\lambda_n^{1/2}} u\left(\frac{-t_n}{\lambda_n}, \frac{x - x_n}{\lambda_n}\right), \frac{1}{\lambda_n^{3/2}} \partial_t u\left(\frac{-t_n}{\lambda_n}, \frac{x - x_n}{\lambda_n}\right)\right).$$

and

$$\lim_{n\to\infty} \frac{|t_n|+|x_n|}{\lambda_n} + |\log(\lambda_n)| = +\infty.$$

By density and energy conservation, we can assume $\vec{u}(0) \in \left(C_0^\infty(\mathbb{R}^3)\right)^2$ and $\vec{v}(0) \in \left(C_0^\infty(\mathbb{R}^3)\right)^2$. By the explicit formulas of Theorem I.5.3 (using e.g. the dispersion inequality), we have

$$\sup_{(t,x)\in\mathbb{R}^4} |u(t,x)| + |\nabla u(t,x)| + |\partial_t u(t,x)| < \infty$$

and similarly for v.

If $\lambda_n \to \infty$, (VI.3.2) is immediate (since the scalar product is bounded by $C/\lambda_n^{1/2}$ for large n). If $\lambda_n \to 0$, one can perform the change of variable $x \mapsto x/\lambda_n$ and obtain (VI.3.2), using that the scalar product is bounded by $C\lambda_n^{1/2}$ for large n. Thus extracting subsequences, we can assume that λ_n and $1/\lambda_n$ are bounded.

If $\lim_{n\to\infty} \frac{|t_n|}{\lambda_n} = \infty$, we see that the two components of $\overrightarrow{u^{g_n}}(0)$ go to 0 uniformly as n goes to infinity, and (VI.3.2) follows.

Extracting subsequences again, we are left with treating the case where $\frac{|t_n|}{\lambda_n}$, λ_n and $1/\lambda_n$ are bounded. In this case, $\lim_{n\to\infty} |x_n| = +\infty$, and (VI.3.2) follows by the strong Huygens principle. □

3.b. Statement. We are now ready to state the profile decomposition result of H. Bahouri and P. Gérard.

Theorem VI.3.7 (Profile decomposition). *Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{LW}$. Then there exists $\mathbf{I} \sqsubset \mathbb{N}$, and, for all $j \geq 1$, there exists $\varphi_j \in \mathcal{LW}$, and a sequence $(g_{j,n})_{n\in\mathbf{I}}$ such that the following properties hold:*

$$j \neq k \Longrightarrow (g_{j,n})_{n\in\mathbf{I}} \perp (g_{k,n})_{n\in\mathbf{I}}, \tag{VI.3.3}$$

$$\forall j, \quad u_n^{g_{j,n}^{-1}} \xrightarrow[n\in\mathbf{I}]{} \varphi_j, \tag{VI.3.4}$$

and, denoting by

$$w_{J,n} = u_n - \sum_{j=1}^{J} (\varphi_j)^{g_{j,n}}, \tag{VI.3.5}$$

we have

$$\lim_{J\to\infty} \limsup_{n\in\mathbf{I}} \|w_{J,n}\|_{L^\infty(\mathbb{R},L^6)} = 0. \tag{VI.3.6}$$

Furthermore, for all J, the following Pythagorean expansion holds:

$$\lim_{n\in\mathbf{I}} \|u_n\|^2 - \sum_{j=1}^{J} \|\varphi_j\|^2 - \|w_{J,n}\|^2 = 0. \tag{VI.3.7}$$

The expansion $u_n = \sum_{j=1}^{J} \varphi_j^{g_{j,n}} + w_{J,n}$ is called a profile decomposition for the sequence $(u_n)_n$. The sequences of modulated solutions $(\varphi_j^{g_{j,n}})_n$ (or equivalently the couples φ_j, $(g_{j,n})_n$, or by a slight simplification the solutions φ_j) are called the (linear) profiles.

Remark VI.3.8. There might be only a finite number of nonzero $(\varphi_j)_j$. In this case, denoting by $(\varphi_j)_{1\leq j\leq J_0}$ the nonzero profiles, we have

$$u_n = \sum_{j=1}^{J_0} \varphi_j + w_{J_0,n},$$

where

$$\limsup_{n\in\mathbf{I}} \|w_{J_0,n}\|_{L^\infty(\mathbb{R},L^6)} = 0.$$

3.c. Quantifying the defect of compactness. To prove Theorem VI.3.7, we will need a somehow more quantitative version of Theorem VI.2.1. For a bounded sequence $(v_n)_{n\in\mathbf{I}}$ in $\mathcal{LW}$, we denote

$$\eta\big((v_n)_{n\in\mathbf{I}}\big) = \sup_{\substack{\varphi\in\mathcal{LW}\\ \|\varphi\|=1}} \limsup_{n\in\mathbf{I}} \left(\sup_{g\in\mathcal{G}} \langle v_n^g|\varphi\rangle\right).$$

Let $\mathcal{A}((v_n)_n)$ be the set of $U \in \mathcal{LW}$ such that there exists $\mathbf{I}' \sqsubset \mathbf{I}$, $(g_n)_n \in \mathcal{G}^{\mathbf{I}'}$ such that

$$v_n^{g_n} \xrightarrow[n\in\mathbf{I}']{} U.$$

Then

$$\eta\big((v_n)_n\big) = \sup_{U \in \mathcal{A}((v_n)_n)} \|U\|. \tag{VI.3.8}$$

Indeed, if $U \in \mathcal{A}\left((v_n)_n\right)$, then

$$v_n^{g_n} \xrightarrow[n\in \mathbf{I}']{} U$$

for a set $\mathbf{I}' \sqsubset \mathbf{I}$ and sequence $(g_n)_{n\in\mathbf{I}'}$ of modulation parameters. This implies that for all $\varphi \in \mathcal{LW}$,

$$\lim_{n\in\mathbf{I}'} \langle v_n^{g_n} | \varphi \rangle = \langle U, \varphi \rangle,$$

and the definition of η gives

$$\langle U, \varphi\rangle \leq \eta\left((v_n)_n\right) \|\varphi\|,$$

and thus $\|U\| \leq \eta\left((v_n)_n\right)$. This yields the inequality $\sup_{U\in\mathcal{A}((v_n)_n)} \|U\| \leq \eta\big((v_n)_n\big)$. We leave the proof of the reverse inequality to the reader. As a corollary to Theorem VI.2.1, we have:

Corollary VI.3.9.

- *Let v_n be a bounded sequence of $\mathcal{LW}$ such that $\eta\left((v_n)_n\right) = 0$. Then*

$$\lim_{n\to\infty} \|v_n\|_{L^\infty(\mathbb{R},L^6)} = 0.$$

- *Let $\left((v_{n,j})_{n\in\mathbf{I}}\right)_{j\geq 1}$ be a family of $\mathcal{LW}$ such that*

$$\sup_{\substack{n\in\mathbf{I}\\ j\geq 1}} \|v_{n,j}\| < \infty \text{ and } \lim_{j\to\infty} \eta\left((v_{n,j})_n\right) = 0. \tag{VI.3.9}$$

Then

$$\lim_{j\to\infty} \limsup_{n\in\mathbf{I}} \|v_{n,j}\|_{L^\infty(\mathbb{R},L^6)} = 0. \tag{VI.3.10}$$

Proof. By (VI.3.8), $\eta\left((v_n)_n\right) = 0$ is equivalent to the fact that $\left(v_n^{g_n}\right)_n$ goes weakly to 0 for all sequence of transformations $(g_n)_n$. Thus the first point of the corollary follows immediately from Theorem VI.2.1.

The second point of the corollary is also a consequence of Theorem VI.2.1, although less immediate. We let $v_{n,j}$ satisfy (VI.3.9). We denote by $D = (Z_p)_{p\in\mathbb{N}}$ a countable dense family of the unit sphere of $\mathcal{LW}$ (it is easy to show the existence of such a family, since $\mathcal{H}$ has a countable Hilbert basis). We let $D_k = (Z_p)_{0\leq p\leq k}$. We prove (VI.3.10) by contradiction, assuming that there exists $\varepsilon > 0$ and a sequence $j_k \to \infty$ such that

$$\limsup_{n\in\mathbf{I}} \|v_{n,j_k}\|_{L^\infty L^6} \geq \varepsilon.$$

For all k, we fix $n_k \in \mathbf{I}$ such that

$$\sup_{g\in\mathcal{G}} \sup_{Z\in D_k} \left|\langle v^g_{n_k,j_k} | Z\rangle\right| \leq \eta\left((v_{n,j_k})_n\right) + 2^{-k}$$

and

$$\|v_{n_k,j_k}\|_{L^\infty L^6} \geq \frac{\varepsilon}{2}. \tag{VI.3.11}$$

Let $\varphi \in \mathcal{L}W$ with $\|\varphi\| = 1$ and $\delta > 0$. Then there exists $p \in \mathbb{N}$ such that $\|\varphi - Z_p\|_{\dot{\mathcal{H}}^1} \leq \delta$. By the choice of n_k and the uniform boundedness of $(v_{n,j})$, we have:

$$\sup_{g\in\mathcal{G}} \left|\langle v^g_{n_k,j_k} | \varphi\rangle\right| \leq C\delta + \eta\left((v_{n,j_k})_n\right) + 2^{-k},$$

and hence, using the assumption that $\eta\left((v_{n,j})_n\right)$ goes to 0 as j goes to infinity,

$$\lim_{k\to\infty} \sup_{g\in\mathcal{G}} \left|\langle v^g_{n_k,j_k}, \varphi\rangle\right| \leq C\delta.$$

Since δ is arbitrary, we obtain

$$\lim_{k\to\infty} \sup_{g\in\mathcal{G}} \left|\langle v^g_{n_k,j_k} | \varphi\rangle\right| = 0.$$

This contradicts Theorem VI.2.1 and (VI.3.11), concluding the proof. □

3.d. Construction of the profile decomposition. We are now ready to prove Theorem VI.3.7.

PROOF. *Step 1: induction argument.*

We construct a nonincreasing sequence $(\mathbf{I}_J)_{J\geq 1}$ of infinite subsets of $\mathbb{N}$, and, for $j\geq 1$, a sequence $(g_{j,n})_{n\in\mathbf{I}_j}$ of elements of $\mathcal{G}$ and a solution $\varphi_j\in\mathcal{LW}$ such that for all $J'\geq 1$,

$$1\leq j\neq k\leq J' \Longrightarrow (g_{j,n})_{n\in\mathbf{I}_{J'}}\perp(g_{k,n})_{n\in\mathbf{I}_{J'}}, \tag{VI.3.12}$$

$$u_n^{g^{-1}_{J',n}} \xrightarrow[n\in\mathbf{I}_{J'}]{} \varphi^{J'}, \tag{VI.3.13}$$

Let $J\geq 1$ and assume that there exist $\mathbf{I}_{J-1}\sqsubset\mathbb{N}$, and, for $1\leq j\leq J-1$, a profile φ_j and a sequence of parameters $(g_{j,n})_{n\in\mathbf{I}_{J-1}}$ such that (VI.3.12) hold for $J'=J-1$, and (VI.3.13) hold for all $J'<J$.

Recall from Subsection 3.c the definitions of $\eta((v_n)_n)$ and $\mathcal{A}((v_n)_n)$, where $(v_n)_n$ is a sequence in n. We distinguish wo cases

Case 1. $\eta\left((w_{J-1,n})_{n\in\mathbf{I}_{J-1}}\right)=0$. In this case we stop the process and let $\varphi_j=0$ for all $j\geq J$.

Case 2. $\eta\left((w_{J-1,n})_{n\in\mathbf{I}_{J-1}}\right)>0$. In this case, there exists an element φ_J of $\mathcal{A}\left((w_{J-1,n})_{n\in\mathbf{I}_{J-1}}\right)$ such that

$$\|\varphi_J\|\geq\frac{1}{2}\eta\left((w_{J-1,n})_{n\in\mathbf{I}_{J-1}}\right), \tag{VI.3.14}$$

and we choose $\mathbf{I}_J\sqsubset\mathbf{I}_{J-1}$ and a sequence $(g_{J,n})_{n\in\mathbf{I}_J}$,

$$w_{J-1,n}^{g^{-1}_{J,n}} \xrightarrow[n\in\mathbf{I}_J]{} \varphi_J \tag{VI.3.15}$$

weakly in $\mathcal{LW}$.

We next prove that the orthogonality conditions (VI.3.12) hold for $J'=J$. By the induction hypothesis ((VI.3.12) with $J'=J-1$), it is sufficient to prove $(g_{k,n})_n\perp(g_{J,n})_n$ for $1\leq k<J$.

We have (by the induction hypothesis (VI.3.13) that hold for $1\leq J'\leq J-1$),

$$\sum_{j=1}^{J-1}\varphi_j^{g^{-1}_{k,n}\circ g_{j,n}}+w_{J-1,n}^{g^{-1}_{k,n}}=u_n^{g^{-1}_{k,n}}\xrightarrow[n\in\mathbf{I}_J]{}\varphi_k,$$

and, by the induction hypothesis and Claim VI.3.6

$$1\leq j\neq k\leq J-1\Longrightarrow\varphi_j^{g^{-1}_{k,n}\circ g_{j,n}}\xrightarrow[n\in\mathbf{I}_J]{}0.$$

Thus

$$w_{J-1,n}^{g^{-1}_{k,n}}\xrightarrow[n\in\mathbf{I}_J]{}0 \tag{VI.3.16}$$

Assume that $(g_{J,n})_{n\in\mathbf{I}_J}$ and $(g_{k,n})_{n\in\mathbf{I}_J}$ are not orthogonal. Then there exists $\mathbf{I}'\sqsubset\mathbf{I}_J$ and $g\in\mathcal{G}$ such that

$$\lim_{n\in\mathbf{I}'}g_{k,n}\circ g^{-1}_{J,n}=g.$$

Hence from Lemma VI.3.2 and (VI.3.16),

$$w_{J-1,n}^{g^{-1}_{J,n}}\xrightarrow[n\in\mathbf{I}_J]{}0$$

which contradicts (VI.3.15). Thus

$$(g_{k,n})_{n\in\mathbf{I}_J}\perp(g_{J,n})_{n\in\mathbf{I}_J},\quad 1\leq k\leq J-1.$$

We have obtained (VI.3.12) for $J'=J$. As a consequence, using the equality

$$\sum_{j=1}^{J-1}\varphi_j^{g^{-1}_{J,n}\circ g_{j,n}}+w_{J-1,n}^{g^{-1}_{J,n}}=u_n^{g^{-1}_{J,n}}$$

and arguing as above we also obtain (VI.3.13) for $J'=J$. This concludes the induction argument.

This general principle of proof is sometimes called "*exhaustion method*".

Step 2. Conclusion of the proof

If there exists a $J\geq 1$ such that Case 1 above holds, then we are done: indeed, in this case, $w_{J,n}$ does not depend on J for large J, and (VI.3.6) is an immediate consequence of the definition of η and $\mathcal{A}$ and Corollary VI.3.9.

It is easy to see, expanding the square of the norm of $u_n=\sum_{j=1}^J\varphi_j^{g_{j,n}}+w_{J,n}$ and using the orthogonality of the modulation parameters and Claim VI.3.6, that the Pythagorean expansion (VI.3.7) holds for all J.

Next assume that Case 2 holds for all $J \geq 1$. Using a diagonal extraction argument, we obtain $\mathbf{I} \sqsubset \mathbb{N}$ and, for all $j \geq 1$, profiles φ_j, and sequences of parameters $(g_{j,n})_{n\in\mathbf{I}}$ such that (VI.3.3) and (VI.3.4) hold. Again, the Pythagorean expansion follows easily for every $J \geq 1$.

It remains to prove (VI.3.6). By the Pythagorean expansion (VI.3.7), we see that

$$\sum_{j\geq 1} \|\varphi_j\|^2 < \infty,$$

and thus

$$\lim_{j\to\infty} \|\varphi_j\| = 0.$$

By the choice of φ_j, (VI.3.14), this implies

$$\lim_{J\to\infty} \eta\left((w_{J,n})_n\right) = 0,$$

and the conclusion follows from Corollary VI.3.9. $\square$

CHAPTER VII

Nonlinear profile decomposition

The goal of this chapter is to extend the decomposition of bounded sequences of solutions of the linear wave equation from the previous chapter to bounded sequences of solutions of the quintic wave equation

$$(\partial_t^2 - \Delta)u = \sigma u^5, \tag{W5}$$

where $\sigma \in \{\pm 1\}$ is fixed. This will be needed for a compactness argument, detailed in Chapter VIII, which is part of the proof of Theorems V.2.5 and V.3.8.

The possibility of finite time blow-up is a major obstruction to this nonlinear profile decomposition. We will see however that it is possible to achieve the decomposition with a natural assumption on the profiles that exclude finite time blow-up.

VII.1. Nonlinear profile

Recall that a *linear profile* is a couple $(\varphi, (g_n)_{n\in\mathbf{I}})$, where $\mathbf{I} \sqsubset \mathbb{N}$, $\varphi \in \mathcal{LW}$ and $g_n \in \mathcal{G}$. Given such a linear profile, we seek for a solution u of the *nonlinear* wave equation (W5) such that

$$\lim_{n\to\infty} \left\| \overrightarrow{u^{g_n}}(0) - \overrightarrow{\varphi^{g_n}}(0) \right\|_{\mathcal{H}} = 0. \tag{VII.1.1}$$

This is possible with an additional assumption on the sequence $(g_n)_n$:

Definition VII.1.1. We say that the sequence $(g_n)_{n\in\mathbf{I}} \in \mathcal{G}^{\mathbf{I}}$ is *tidy* when there exists $\tau \in \mathbb{R}\cup\{+\infty\}\cup\{-\infty\}$ such that

$$\lim_{n\in\mathbf{I}} \frac{-t_n}{\lambda_n} = \tau.$$

Lemma VII.1.2. *Let $\varphi \in \mathcal{LW}$ and $(g_n)_{n\in\mathbf{I}} \in \mathcal{G}^{\mathbf{I}}$ a tidy sequence. Then there exists a solution u of* (W5), *defined in a neighborhoud of $\tau = \lim_n -t_n/\lambda_n$, such that* (VII.1.1) *holds.*

Definition VII.1.3. The solution u given by Lemma VII.1.2 is called the *nonlinear profile* associated to φ, $(g_n)_{n\in\mathbf{I}}$.

Proof. By the definition of the action $\varphi \mapsto \varphi^g$ and a simple change of variable, the condition (VII.1.1) is equivalent to

$$\lim_{n\in\mathbf{I}} \left\| \vec{u}\left(-\frac{t_n}{\lambda_n}\right) - \vec{\varphi}\left(-\frac{t_n}{\lambda_n}\right) \right\|_{\mathcal{H}} = 0. \tag{VII.1.2}$$

We distinguish two cases according to the value of $\tau = \lim_n -t_n/\lambda_n$.

Case 1: $\tau \in \mathbb{R}$. Let u be the solution of (W5) such that $\vec{u}(\tau) = \vec{\varphi}(\tau)$. Then by continuity of the $\mathcal{H}$-valued functions $\vec{u}$ and $\vec{\varphi}$, (VII.1.2) holds.

Case 2: $\tau \in \{\pm\infty\}$. We know by the existence of wave operators (see Theorem V.2.4) that there exists a solution u of (W5), defined in a neighborhood of τ, such that

$$\lim_{t\to\tau} \|\vec{u}(t) - \vec{\varphi}(t)\|_{\mathcal{H}} = 0,$$

and (VII.1.1) follows. □

VII.2. Nonlinear profile decomposition

Consider a sequence of initial data $(\vec{u}_{0,n})_{n\in\mathbb{N}}$ which is bounded in $\mathcal{H}$. Let $(u_{Ln})_{n\in\mathbb{N}}$ be the corresponding sequence of solutions of the linear wave equation, with $\vec{u}_{Ln}(0) = \vec{u}_{0,n}$. By Theorem VI.3.7, there exists $\mathbf{I}' \sqsubset \mathbb{N}$ such that $(u_{Ln})_{n\in\mathbf{I}'}$ has a profile decomposition:

$$u_{Ln} = \sum_{j=1}^{J} \varphi_j^{g_{j,n}} + w_{J,n}, \tag{VII.2.1}$$

where $(g_{j,n})_n \perp (g_{k,n})_n$ for $j \neq k$, and

$$\lim_{J\to\infty} \limsup_{n\in\mathbf{I}'} \|w_{J,n}\|_{L^\infty L^6} = 0,$$

which implies

$$\lim_{J\to\infty} \limsup_{n\in\mathbf{I}'} \|w_{J,n}\|_{L^5 L^{10}} = 0.$$

Extracting subsequences at fixed j, and using a diagonal extraction argument, we can assume that there exists $\mathbf{I} \sqsubset \mathbf{I}'$ such that for all j, the sequence $(g_{j,n})_{n\in\mathbf{I}}$ is tidy. By Lemma VII.1.2, we can associate to each linear profile φ_j, $(g_{j,n})_{n\in\mathbf{I}}$ a nonlinear profile U_j, that is a solution of the nonlinear wave equation (W5) such that

$$\lim_{n\to\infty} \left\| \overrightarrow{\varphi_j^{g_{j,n}}}(0) - \overrightarrow{U_j^{g_{j,n}}}(0) \right\|_{\mathcal{H}} = 0.$$

The linear expansion (VII.2.1) gives

$$\vec{u}_{0,n} = \sum_{j=1}^{J} \overrightarrow{U_j^{g_{j,n}}}(0) + \vec{w}_{J,n}(0) + o_n(1), \text{ in } \mathcal{H}, \quad n \in \mathbf{I}. \tag{VII.2.2}$$

It is reasonable to think that the $U_j^{g_{j,n}}$ are decoupled at first order because of the orthogonality of the parameters, and that (VII.2.2) implies for the solution u_n of the nonlinear wave equation (W5) with initial data $\vec{u}_n(0) = \vec{u}_{0,n}$, an expansion of the form: $u_n(t) = \sum_{j=1}^J U_j^{g_{j,n}}(t) + w_{J,n}(t) + o_{J,n}(1)$, where $o_{J,n}(1)$ is small when J and n are large, in a sense to be specified. It turns out that such an expansion can be obtained with an additional assumption on the nonlinear profiles:

Theorem VII.2.1 (Nonlinear profile decomposition). *Let $(\vec{u}_{0,n})_{n\in\mathbf{I}}$ be a bounded sequence in $\mathcal{H}$. Let $u_{L,n}(t) = S_L(t)\vec{u}_{0,n}$, and assume that $(u_{L,n})_{n\in\mathbf{I}}$ admits a profile decomposition with profiles φ_j, $(g_{j,n})_n$. Assume that for all j, the sequence $(g_{j,n})_n$ is tidy, and consider U_j, the nonlinear profile associated to φ_j, $(g_{j,n})_n$. Assume that U_j scatters in both time directions. Let u_n be the solution of* (W5) *with initial data $\vec{u}_{0,n}$. Then for large n, u_n is global and scatters in both time directions. Moreover, for large J, we have the expansion as $n\to\infty$ in $\mathbf{I}$:*

$$u_n = \sum_{j=1}^{J} U_j^{g_{j,n}} + w_{J,n} + r_{J,n}, \tag{VII.2.3}$$

where

$$\lim_{J\to\infty} \limsup_{n\in\mathbf{I}} \left(\|r_{J,n}\|_{L^5(\mathbb{R},L^{10})} + \sup_{t\in\mathbb{R}} \|\vec{r}_{J,n}(t)\|_{\mathcal{H}} = 0 \right).$$

Remark VII.2.2. Theorem VII.2.1 implies

$$\limsup_{n\in\mathbf{I}} \|u_n\|_{L^5L^{10}} < \infty.$$

Indeed, letting J large enough so that the expansion (VII.2.3) holds for large n and

$$\limsup_{n\in\mathbf{I}} \|w_{J,n} + r_{J,n}\|_{L^5L^{10}} \leq 1,$$

we have

$$\limsup_{n\in\mathbf{I}} \|u_n\|_{L^5L^{10}} \leq \sum_{j=1}^{J} \|U_j\|_{L^5L^{10}} + 1.$$

Theorem VI.3.7 is due to Bahouri and Gérard (see the main result of [**3**]). In [**3**], the authors consider the defocusing equation ($\sigma = -1$), but the proof works as well in the focusing case, assuming that all the nonlinear profiles U^j scatter.

Remark VII.2.3. The assumption that U^j scatters in both time direction is equivalent to the fact that the maximal interval of existence of U^j is $\mathbb{R}$ and that $U^j \in L^5(\mathbb{R}, L^{10})$.

The proof of Theorem VII.2.1 is based on a refinement of the stability result (Theorem IV.5.1 above) that we will now state.

VII.3. Long time perturbation theory

THEOREM VII.3.1. *Let $M > 0$. There exist a small $\varepsilon_M > 0$ and a large $C_M > 0$ with the following properties. Let I be an interval and $t_0 \in I$. Let $v \in L^5(I, L^{10})$ such that*

$$\|v\|_{L^5(I,L^{10})} \leq M \tag{VII.3.1}$$

and, letting $e = \partial_t^2 v - \Delta v - \sigma v^5$, one has

$$\|e\|_{L^1(I,L^2)} \leq \varepsilon \leq \varepsilon_M. \tag{VII.3.2}$$

Let u be a solution of (W5) *such that*

$$\|\vec{u}(t_0) - \vec{v}(t_0)\|_{\mathcal{H}} \leq \varepsilon \leq \varepsilon_M. \tag{VII.3.3}$$

Then u is defined on I and

$$\|u - v\|_{L^5(I,L^{10})} + \sup_{t \in I} \|\vec{u}(t) - \vec{v}(t)\|_{\mathcal{H}} \lesssim C_M \varepsilon.$$

SKETCH OF PROOF. The proof of this long-time perturbation theorem is a refinement of the proof of Theorem IV.5.1. To lighten notations, we assume $t_0 = 0$ and consider only positive times, letting $I = [0, T]$ (without loss of generality due to the reversibility of the equation). We divide the interval $[0, T]$ as follows:

$$a_0 = 0 < a_1 < \ldots < a_n = T,$$

where

$$\|v\|_{L^5([a_j,a_{j+1}],L^{10})} \leq \delta, \quad 0 \leq j \leq n-1$$

and δ is a small universal constant to be specified. We choose n in order to minimize the number n of subdivisions. This can be done, for example, by choosing a_j such that $\|v\|_{L^5([a_j,a_{j+1}],L^{10})} = \delta$ for $0 \leq j \leq n-2$. We thus have

$$M^5 \geq \|v\|^5_{L^5(I,L^{10})} \geq (n-1)\delta^5,$$

which gives $n \lesssim M^5$.

Let $t \in I_{\max}(u) \cap [a_j, a_{j+1}]$. Then by Strichartz estimates and the equations satisfied by u and v,

$$\begin{aligned}\|u - v\|_{L^5([a_j,t],L^{10})} &\lesssim \|e\|_{L^1([0,t],L^{10})} + \left\|u^5 - v^5\right\|_{L^1([0,t],L^2)} + \|\vec{u}(a_j) - \vec{v}(a_j)\|_{\mathcal{H}} \\ &\lesssim \varepsilon + \|\vec{u}(a_j) - \vec{v}(a_j)\|_{\mathcal{H}} + \|u - v\|_{L^5([a_j,t],L^{10})} \left(\|u - v\|^4_{L^5([a_j,t],L^{10})} + \delta^4\right).\end{aligned}$$

Assuming that $\|\vec{u}(a_j) - \vec{v}(a_j)\|_{\mathcal{H}}$ is small, and using also the smallness of δ, we obtain, by a bootstrap argument,

$$\forall t \in [a_j, a_{j+1}] \cap I_{\max}(u), \quad \|u - v\|_{L^5([a_j,t],L^{10})} \leq C\varepsilon + C\left\|\vec{u}(a_j) - \vec{v}(a_j)\right\|_{\mathcal{H}},$$

and thus, using energy estimates

$$\forall t \in [a_j, a_{j+1}] \cap I_{\max}(u), \quad \|\vec{u}(t) - \vec{v}(t)\|_{\mathcal{H}} \leq C\varepsilon + C\left\|\vec{u}(a_j) - \vec{v}(a_j)\right\|_{\mathcal{H}}.$$

By induction on j, provided that ε is small enough, we deduce that u is defined on $[a_j, a_{j+1}]$, for all $j \in [0, n-1]$, and that

$$\|\vec{u}(a_j) - \vec{v}(a_j)\|_{\mathcal{H}} \leq 10C^j\varepsilon.$$

Finally, we obtain

$$\sup_{0 \leq t \leq T} \|\vec{u}(t) - \vec{v}(t)\|_{\mathcal{H}} \leq 20C^n\varepsilon,$$

and

$$\|u - v\|_{L^5([0,T],L^{10})} \leq 20nC^n\varepsilon.$$

This concludes the proof. Note that the argument needs $C^n\varepsilon \ll 1$, which is a smallness assumption on ε depending only on n, thus depending only on M since $n \lesssim M^5$. □

VII.4. Proof of the nonlinear profile decomposition

We next give a sketch of proof of Theorem VII.2.1.

We let

$$v_{J,n} = \sum_{j=1}^{J} U_j^{g_{j,n}} + w_{J,n}.$$

We will check that for large J and n, $v_{J,n}$ and u_n satisfy the assumptions of Theorem VII.3.1. More precisely, we will prove that there exists $M > 0$ such that for all $J \geq 1$,

$$\limsup_{n\to\infty} \|v_{J,n}\|_{L^5(\mathbb{R},L^{10})} \leq M, \tag{VII.4.1}$$

and letting

$$e_{J,n} = (\partial_t^2 - \Delta)v_{J,n} - \sigma v_{J,n}^5,$$

one has

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|e_{J,n}\|_{L^1(\mathbb{R},L^2)} = 0. \tag{VII.4.2}$$

Since

$$\lim_{n\to\infty} \|\vec{u}_n(0) - \vec{v}_{J,n}(0)\|_{\mathcal{H}} = 0$$

by the definition of the nonlinear profiles, we see that for a large fixed J, we can apply Theorem VII.3.1 for large n, and that the conlusion of Theorem VII.2.1 hold.

Proof of (VII.4.2). We have

$$e_{J,n} = \sigma \sum_{j=1}^{J} \left(U_j^{g_{j,n}}\right)^5 - \sigma \left(\sum_{j=1}^{J} U_j^{g_{j,n}} + w_{J,n}\right)^5.$$

Using the elementary inequality

$$\left|\sum_j a_j^5 - \left(\sum_j a_j\right)^5\right| \leq \sum_{1\leq j\neq k\leq J} a_j^4|a_k|,$$

(see Claim C.0.1 in the appendix), we obtain

$$\|e_{J,n}\|_{L^1(\mathbb{R},L^2)} \leq C_J \sum_{j\neq k} \|(U_j^{g_{j,n}})^4 U_k^{g_{k,n}}\|_{L^1L^2} + \|w_{J,n}\|^5_{L^5L^{10}}.$$

By the orthogonality of the parameters, we have

$$\lim_{n\to\infty} \|(U_j^{g_{j,n}})^4 U_k^{g_{k,n}}\|_{L^1L^2} = 0$$

for $j \neq k$ (see Proposition C.0.2 in the appendix). Since

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|w_{J,n}\|_{L^5L^{10}} = 0,$$

we obtain (VII.4.2).

Proof of (VII.4.1). The same argument as above gives

$$\lim_n \left\| \sum_{j=1}^{J} \left(U_j^{g_{j,n}}\right)^5 - \left(\sum_{j=1}^{J} U_j^{g_{j,n}}\right)^5 \right\|_{L^1L^2} = 0.$$

This proves

$$\limsup_n \left\| \sum_{j=1}^{J} U_j^{g_{j,n}} \right\|^5_{L^5L^{10}} = \limsup_n \left\| \sum_{j=1}^{J} (U_j^{g_{j,n}})^5 \right\|_{L^1L^2} \leq \sum_{j=1}^{J} \|U_j\|^5_{L^5L^{10}},$$

by the triangle inequality in L^1L^2.

Let $J_0 \geq 1$ such that for $j \geq J_0 + 1$

$$\|\varphi_j\| = \|\vec{\varphi}_j(0)\|_{\mathcal{H}} \leq \delta/2,$$

where the small constant $\delta > 0$ is given by the well-posedness theory. By Theorem IV.3.1, for $j \geq J_0 + 1$, the corresponding nonlinear profile U_j is global and satisfies $\|U_j\|_{L^5 L^{10}} \leq C\|\varphi_j\|$. Since $U_j \in L^5 L^{10}$ for all j by our assumptions, we obtain, for $J \geq J_0 + 1$,

$$\sum_{j=1}^{J} \|U_j\|_{L^5 L^{10}}^5 \leq \sum_{j=1}^{J_0} \|U_j\|_{L^5 L^{10}}^5 + C \sum_{j=J_0+1}^{J} \|\varphi_j\|^5.$$

By the Pythagorean expansion (VI.3.7),

$$\sum_{j=J_0+1}^{J} \|\varphi_j\|^5 \leq \Big(\sum_{j=1}^{\infty} \|\varphi_j\|^2 \Big)^{\frac{5}{2}} \leq \limsup_n \|\vec{u}_n(0)\|_{\mathcal{H}}^5.$$

Combining the inequalities above, we obtain (VII.4.1) with

$$M = \sum_{j=1}^{J_0} \|U_j\|_{L^5 L^{10}}^5 + C \sum_{j=J_0+1}^{\infty} \|\varphi_j\|^5 + \sup_{J \geq 1} \limsup_n \|w_n^J\|_{L^5 L^{10}},$$

which is finite. □

CHAPTER VIII

Compactness/Rigidity method

In this chapter we give an application of the nonlinear profile decomposition, the compactness/rigidity method, which can be used to prove Theorems V.2.5 and V.3.8. It was introduced, for the nonlinear focusing energy-critical wave equation, by Kenig and Merle in [**23**] to prove Theorem V.3.8. The scheme of proof introduced in [**23**] also simplifies the proof of scattering in the defocusing case (i.e. Theorem V.2.5). The ideas introduced here are also useful for the proof of Theorem V.4.1 (the resolution into bubbles in the radial case).

The compactness step of the argument, detailed below, was also introduced by Tao, Vişan and Zhang [**35**] for the mass-critical Schrödinger equation (see also the work of Keraani [**24**] on the same equation). The general philosophy leading to this strategy of proof for nonlinear dispersive equations can be traced back at least to the work of Martel-Merle [**27**] on the generalized KdV equation, where a rigidity theorem is proved, and the work of Bourgain [**6**] where the idea of working inductively on the energy is introduced.

Let us introduce a notation. We let $\widetilde{\mathcal{G}} = \{\tilde{g} = (\lambda, X) \in (0,\infty)\times\mathbb{R}^3\}$ acting on $\mathcal{H} = \dot{H}^1 \times L^2$ by

$$(u_0,u_1)^{\tilde{g}}(x) = \left(\frac{1}{\lambda^{1/2}}u_0\left(\frac{x-X}{\lambda}\right), \frac{1}{\lambda^{3/2}}u_1\left(\frac{x-X}{\lambda}\right)\right).$$

Note that this action is different from the action of the group $\mathcal{G}$ defined in the previous chapters, which includes a time translation. To distinguish between the two, we will always use a tilde to denote an element of $\widetilde{\mathcal{G}}$.

VIII.1. Setting of the argument

Let us consider to fix ideas the defocusing wave equation:

$$\text{(W5d)} \qquad \partial_t^2 u - \Delta u = -u^5, \quad x\in\mathbb{R}^3,$$

with initial data $(u_0,u_1)\in \dot{H}^1\times L^2 = \mathcal{H}$. Our goal is to prove that all solutions of (W5d) scatter. We will prove a slightly stronger result. Recall that the energy E of the solution, defined by

$$E(\vec{u}(t,x)) = \frac{1}{2}\int|\nabla u(t,x)|^2dx + \frac{1}{2}\int(\partial_t u(t,x))^2dx + \frac{1}{6}\int u^6(t,x)dx,$$

is independent of t. We will often denote it by $E(u)$ to lighten notations.

Theorem VIII.1.1. *Let $M>0$ and $\mathcal{N}(M)$ the set of solutions u of* (NLW) *with energy $E(u)\le M$. Then*

$$\text{(VIII.1.1)} \qquad \sup_{u\in\mathcal{N}(M)} \|u\|_{L^5(I_{\max},L^{10})} < \infty.$$

Note that (VIII.1.1) implies that all solutions with energy $\le M$ scatter in both time directions by Proposition V.2.2.

The proof of Theorem VIII.1.1 is by contradiction. By the small data theory, we know that the conclusion of Theorem VIII.1.1 is true for small $M>0$. Indeed, if M is small and $u\in\mathcal{N}(M)$, then $\|\vec{u}(0)\|^2_{\mathcal{H}} \le 2E(u)\le 2M$ and Theorem IV.3.1 implies that u scatters and that $\|u\|_{L^5L^{10}}\lesssim\sqrt{E}\lesssim M$. We assume that the conclusion of the theorem is false. Then there exists E_c such that (VIII.1.1) holds for all $M<E_c$, and (VIII.1.1) is false for all $M>E_c$.

We will then prove the following:

Theorem VIII.1.2. *Assume that the conclusion of Theorem V.2.5 is false and let E_c be as above. Then there exists a solution u_c with $E(u_c)=E_c$, a family of transformations $(\tilde{g}(t))_{t\in I_{\max}(u_c)}$, with $\tilde{g}\in\widetilde{\mathcal{G}}$ such that*

$$\text{(VIII.1.2)} \qquad K = \left\{(\vec{u}_c(t))^{\tilde{g}(t)}\ :\ t\in I_{\max}(u_c)\right\}$$

has compact closure in $\mathcal{H}$.

Remark VIII.1.3. By definition, denoting $\tilde{g}(t) = (\lambda(t),x(t))$, we have

$$(\vec{u}_c(t))^{\tilde{g}(t)}(x) = \left(\frac{1}{\lambda(t)^{1/2}}u_c\left(t,\frac{x-x(t)}{\lambda(t)}\right), \frac{1}{\lambda(t)^{3/2}}\partial_t u_c\left(t,\frac{x-x(t)}{\lambda(t)}\right)\right).$$

The solution u_c of Theorem VIII.1.2 is called a *critical solution*.

Theorem VIII.1.2 is the first step of the proof of Theorem V.2.5 on the defocusing equation (W5d). The first step of the proof of Theorem V.3.8 on the focusing equation (W5f) is similar. In this case, we prove by contradiction that for all $M < E(W,0)$,

$$\sup_{u\in\mathcal{N}_f(M)} \|u\|_{L^5(I_{\max},L^{10})} < \infty, \tag{VIII.1.3}$$

where $\mathcal{N}_f(M)$ is the set of solutions of (W5f) such that $E(u) \leq E(W,0)$ and $\int |\nabla u(0)|^2 + \int(\partial_t u(t))^2 < \int |\nabla W|^2$. If this is not true, then there exists a critical solution $u_c \in \mathcal{N}_f(M)$, with $M < E(W,0)$ and such that there exists a family of transformation $(\tilde{g}(t))_{t\in I_{\max}(u_c)}$ such that K defined by (VIII.1.2) has compact closure in $\mathcal{H}$.

Theorem VIII.1.2 is the "compactness step" of the compactness/rigidity method. To complete the proof of Theorem V.2.5, one must exclude the existence of the critical element by a rigidity theorem (sometimes called Liouville theorem):

THEOREM VIII.1.4. *Let u_c be a solution of* (W5d)*) such that there exists $(\tilde{g}(t))_{t\in I_{\max}(u_c)}$ such that K defined by* (VIII.1.2) *has compact closure in $\mathcal{H}$. Then $u_c = 0$.*

THEOREM VIII.1.5. *Let u_c be a solution of* (W5f) *such that there exists $(\tilde{g}(t))_{t\in I_{\max}(u_c)}$ such that K defined by* (VIII.1.2) *has compact closure in $\mathcal{H}$. Assume furthermore $E(u_c,0) < E(W,0)$, $\int |\nabla u_c(0)|^2 + \int(\partial_t u(0))^2 < \int |\nabla W|^2$. Then $u_c = 0$.*

This compactness/rigidity scheme can be used on many nondispersive equations (Nonlinear Schrödinger equation, Klein-Gordon equation...). The "compactness step" is very robust and similar on all the equations. On the other hand, the proof of the rigidity theorem strongly depends on the equation that is considered. For example, the proof of Theorem VIII.1.4 on the defocusing equation is much easier than the proof of Theorem VIII.1.5 on the focusing wave equation. See Exercise E.1 in the appendix for the proof of Theorem VIII.1.4 in a particular case.

VIII.2. Proof of the existence and compactness of a critical element.

In this section we prove Theorem VIII.1.2. The proof of the analogous theorem for the focusing equation (W5f) is very close and left as an exercise to the reader. We divide the proof into a few steps.

2.a. Expansion of the energy of the nonlinear equation. We will prove Theorem VIII.1.2 using the linear and nonlinear profile decompositions introduced in the previous two chapters. For this, we need to show that the energy of the nonlinear wave equation (W5) satisfies an expansion which is similar to the Pythagorean expansion (VI.3.7) of the square of the $\mathcal{H}$ norm. More precisely, we will prove:

LEMMA VIII.2.1. *Let $(u_n)_n$ be a bounded sequence in $\mathcal{LW}$ that admits a profile decomposition with profiles $(\varphi_j)_{j\geq 1}$, $(g_{j,n})_n$. Then for all $J \geq 1$,*

$$\lim_n \left(E(\vec{u}_n(0)) - \sum_{j=1}^{J} E\left(\overrightarrow{\varphi_j^{g_{j,n}}}(0)\right) - E\left(\vec{w}_{J,n}(0)\right)\right) = 0.$$

In the lemma, E refers to the energy for the nonlinear wave equation

$$E(f,g) = \frac{1}{2}\int |\nabla f|^2 + \frac{1}{2}\int g^2 - \frac{\sigma}{6}\int f^6,$$

for one of the signs $\sigma \in \{\pm 1\}$.

PROOF. As seen before, we have, for $J \geq 1$,

$$\lim_n \left(\|u_n\|^2 - \sum_{j=1}^{J} \|\varphi_j\|^2 - \|w_{J,n}\|^2 \right) = 0.$$

By definition of the norm $\|\cdot\|$ and since the transformations $g_{j,n}$ are isometries, this is equivalent to

$$\lim_n \left(\|\vec{u}_n(0)\|^2_{\mathcal{H}} - \sum_{j=1}^{J} \left\|\overrightarrow{\varphi_j^{g_{j,n}}}(0)\right\|^2_{\mathcal{H}} - \|\vec{w}_{J,n}(0)\|^2_{\mathcal{H}} \right) = 0.$$

We are thus left with proving:

$$\lim_n \left(\|u_n(0)\|_{L^6}^6 - \sum_{j=1}^J \left\|\varphi_j^{g_{j,n}}(0)\right\|_{L^6}^6 - \|w_{J,n}(0)\|_{L^6}^6 \right) = 0.$$

Expanding the equality:

$$u_n(0) = \sum_{j=1}^J \varphi_j^{g_{j,n}}(0) + w_{J,n}(0),$$

we obtain that

$$\left| \|u_n(0)\|_{L^6}^6 - \sum_{j=1}^J \left\|\varphi_j^{g_{j,n}}(0)\right\|_{L^6}^6 - \|w_{J,n}(0)\|_{L^6}^6 \right|$$
$$\lesssim \sum_{1\leq j\neq k\leq J} \int |\varphi^{g_{j,n}}(0,x)|^5 \, |\varphi^{g_{k,n}}(0,x)| \, dx$$
$$+ \sum_{1\leq k\leq J} \int |\varphi^{g_{k,n}}(0,x)|^5 \, |w_{J,n}(0,x)| \, dx + \sum_{1\leq k\leq J} \int |\varphi^{g_{k,n}}(0,x)| \, |w_{J,n}(0,x)|^5 \, dx.$$

The orthogonality of the sequences of parameters imply

$$j \neq k \Longrightarrow \int |\varphi^{g_{j,n}}(0,x)|^5 \, |\varphi^{g_{k,n}}(0,x)| \, dx = 0$$

(see Claim VIII.2.2 below). For $J' > J$, writing

$$w_{J',n} = w_{J,n} + \sum_{j=J+1}^{J'} \varphi_j^{g_{j,n}},$$

we see that it also implies

$$\lim_n \int |\varphi^{g_{k,n}}(0,x)|^5 \, |w_{J,n}(0,x)| = \lim_n \int |\varphi^{g_{k,n}}(0,x)|^5 \, |w_{J',n}(0,x)|$$

and

$$\lim_n \int |\varphi^{g_{k,n}}(0,x)| \, |w_{J,n}(0,x)|^5 \, dx = \lim_n \int |\varphi^{g_{k,n}}(0,x)| \, |w_{J',n}(0,x)|^5 \, dx.$$

Using

$$\limsup_{J'} \lim_{n\to\infty} \|w_{J',n}(0)\|_{L^6} = 0$$

we obtain the desired conclusion. ☐

In the preceding proof we used:

Claim VIII.2.2. *Let φ, $\psi \in \mathcal{LW}$. Let $(g_n)_{n\in\mathbf{I}}$ and $(h_n)_{n\in\mathbf{I}}$ be two* orthogonal *sequences of $\mathcal{G}$. Then*

$$\lim_{n\in\mathbf{I}} \int (\varphi^{g_n}(0,x))^5 \left(\psi^{h_n}(0,x)\right) dx = 0.$$

Proof. We denote $g_n = (\lambda_n, t_n, x_n)$, and $h_n = (\mu_n, s_n, y_n)$.

By conservation of the energy, $\varphi(t)$ and $\psi(t)$ are bounded in L^6, uniformly for $t \in \mathbb{R}$. Furthermore it follows from the dispersive estimate (Theorem I.5.6) that

$$\lim_{t\to\pm\infty} \|\varphi(t)\|_{L^6} = \lim_{t\to\pm\infty} \|\psi(t)\|_{L^6} = 0.$$

(Indeed, Theorem I.5.6 that this holds when the initial data of φ (resp. ψ) is smooth and compactly supported. The general case follows by density). Thus if $\lim_{n\to\infty} -\frac{t_n}{\lambda_n} = \pm\infty$ or $\lim_{n\to\infty} -\frac{s_n}{\mu_n} = \pm\infty$ then $\lim_{n\to\infty} \|\varphi^{g_n}(0)\|_{L^6} = 0$ or $\lim_{n\to\infty} \|\psi^{h_n}(0)\|_{L^6} = 0$, and the conclusion of the claim follows.

Arguing by contradiction, we can thus assume that the following limits are finite

$$\lim_n -\frac{t_n}{\lambda_n} = \tau, \quad \lim_n -\frac{s_n}{\mu_n} = \sigma.$$

We are thus reduced to prove

$$\lim_n \int \frac{1}{\lambda_n^{5/2}} \varphi\left(\tau, \frac{x-x_n}{\lambda_n}\right) \frac{1}{\mu_n^{\frac{1}{2}}} \psi\left(\sigma, \frac{x-y_n}{\mu_n}\right) dx = 0, \tag{VIII.2.1}$$

assuming
$$\lim_n \frac{\lambda_n}{\mu_n} + \frac{\mu_n}{\lambda_n} + \frac{|x_n - y_n|}{\lambda_n} = \infty.$$
To prove (VIII.2.1), one can assume by density that $\varphi(\tau)$ and $\psi(\sigma)$ are continuous and compactly supported. In this case, the proof of (VIII.2.1) is quite easy and we leave it to the reader. □

2.b. The compactness argument. The core of the proof of Theorem VIII.1.2 is the following compactness property for sequences of solutions of (W5d) whose energies converge to the critical energy E_c:

PROPOSITION VIII.2.3. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of solutions to the nonlinear wave equation with maximal interval of existence $(T_n^-, T_n^+) \ni 0$ and initial data $\vec{u}_{0,n} = \vec{u}_n(0)$. We assume*

$$\limsup_{n\to\infty} \|\vec{u}_{0,n}\|_{\mathcal{H}} < \infty \tag{VIII.2.2}$$

$$\lim_{n\to\infty} E(\vec{u}_{0n}) = E_c \tag{VIII.2.3}$$

$$\lim_{n\to\infty} \|u_n\|_{L^5((0,T_n^+),L^{10})} = \lim_{n\to\infty} \|u_n\|_{L^5((T_n^-,0),L^{10})} = +\infty. \tag{VIII.2.4}$$

Then there exists $\mathbf{I} \sqsubset \mathbb{N}$ and a sequence $(\tilde{g}_n)_{n\in\mathbf{I}}$ in $\widetilde{\mathcal{G}}$ such that $\left((\vec{u}_{0n})^{\tilde{g}_n}\right)_{n\in\mathbf{I}}$ converges strongly in $\mathcal{H}$.

REMARK VIII.2.4. In the assumptions of the proposition, we allow $u_n \notin L^5((0,T_n^+),L^{10})$: in this case we make the convention
$$\|u_n\|_{L^5((0,T_n^+),L^{10})} = \infty.$$
We make a similar convention on the interval $(T_n^-, 0)$.

PROOF. Let $u_{L,n}$ be the solution of the linear wave equation with initial data $\vec{u}_{0,n}$ at $t = 0$. By Theorem VI.3.7, there exists $\mathbf{I} \sqsubset \mathbb{N}$ such that $(u_{L,n})_{n\in\mathbf{I}}$ admits a profile decomposition
$$u_{L,n} = \sum_{j=1}^{J} \varphi_j^{g_{j,n}} + w_{J,n}, \quad n \in \mathbf{I},$$
and the sequences $(g_{j,n})_{n\in\mathbf{I}}$ are tidy (see Definition VII.1.1). Let U_j be the nonlinear profile associated to φ_j, $(\varphi_{j,n})_{n\in\mathbf{I}}$ (see Definition VII.1.3). By Lemma VIII.2.1,
$$E(\vec{u}_{0,n}) = \sum_{j=1}^{J} E\left(\overrightarrow{\varphi_j^{g_{j,n}}}(0)\right) + E(\vec{w}_{J,n}(0)) + o(1)$$
as $n \to \infty$, $n \in \mathbf{I}$.

We note that by definition of the nonlinear profile:
$$\lim_{n\in\mathbf{I}} E\left(\overrightarrow{\varphi_j^{g_{j,n}}}(0)\right) = E(U_j),$$
where $E(u)$ denotes the conserved energy of a solution u of the nonlinear equation (W5d). Thus

$$\sum_{j=1}^{J} E\left(U_j\right) + \lim_{n\to\infty} E(\vec{w}_{J,n}(0)) = E_c \tag{VIII.2.5}$$

Step 1. We prove that there is at least one nonzero profile φ_j. If not, then
$$\lim_{n\to\infty} \|u_{Ln}\|_{L^5L^{10}} = 0.$$
This proves by the small data theory that for large n, u_n scatters and that
$$\lim_{n\to\infty} \|u_n\|_{L^5L^{10}} = 0,$$
contradicting the assumption (VIII.2.4).

Step 2. We prove that there is exactly one nonzero profile φ_j. If not, since $E(U_j) > 0$ if φ_j is not 0, we see by (VIII.2.5) that $E(U_j) < E_c$ for all $j \geq 1$. By the definition of E_c, it means that U_j scattes for all $j \geq 1$. By Theorem VII.2.1 and Remark VII.2.2, we have that u_n scatters for large n and
$$\lim_n \|u_n|\|_{L^5L^{10}} < \infty,$$
contradicting the assumption (VIII.2.4).

Step 3. Reordering the profiles so that φ_1 is the only nonzero profile, we prove that

$$\lim_{n\to\infty} \|\vec{w}_{1,n}(0)\|_{\mathcal{H}} = 0. \tag{VIII.2.6}$$

Indeed, we have $U_j = 0$ for $j \geq 2$ and

$$E_c = E(U_1) + E(\vec{w}_{1,n}(0)) + o_n(1), \quad n \to \infty,\ n \in \mathbf{I}.$$

Thus if (VIII.2.6) does not hold, then $E(U_1) < E_c$, which implies that all nonlinear profiles U^j scatter. Again, the nonlinear profile decomposition Theorem VII.2.1 gives a contradiction.

Step 4. We have proved so far

$$\vec{u}_{0,n} = \left(\frac{1}{\lambda_{1,n}^{1/2}}\varphi_1\left(\frac{-t_{1,n}}{\lambda_{1,n}}\right), \frac{1}{\lambda_{1,n}^{3/2}}\partial_t\varphi_1\left(\frac{-t_{1,n}}{\lambda_{1,n}}\right)\right) + o(1), \quad n \to \infty,\ n \in \mathbf{I}.$$

Let $\tau_1 = \lim_{n\in\mathbf{I}} -t_{1,n}/\lambda_{1,n}$. We prove in this step that $\tau_1 \in \mathbb{R}$, which will conclude the proof of Proposition VIII.2.3.

We argue by contradiction. We have

$$u_{L,n}(t,x) = \frac{1}{\lambda_{1,n}^{1/2}}\varphi_1\left(\frac{t-t_{1,n}}{\lambda_{1,n}}, \frac{x-x_{1,n}}{\lambda_{1,n}}\right) + o(1).$$

Thus

$$\|u_{L,n}\|_{L^5([0,\infty),L^{10})} = \|\varphi_1\|_{L^5([-t_{1,n}/\lambda_{1,n},L^{10})} + o_n(1).$$

As a consequence, if $\tau_1 = \infty$, we obtain

$$\lim_{n\to\infty} \|u_{L,n}\|_{L^5([0,\infty),L^{10})} = 0.$$

By the small data theory (Theorem IV.3.1), we deduce that $T_+(u_n) = +\infty$ for large n and

$$\lim_{n\to\infty} \|u_n\|_{L^5([0,\infty),L^{10})} = 0,$$

contradicting the assumption (VIII.2.4).

Assuming that $\tau_1 = -\infty$, we obtain similarly $T_-(u_n) = -\infty$ for large n and

$$\lim_{n\to\infty} \|u_n\|_{L^5((-\infty,0],L^{10})} = 0,$$

which also give a contradiction, concluding the proof of Proposition VIII.2.3. □

2.c. Existence of the critical element. We next prove

LEMMA VIII.2.5. *With the same assumptions as in Theorem VIII.1.2, there exists a solution u_c of* (W5d)*, defined on a maximal interval of existence $I_c = (T_-, T_+) \ni 0$, such that $E(u_c) = E_c$ and $u_c \notin L^5([0,T_+), L^{10})$, $u_c \notin L^5((T_-,0], L^{10})$.*

PROOF. By the definition of E_c, for all $n \geq 1$, there exists a solution $\tilde{u}_n$ of (W5d) with

$$E(\tilde{u}_n) \leq E_c + \frac{1}{n}$$

and

$$\|\tilde{u}_n\|_{L^5(I_n,L^{10})} \geq n,$$

where $I_n = (\widetilde{T}_n^-, \widetilde{T}_n^+)$ is the maximal interval of existence of $\tilde{u}_n$. We let $t_n \in (\widetilde{T}_n^-, \widetilde{T}_n^+)$ such that

$$\|\tilde{u}_n\|_{L^5((t_n,\widetilde{T}_n^+),L^{10})} \geq n/2 \text{ and } \|\tilde{u}_n\|_{L^5((\widetilde{T}_n^-,t_n),L^{10})} \geq n/2,$$

and $u_n(t,x) = \tilde{u}_n(t+t_n,x)$. Then $E_c(u_n) \leq E_c + \frac{1}{n}$, u_n has maximal interval of existence (T_n^-, T_n^+), where $T_n^{\pm} = \widetilde{T}_n^{\pm} - t_n$, so that $T_n^- < 0 < T_n^+$ and

$$\|u_n\|_{L^5((0,T_n^+),L^{10})} \geq \frac{n}{2}, \quad \|u_n\|_{L^5((T_n^-,0),L^{10})} \geq \frac{n}{2}.$$

Thus $(u_n)_{n\in\mathbb{N}}$ satisfies the assumptions of Proposition VIII.2.3. By this proposition, there exists $\mathbf{I} \subset \mathbb{N}$ and $(\tilde{g})_{n\in\mathbf{I}} \in \widetilde{\mathcal{G}}^{\mathbf{I}}$ such that $(\vec{u}_n(0)^{\tilde{g}_n})_{n\in\mathbf{I}}$ converges strongly in $\mathcal{H}$. Let

$$(u_0,u_1) = \lim_{n\in\mathbf{I}} \vec{u}_n(0)^{\tilde{g}_n} \text{ in } \mathcal{H}.$$

Let u_c be the solution of (W5d) with initial data $\vec{u}_c(0) = (u_0,u_1)$, and (T_-,T_+) its maximal interval of existence.

Then $E(u_c) \leq E_c$, passing to the limit in the inequality $E_c(u_n) \leq E_c + \frac{1}{n}$. Furthermore we have

$$u_c \notin L^5((0,T_+),L^{10}) \text{ and } u_c \notin L^5((T_-,0),L^{10}). \tag{VIII.2.7}$$

Indeed, we prove this property by contradiction. If for example $u_c \in L^5((0,T_+),L^{10})$, then $T_+ = \infty$ and we can apply Theorem VII.3.1 with $I = [0,\infty)$, $v = u_c$ and $u = u_n$ (for large n) proving $T_+(u_n) = +\infty$ and

$$\lim_{n\infty} \|u_n\|_{L^5([0,+\infty),L^{10})} = \|u_c\|_{L^5([0,+\infty),L^{10})} < \infty,$$

in contradiction with the properties of u_n.

Using the same argument on the interval $(T_-,0)$, we obtain (VIII.2.7), concluding the proof of Lemma VIII.2.5 $\square$

2.d. Compactness of the critical solution. We next conclude the proof of Theorem VIII.1.2 with the following lemma:

LEMMA VIII.2.6. *Let u_c be given by Lemma VIII.2.5. Then there exist a family $(\tilde{g}(t))_{t\in I_c}$ of $\widetilde{\mathcal{G}}$ such that*

$$K = \{(\vec{u}_c(t))^{\tilde{g}(t)} \;:\; t \in I_c\}$$

has compact closure in $\mathcal{H}$.

PROOF. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence in I_c and consider the sequence $(u_n)_{n\in\mathbb{N}}$ of solutions of (W5d) defined by $u_n(t) = u_c(t+t_n)$. It follows from the properties of u_c that $(u_n)_n$ satisfies the assumptions of Proposition VIII.2.3. By this proposition, there exists a subsequence $(t_n)_{n\in\mathbf{I}}$ of $(t_n)_{n\in\mathbb{N}}$ and a sequence of transformations $(\tilde{g}_n)_{n\in\mathbf{I}}$ in $\widetilde{\mathcal{G}}$ such that $\left((\vec{u}_n(t_n))^{\tilde{g}_n}\right)_{n\in\mathbf{I}}$.

By a general lifting lemma, proved in Appendix D, the preceding property implies the existence of $(\tilde{g}(t))_{t\in I_c}$ such that K has compact closure in $\mathcal{H}$ $\square$

APPENDIX A

Bochner integrals

(See e.g. section 1.2 in the book [**7**].)

Using the Lebsegue integral for scalar valued functions, we can define integrals of functions defined in a Banach space. Fix a Banach space B and an interval I. A *simple function* $s : I \to B$ is a finite sum of the form:

$$s(x) = \sum_{i=1}^{p} \mathbb{1}_{E_i} b_i,$$

where $b_i \in B$ and E_i is a measurable subset of I.

The integral of f on I is then defined as $\int_I f = \sum_{i=1}^{p} |E_i| b_i$, where $|E_i|$ is the Lebesgue measure of E_i.

One says that a function $f : I \to B$ is *measurable* if there exists a sequence $(s_n)_n$ of simple functions such that

$$\lim_{n\to\infty} \|s_n(t) - f(t)\|_B \text{ for almost all } t \in I.$$

If f is measurable, the function $t \mapsto \|f(t)\|_B$ is measurable. One says that f is *Bochner-integrable* if there exists a sequence s_n of simple functions such that

$$\lim_n \int_I \|s_n - f\|_B = 0.$$

In this case, the integral of f on I is defined by

$$\int_I f = \lim_{n\to\infty} \int s_n,$$

where s_n is any sequence of simple functions such that $\lim_{n\to\infty} \int_I \|s_n - f\|_B = 0$.

S. Bochner has proved that Bochner-integrability is equivalent to the condition:

$$\int_I \|f\|_B < \infty,$$

and that

$$\left\| \int_I f \right\|_B \leq \int_I \|f\|_B.$$

More generally, for $1 \leq p < \infty$ the space $\mathcal{L}^p(I, B)$ is defined as the space of measurable functions such that $\int_I \|f\|^p < \infty$. In this case, we denote by

$$\|f\|_{L^p(I,B)} = \left(\int_I \|f(t)\|^p dt \right)^{1/p}.$$

We will mainly use these spaces with $B = L^q(\mathbb{R}^N)$ for some $q \in [1, \infty)$. One can prove that a function $f \in L^p(I, L^q(\mathbb{R}^N))$ can be identified with a L^1_{loc} function on $I \times \mathbb{R}^N$, and that the space $C_c^\infty(I \times \mathbb{R}^N)$ is dense in $L^p(I, L^q(\mathbb{R}^N))$. In these notes, using these properties, we will mostly see $L^p(I, L^q(\mathbb{R}^N))$ as the closure of $C_c^\infty(I \times \mathbb{R}^N)$ for the $L^p(I, L^q(\mathbb{R}^N))$ norm.

APPENDIX B

Completion of the proof of Strichartz estimates

This appendix is dedicated to the proof of the $L^4/L^{4/3}$ dispersion inequality III.3.5. This proof uses the Riesz-Thorin interpolation Theorem, and some elements of Littlewood-Paley theory which we recall in Sections B.1 and B.2?

B.1. Interpolation

We first recall an interpolation Theorem for a linear operator between L^p space.

THEOREM B.1.1 (Riesz–Thorin interpolation Theorem). *Let* (X,μ), (Y,ν) *be measure spaces. Let*

$$\theta \in]0,1[, \quad (p_0,p_1,q_0,q_1,p,q)\in[1,\infty]^6$$

with

$$\frac{1}{p}=\frac{\theta}{p_0}+\frac{1-\theta}{p_1}, \quad \frac{1}{q}=\frac{\theta}{q_0}+\frac{1-\theta}{q_1}. \tag{B.1.1}$$

Let A *be a linear operator defined on* $L^{p_0}(X)+L^{p_1}(X)$ *which is bounded from* $L^{p_0}(X)$ *to* $L^{q_0}(Y)$ *and from* $L^{p_1}(X)$ *to* $L^{q_1}(Y)$. *Then* A *is a bounded linear operator from* $L^p(X)$ *to* $L^q(Y)$, *and*

$$\|A\|_{L^p(X)\to L^q(Y)} \le \|A\|^{\theta}_{L^{p_0}(X)\to L^{q_0}(Y)}\|A\|^{1-\theta}_{L^{p_1}(X)\to L^{q_1}(Y)}.$$

In the theorem, $\|A\|_{E\to F}$ denotes the operator norm of the bounded operator $A: E\to F$, where E and F are Banach spaces. Theorem B.1.1 is classical. See e.g. [**31**, Chapter 2, Section 2].

B.2. Littlewood-Paley theory

We next give a few elements of Littlewood-Paley theory, which is a useful Fourier analysis tool to study L^p spaces with $p\neq 2$. What follows is by no mean a complete account on Littlewood-Paley theory: we will just state the needed results, and will give only some of the proofs. We refer to [**2**, Chapter 2] for a complete introduction to the subject.

We will need Young's inequality for the convolution

THEOREM B.2.1. *Let* $f\in L^q(\mathbb{R}^N)$, $g\in L^r(\mathbb{R}^N)$ *with* $1/q+1/r\ge 1$, *and* p *defined by* $\frac{1}{p}+1=\frac{1}{q}+\frac{1}{r}$. *Then*

$$f*g(x)=\int f(x-y)g(y)dy$$

is defined for almost every $x\in\mathbb{R}^N$ *and*

$$\|f*g\|_{L^p}\le \|f\|_{L^q}\|g\|_{L^r}, \tag{B.2.1}$$

EXERCISE B.1. Prove Young's inequality. Hint: start with the cases $(q,r)=(1,1)$, $(q,r)=(\infty,1)$, $(q,r)=(1,\infty)$ and use the interpolation theorem B.1.1.

We start with some inequalities on frequency localized function.

THEOREM B.2.2 (Berstein-type estimates). *Let* $\psi\in C_0^\infty(\mathbb{R}^N)$. *Then if* $1\le q\le p\le\infty$

$$\forall f\in\mathcal{S}(\mathbb{R}^N),\quad \forall\lambda>0,\quad \|\psi(\lambda D)f\|_{L^p}\lesssim \lambda^{\left(\frac{N}{p}-\frac{N}{q}\right)}\Big\|f\Big\|_{L^q} \tag{B.2.2}$$

Assume furthermore $\psi(\xi)=0$ *for* ξ *close to* 0. *Then, if* $s\in\mathbb{R}$ *and* $p\in[1,\infty]$,

$$\forall f\in\mathcal{S}(\mathbb{R}^N),\quad \forall\lambda>0,\quad \Big\||D|^s\psi(\lambda D)f\Big\|_{L^p}\approx\lambda^{-s}\Big\|\psi(\lambda D)f\Big\|_{L^p}. \tag{B.2.3}$$

Moreover, if $s\in\mathbb{N}$,

$$\forall f\in\mathcal{S}(\mathbb{R}^N),\quad \forall\lambda>0,\quad \sup_{|\alpha|=s}\Big\|\partial_x^\alpha\left(\psi(\lambda D)f\right)\Big\|_{L^p}\approx\lambda^{-s}\Big\|\psi(\lambda D)f\Big\|_{L^p}. \tag{B.2.4}$$

In the theorem, the implicit constants might depend on ψ, but of course not on f and $\lambda > 0$.

PROOF. *Step 1.*

We first prove (B.2.2) for $\lambda = 1$. We have

$$\psi(D)u = (\overline{\mathcal{F}}\psi) * u, \tag{B.2.5}$$

where $f * g$ is the convolution of f and g. This is a classical property of the Fourier transform, which can be checked by an explicit computation of $\mathcal{F}(\psi(D)u)$. Note that $\overline{\mathcal{F}}\psi \in \mathcal{S} \subset \bigcap_{1\leq p\leq\infty} L^p$. Using Young's inequality

we obtain that (B.2.2) holds for $\lambda = 1$, i.e. that there exists $C > 0$ such that

$$\forall f \in \mathcal{S}(\mathbb{R}^N), \quad \|\psi(D)f\|_{L^p} \leq \|f\|_{L^q}.$$

Step 2: rescaling. Denote by $T_\lambda u(x) = u(\lambda x)$. By a simple change of variable, one can prove

$$\Psi(D)(T_\lambda u) = T_\lambda \left(\psi(\lambda D)u\right)$$

Thus by Step 1,

$$\|T_\lambda \left(\psi(\lambda D)u\right)\|_{L^p} \lesssim \|T_\lambda u\|_{L^q}.$$

Since $\|T_\lambda f\|_{L^p} = \frac{1}{\lambda^{N/p}}\|f\|_{L^p}$, we obtain (B.2.2) for any $\lambda > 0$.

Step 3: proof of (B.2.3).

Let $\chi \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$, such that $\chi(\xi) = 1$ if $\xi \in \operatorname{supp}(\psi)$. Then

$$|D|^s\psi(\lambda D)u = |D|^s\chi(\lambda D)\psi(\lambda D)u = \frac{1}{\lambda^s}\Xi(\lambda D)\psi(\lambda D)u,$$

where $\Xi(\xi) = |\xi|^s\chi(\xi)$. Using (B.2.2) with $p = q$, we obtain

$$\left\||D|^s\psi(\lambda D)u\right\|_{L^p} \lesssim \frac{1}{\lambda^s}\left\|\psi(\lambda D)u\right\|_{L^p}. \tag{B.2.6}$$

Using (B.2.6), with s replaced by $-s$ and u replaced by $|D|^s\chi(\lambda D)u$, we obtain

$$\left\|\psi(\lambda D)u\right\|_{L^p} = \left\||D|^{-s}\psi(\lambda D)|D|^s u\right\|_{L^p} \lesssim \lambda^s\left\|\psi(\lambda D)|D|^s u\right\|_{L^p}.$$

This concludes the proof of (B.2.3).

Step 4: proof of (B.2.4). First, we have

$$\left\|\psi(\lambda D)\partial_x^\alpha f\right\|_{L^p} = \left\|\partial_x^\alpha\chi(\lambda D)\psi(\lambda D)f\right\|_{L^p} = \frac{1}{|\lambda|^{|\alpha|}}\left\|\Xi_\alpha(\lambda D)\psi(\lambda D)f\right\|_{L^p}, \tag{B.2.7}$$

where χ is as above and $\Xi_\alpha(\xi) = (i\xi)^\alpha\chi(\xi)$. The estimate $\lesssim$ in (B.2.4) then follows from (B.2.2) with $q = p$.

Next, if s is even, we have $|D|^s = (-\Delta)^{s/2}$, which shows that (B.2.3) implies the other estimate in (B.2.4).

If s is odd, we write

$$\left\|\psi(\lambda D)|D|^s f\right\| = \|\psi(\lambda D)|D|^{s+1}\frac{1}{|D|}f\|_{L^p} \lesssim \sup_{|\alpha|=s+1}\left\|\partial_x^\alpha|D|^{-1}\psi(\lambda D)f\right\|_{L^p}$$

$$\approx \frac{1}{\lambda}\sup_{|\alpha|=s+1}\|\partial_x^\alpha\psi(\lambda D)f\|_{L^p},$$

and we conclude with (B.2.7) that the inequality $\gtrsim$ in (B.2.4) holds in this case also. □

The Littlewood-Paley theory is based on a dyadic decomposition of a distribution $f \in \mathcal{S}'(\mathbb{R}^N)$. We fix once and for all a radial function $\varphi \in C_0^\infty(\mathbb{R}^N)$ with $\varphi(\xi) = 1$ if $|\xi| \leq 1/2$, and $\varphi(x) = 0$ if $|x| \geq 1$. We let

$$\Theta_j(\xi) = \varphi\left(\frac{\xi}{2^{j+1}}\right) - \varphi\left(\frac{\xi}{2^j}\right) = \Theta\left(\frac{\xi}{2^j}\right), \quad \Theta(\xi) = \varphi(\xi/2) - \varphi(\xi).$$

We have

$$\operatorname{supp}\Theta_j \subset \left\{2^{j-1} \leq |\xi| \leq 2^{j+1}\right\}, \quad \sum_{j=-\infty}^{+\infty}\Theta_j(\xi) = 1,\ (\xi \neq 0),$$

where the sum is, for any fixed ξ, a finite sum. We denote

$$\Delta_j f = \Theta_j(D),$$

so that (at least formarly) $f = \sum_{j\in\mathbb{Z}}\Theta_j(D)f$ (*Dyadic decomposition of f in frequencies*). If $f \in \mathcal{S}_0$, it is easy to prove that this sum converges in $\mathcal{S}$.

We have the inequality

$$\forall \xi \neq 0, \quad \frac{1}{2} \leq \sum_{j\in\mathbb{Z}} \Theta_j^2(\xi) \leq 1. \tag{B.2.8}$$

Exercise B.2. Prove (B.2.8). *Hint:* Let

$$A(\xi) = \sum_{j \text{ odd}} \Theta_j(\xi), \quad B(\xi) = \sum_{j \text{ even}} \Theta_j(\xi).$$

Check that if $\xi \neq 0$,

$$A(\xi) + B(\xi) = 1, \quad A^2(\xi) = \sum_{j \text{ odd}} \Theta_j^2(\xi), \quad B^2(\xi) = \sum_{j \text{ even}} \Theta_j^2(\xi).$$

Combining with Plancherel identity, it follows that if $f \in \mathcal{S}(\mathbb{R}^N)$,

$$\|f\|^2_{L^2(\mathbb{R}^N)} \approx \sum_{j\in\mathbb{Z}} \|\Delta_j f\|^2_{L^2(\mathbb{R}^N)}, \tag{B.2.9}$$

and more generally,

$$\|f\|^2_{\dot{H}^s} \approx \sum_{j\in\mathbb{Z}} \|\Delta_j |D|^s f\|^2_{L^2} \approx \sum_{j\in\mathbb{Z}} (2^{2j})^s \|\Delta_j f\|^2_{L^2}. \tag{B.2.10}$$

The situation is more complicated for $p \neq 2$. Nevertheless, we have the following estimates:

Theorem B.2.3. *For all $p \in (1,2]$, for any $f \in \mathcal{S}$*

$$\sum_{j\in\mathbb{Z}} \|\Delta_j f\|^2_{L^p} \lesssim \|f\|^2_{L^p} \tag{B.2.11}$$

For all $p \in [2,\infty)$, for any $f \in L^p$,

$$\|f\|^2_{L^p} \lesssim \sum_{j\in\mathbb{Z}} \|\Delta_j f\|^2_{L^p}\,. \tag{B.2.12}$$

We omit the proof refering the interested reader to [**2**, Theorem 2.40].

Exercise B.3. Prove:

- For all $p \in [1,2]$, for any $f \in \mathcal{S}$

$$\|f\|^p_{L^p} \lesssim \sum_{j\in\mathbb{Z}} \|\Delta_j f\|^p_{L^p} \tag{B.2.13}$$

- For all $p \in [2,\infty]$, for any $f \in L^p$,

$$\sum_{j\ \in\mathcal{Z}} \|\Delta_j f\|^p_{L^p} \lesssim \|f\|^p_{L^p} \tag{B.2.14}$$

(where the sum has to be interpreted as $\sup_j \|\Delta_j f\|_{L^\infty}$ when $p = \infty$).

Hint: Start with the cases $p = 1$ and $p = 2$ for (B.2.13) and $p = \infty$ and $p = 2$ for (B.2.14), then use an interpolation argument.

The two estimates of Exercise B.3 complete the estimates of Theorem B.2.3. The proofs are simpler than the proof of Theorem B.2.3, but are not detailed here since we will not need these estimates below.

Note that there is no perfect equivalence between the norm $\|f\|_{L^p}$ and a norm defined as a ℓ^q norm of the sequence $(\|\Delta_j f\|_{L^p})_j$ if $p \neq 2$.

Let us mention that the quantities

$$\|f\|^q_{\dot{B}^0_{p,q}} = \sum_{j\in\mathbb{Z}} \|\Delta_j f\|^q_{L^p} \tag{B.2.15}$$

appearing in (B.2.11), (B.2.12), (B.2.13) and (B.2.14) defines the norm of the so-called Besov space $\dot{B}^0_{p,q}$. See Sections 2.3, 2.4 and 2.5 of [**2**] for more details on Besov spaces.

B.3. Proof of a dispersion inequality

In this Section, we prove the dispersion inequality (III.3.5), which we recall:

$$\left\| e^{it|D|} \frac{1}{|D|} \varphi \right\|_{L^4} \lesssim \frac{1}{|t|^{1/2}} \|\varphi\|_{L^{4/3}}, \tag{B.3.1}$$

using the tools introduced in Sections B.1 and B.2.
Step 1: frequency-localized dispersion estimate.

We will use the Littlewood-Paley decomposition of φ, $\varphi = \sum_{j\in\mathbb{Z}} \Delta_j \varphi$. In this step we prove the following frequency localized version of the dispersion inequality for the wave equation

$$\forall j, \quad \left\| \frac{e^{it|D|}}{|D|} \Delta_j \varphi \right\|_{L^\infty} \lesssim \frac{2^j}{t} \|\Delta_j \varphi\|_{L^1}. \tag{B.3.2}$$

We let $\varphi_j = \Delta_j \varphi$. By the dispersion inequality for the full wave equation and Theorem B.2.2, we have

$$\left\| \frac{\sin(t|D|)}{|D|} \varphi_j \right\|_{L^\infty} \lesssim \frac{1}{|t|} \|\varphi_j\|_{\dot{W}^{1,1}} \approx \frac{2^j}{|t|} \|\varphi_j\|_{L^1}$$

and

$$\left\| \frac{\cos(t|D|)}{|D|} \varphi_j \right\|_{L^\infty} \approx \frac{1}{2^j} \|\cos(t|D|)\varphi_j\|_{L^\infty} \lesssim \frac{1}{2^j|t|} \|\varphi_j\|_{\dot{W}^{2,1}} \approx \frac{2^j}{|t|} \|\varphi_j\|_{L^1}.$$

Step 2. A $L^4/L^{4/3}$ dispersion inequality

We next introduce $\widetilde{\Delta}_j f = \Delta_{j-1} f + \Delta_j f + \Delta_{j+1} f$. Noting that $\Theta_{j-1} + \Theta_j + \Theta_{j+1} = 1$ on the support of Θ_j, we see that $\widetilde{\Delta}_j \Delta_j f = \Delta_j f$. For fixed $t > 0$ and j, consider the operator $e^{it|D|}|D|^{-1}\widetilde{\Delta}_j$. By Step 1, it is a bounded operator from L^1 to L^∞, with operator norm $\lesssim 2^j/t$. By Plancherel and Theorem B.2.2, it is bounded from L^2 to L^2 with operator norm $\lesssim 2^{-j}$. Using the interpolation Theorem B.1.1, we obtain that $e^{it|D|}|D|^{-1}\widetilde{\Delta}_j$ is a bounded operator from $L^{4/3}$ to L^4 with operator norm $\lesssim t^{-1/2}$. Using that $\widetilde{\Delta}_j \Delta_j = \Delta_j$, we deduce

$$\left\| e^{it|D|} \frac{1}{|D|} \Delta_j \varphi \right\|_{L^4} \lesssim \frac{1}{|t|^{1/2}} \|\Delta_j \varphi\|_{L^{4/3}}.$$

Taking the square and summing up, we deduce (using Theorem B.2.3) the inequality (B.3.1).

□

B.4. Alternative proof of the Strichartz estimate for the half-wave equation

To illustrate the tools introduce Section B.2, we give here a proof of the Strichartz inequality (III.3.1) from (III.3.7) that does not use the Sobolev inequality (III.3.8), but rather Theorems B.2.2 and B.2.3. By the preceding step, applied to $\Delta_j |D| \varphi$, we have

$$2^j \left\| e^{i\cdot|D|} \Delta_j \varphi \right\|^2_{L^4(\mathbb{R}\times\mathbb{R}^3)} \leq \left\| \Delta_j |D| \varphi \right\|^2_{L^2}. \tag{B.4.1}$$

By Theorem B.2.2 (Bernstein inequalities), at fixed t,

$$\|e^{it|D|}\Delta_j \varphi\|_{L^{12}} \lesssim 2^{j/2} \left\| e^{it|D|} \Delta_j \varphi \right\|_{L^4(\mathbb{R}^3)}.$$

Taking the L^4 norm in time, then summing up the squares, we obtain

$$\sum_j \|e^{i\cdot|D|}\Delta_j\varphi\|^2_{L^4L^{12}} \lesssim \sum_j 2^j \left\| e^{it|D|}\Delta_j \varphi \right\|^2_{L^4(\mathbb{R}\times\mathbb{R}^3)} \lesssim \sum_j \|\Delta_j |D| \varphi\|^2_{L^2}, \tag{B.4.2}$$

where we have used (B.4.1) to obtain the last inequality. The right-hand side of (B.4.2) is $\approx \|\varphi\|^2_{\dot{H}^1}$ by Plancherel equality (see (B.2.9)). We must prove that the left-hand side dominates $\left\| e^{it|D|}\varphi \right\|_{L^4L^{12}}$. Let $u = e^{it|D|}\varphi$ and $u_j = \Delta_j u$. By Minkowski inequality (i.e. the triangle inequality for the $L^2(\mathbb{R})$ norm), we see that

$$\sum_{j\in\mathbb{Z}} \|u_j\|^2_{L^4L^{12}} = \sum_{j\in\mathbb{Z}} \left\| \|u_j(t)\|^2_{L^{12}(\mathbb{R}^3)} \right\|_{L^2(\mathbb{R})} \geq \left\| \sum_{j\in\mathbb{Z}} \|u_j(t)\|^2_{L^{12}} \right\|_{L^2(\mathbb{R})}$$

By Theorem B.2.3, at fixed t,

$$\|u(t)\|^2_{L^{12}} \lesssim \sum_{j\in\mathbb{Z}} \|u_j(t)\|^2_{L^{12}}.$$

This shows

$$\sum_{j\in\mathbb{Z}} \|u_j\|^2_{L^4L^{12}} \gtrsim \Big\| \|u(t)\|^2_{L^{12}(\mathbb{R}^3)} \Big\|_{L^2(\mathbb{R})} = \|u\|^{1/2}_{L^4L^{12}},$$

which together with (B.4.2) concludes the proof of Proposition III.3.1.

APPENDIX C

Proof of two technical results

In this appendix we prove two results that are needed in the proof of the nonlinear profile decomposition (Theorem VII.2.1).

Claim C.0.1. *Let $J\geq 2$ There exists $C_J>0$ such that for all $(a_j)_j\in\mathbb{R}^J$, one has*

$$\left|\sum_{j=1}^J a_j^5-(\sum_{j=1}^J a_j)^5\right|\leq C_J\sum_{1\leq j\neq k\leq J} a_j^4|a_k|.$$

Proof. Expending $(\sum_{j=1}^J a_j)^5$ we obtain

$$\left|\sum_{j=1}^J a_j^5-\Big(\sum_{j=1}^J a_j\Big)^5\right|\leq \sum_{(j_k)_k}\prod_{k=1}^5 |a_{j_k}|,$$

where the sum is taken over all 5-uple $(j_k)_k$ with $1\leq j_k\leq J$ such that at least two of the indices j_k are distinct. Let $1\leq \ell_1\neq\ell_2\leq J$ such that

$$|a_{\ell_1}|=\max_{1\leq j\leq J}|a_j|,\quad |a_{\ell_2}|=\max_{\substack{1\leq j\leq J\\ j\neq \ell_1}}|a_j|.$$

Then for all 5-uple $(j_k)_k$ with at least two distinct indices j_k, we have

$$\prod_{k=1}^5|a_{j_k}|\leq a_{\ell_1}^4|a_{j_2}|.$$

This yields, for a constant C_J,

$$\left|\sum_{j=1}^J a_j^5-\Big(\sum_{j=1}^J a_j\Big)^5\right|\leq C_J|a_{\ell_1}^4|a_{\ell_2}|,$$

and hence the conclusion of the Claim. ☐

Proposition C.0.2. *Let $(g_n)_n$, $(h_n)_n$ be two orthogonal sequences of $\mathcal{G}$. Let $u,v\in L^5(\mathbb{R},L^{10})$. Then*

$$\lim_{n\to\infty}\left\|u^{g_n}(v^{h_n})^4\right\|_{L^1L^2}=0.$$

Proof. Let $U,V\in L^5L^{10}$. Writing

$$u^{g_n}(v^{h_n})^4-U^{g_n}(V^{h_n})^4=u^{g_n}\left((v^{h_n})^4-(V^{h_n})^4\right)+(u^{g_n}-U^{g_n})(V^{h_n})^4,$$

we see that

$$\begin{aligned}&\left\|u^{g_n}(v^{h_n})^4-U^{g_n}(V^{h_n})^4\right\|_{L^1L^2}\\&\qquad\leq 10\|v-V\|_{L^5L^{10}}\|u\|_{L^5L^{10}}\left(\|v\|_{L^5L^{10}}+\|V\|_{L^5L^{10}}\right)+\|u-U\|_{L^5L^{10}}\|V\|^4_{L^5L^{10}}.\end{aligned}$$

Using the density of $C_c^\infty(\mathbb{R}^4)$ into L^5L^{10}, we deduce that it is sufficient to prove the conclusion of the claim assuming $u,v\in C_c^\infty(\mathbb{R}^4)$. The proof is elementary in this case and we leave it to the reader. ☐

APPENDIX D

A lifting lemma

In this appendix we consider a separable Hilbert space H (with scalar product $\langle\cdot|\cdot\rangle$ and norm $\|\cdot\|$) and a topological group $(G,\circ)$ acting on H. If $x\in H$ and $g\in G$ we will denote x^g the action of g on x, with the convention $(x^h)^g = x^{g\circ h}$. We will assume the following properties:

$$\forall g\in G,\ \forall x\in H,\quad \|x\| = \|x^g\| \tag{D.0.1}$$

$$\forall x\in H,\ \forall (g_n)_n\in G^{\mathbb{N}},\quad \lim_n g_n = g \Longrightarrow \lim_{n\to\infty}\|x^{g_n}-x^g\| = 0. \tag{D.0.2}$$

We will also assume that for all sequence $(g_n)_{n\in\mathbb{N}}$ of elements of G one of the following holds:

(1) For all $x\in H$, $x^{g_n} \underset{n\to\infty}{\rightharpoonup} 0$.

(2) There exists a subsequence $(g_n)_{n\in\mathbf{I}}$ of $(g_n)_{n\in\mathbb{N}}$ that converges in G.

We note that in the second case, if $x\in H\setminus\{0\}$, (x^{g_n}) does not converge weakly to 0, since $(x^{g_n})_{n\in\mathbf{I}}$ converges strongly to $x^g\neq 0$ where $g=\lim_n g_n$. Thus we can replace (1) by

(1') There exists $x\in H\setminus\{0\}$ such that $x^{g_n} \underset{n\to\infty}{\rightharpoonup} 0$.

We have already checked these properties for the action of the group $\mathcal{G}$ on $\mathcal{LW}$ in Chapter VI: see Claims VI.3.1 and VI.3.6, Remark VI.3.5, Lemma VI.3.2 adn the following exercise:

Exercise D.1. Using Lemma VI.3.2 prove that if $(u_n)_n$ converges strongly to u in $\mathcal{LW}$ and g_n converges to g in $\mathcal{G}$, then $(u_n^{g_n})_n$ converges strongly to u^g in $\mathcal{LW}$.

The properties above also hold for the action of the group $\widetilde{\mathcal{G}}$ on $\mathcal{H}$ defined in Chapter VIII. This can be proved directly, or using the properties of $\mathcal{G}$: indeed $\widetilde{\mathcal{G}}$ identifies with the subset of $\mathcal{G}$: $\big\{g=(\lambda,0,X)\ :\ \tilde{g}=(\lambda,X)\in\widetilde{\mathcal{G}}\big\}$, remarking that

$$(\vec{u}_0)^{\tilde{g}} = \overrightarrow{u_L^g}(0),$$

where $\vec{u}_0\in\mathcal{H}$ and $u_L(t)=S_L(t)\vec{u}_0$.

We will prove the following lifting lemma, that completes the proof of Lemma VIII.2.6.

Lemma D.0.1. *Let A be a subset of H such that*

$$\inf_{x\in A}\|x\| > 0. \tag{D.0.3}$$

and for all sequence $(x_n)_{n\in\mathbb{N}}$ in A, there exists $\mathbf{I}\sqsubset\mathbb{N}$ and a sequence $(g_n)_{n\in\mathbf{I}}$ in G such that $(x_n^{g_n})_{n\in\mathbf{I}}$ converges strongly in H. Then there exists a family $(g(x))_{x\in A}$ of elements of G such that

$$K = \Big\{x^{g(x)},\ x\in A\Big\}$$

has compact closure in H.

Proof. The proof relies on the separability of H. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a Hilbert basis of H. Let, for $n\geq 0$, and $x\in H$,

$$\|x\|_n^2 = \sum_{k=0}^{n} |\langle x|\varphi_k\rangle|^2.$$

Step 1. Lower bound for a finite-dimensional norm. We prove by contradiction that there exists $d\geq 1$ such that

$$\inf_{x\in A}\sup_{g\in G}\|x^g\|_d > 0.$$

If not, for all $n\geq 1$, there exists $x_n\in A$ such that

$$\sup_{g\in G}\|x_n^g\|_n \leq \frac{1}{n}.$$

By the assumption on A, there exists $\mathbf{I} \sqsubset \mathbb{N}$ and a sequence $(g_n)_{n\in\mathbf{I}}$ in G such that $\lim_{n\in\mathbf{I}} x_n^{g_n} = x$ strongly in H. This implies:

$$\lim_{n\in\mathbf{I}} \|x_n\| = \|x\| > 0$$

and thus $x \neq 0$. On the other hand, if k is fixed, then

$$\forall n \in \mathbf{I}, \quad n \geq k \Longrightarrow \|x_n^{g_n}\|_k \leq \|x_n^{g_n}\|_n \leq \frac{1}{n},$$

and letting $n \to \infty$ in $\mathbf{I}$ we see that $\|x\|_k = 0$. Since it is true for all k, we have $\langle x|\varphi_k\rangle = 0$ for all k and thus $x = 0$. This is a contradiction.

Step 2: end of the proof. We let d be as in Step 1 and

$$c = \inf_{x\in A} \sup_{g\in G} \|x^g\|_d > 0.$$

For each $x \in A$, we choose $g(x) \in G$ such that $\|x^{g(x)}\|_n \geq \frac{c}{2}$. We will show that with this choice of $g(x)$, K has compact closure in H.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in A. By the assumption on A, there exists $\mathbf{I} \sqsubset \mathbb{N}$, $x \in H$ and a sequence $(g_n)_{n\in\mathbf{I}}$ in G such that

$$\lim_n x_n^{g_n} = x$$

strongly in H. We have

$$x_n^{g(x_n)} = (x_n^{g_n})^{g(x_n)\circ g_n^{-1}} = x^{g(x_n)\circ g_n^{-1}} + o(1)$$

strongly in H, as $n \to \infty$ in $\mathbf{I}$. Since

$$\|x_n^{g(x_n)}\|_d \geq \frac{c}{2},$$

we see that $(x_n^{g(x_n)})_{n\in\mathbf{I}}$ cannot converge to 0 weakly in H. Thus $(x^{g(x_n)\circ g_n^{-1}})_{n\in\mathbf{I}}$ does not converge weakly to 0 in H. By our assumption of the action of G, there exists $\mathbf{I}' \sqsubset \mathbf{I}$ and $g \in G$ such that

$$\lim_{n\in\mathbf{I}'} g(x_n) \circ g_n^{-1} = g.$$

This shows that

$$\lim_{n\in\mathbf{I}'} x_n^{g(x_n)} = x^g$$

strongly in $\mathcal{H}$, concluding the proof of the lemma. □

APPENDIX E

Complement: asymptotic completeness in the radial defocusing case

We prove here a particular case of Theorem V.2.5, assuming that the initial data (u_0, u_1) is radial. In this case the proof of scattering is much simpler:

THEOREM E.0.1. *Let u be a solution of* (W5d) *and radial initial data. Then u scatters.*

This proof is due to J. Ginibre, A. Soffer, G. Velo (see [**17**]). We divide it into a few Lemmas.

LEMMA E.0.2 (Morawetz inequality). *There exists $C > 0$ with the following property. Let u be a solution of* (W5). *Assume that $\vec{u}_0$ is radial, compactly supported and smooth. Let E be the energy of u and $I_{\max}$ its maximal interval of existence. Then*

$$\int_{I_{\max}} \int_{\mathbb{R}^3} \frac{1}{|x|} |u(t,x)|^6 dx dt \leq CE.$$

REMARK E.0.3. To simplify, we prove Lemma E.0.2 assuming that the solution is radial, but the result remains valid without symmetry assumption. In this general case, it is an ingredient of the proof of the rigidity Theorem VIII.1.4 (which implies Theorem V.2.5 by Chapter VIII.1.2): see the next exercise for the proof of a particular case of this theorem.

EXERCISE E.1. Let u be a global solution of (W5d) such that

$$\{\vec{u}(t) \ : \ t \in \mathbb{R}\}$$

has compact closure in $\mathcal{H}$. Using that

$$\int_{I_{\max}} \int_{\mathbb{R}^3} \frac{1}{|x|} |u(t,x)|^6 dx dt < \infty$$

Prove that $u = 0$.

PROOF. By stability, persistence of regularity and finite speed of propagation, the solution u is C^∞ on $I_{\max} \times \mathbb{R}^3$, and there exists $R > 0$ such that $|x| \leq R + |t|$ on the support of u.

Let

$$M(t) = \int_0^\infty \partial_t u(t,r) \partial_r u(t,r) r^2 dr + \int_0^\infty \partial_t u(t,r) u(t,r) r dr.$$

Then

$$M'(t) = \int_0^\infty \partial_t^2 u (u + r\partial_r u) r dr + \underbrace{\int_0^\infty \partial_t u \partial_r (\partial_t u) r^2 dr + \int_0^\infty (\partial_t u)^2 r dr}_{=0},$$

where we use a straightforward integration by parts to prove that the two last terms cancel each other. Using the equation, we have

$$\begin{aligned} M'(t) = \int_0^\infty & \left(\partial_r^2 u + \frac{2}{r} \partial_r u - u^5 \right) (u + r\partial_r u)\, r dr \\ &= \int_0^\infty \frac{1}{2} \frac{\partial}{\partial r} (u + r\partial_r u)^2 \, dr - \int_0^\infty u^5 (u + r\partial_r u) r dr \\ &= -\frac{1}{2} u^2(t,0) - \int_0^\infty u^6 r dr - \int_0^\infty \frac{1}{6} \frac{\partial}{\partial r} u^6 \, r^2 dr = -\frac{1}{2} u^2(t,0) - \frac{2}{3} \int_0^{+\infty} u^6 r dr. \end{aligned}$$

Next, we notice that $M(t) \lesssim E$. Indeed, this follows easily by the Cauchy-Schwarz inequality and Hardy's inequality

$$\int_0^\infty u^2 dr \leq 4 \int_0^\infty (\partial_r u)^2 r^2 dr, \tag{E.0.1}$$

which follows from Cauchy-Schwarz and the equality

$$2\int_0^\infty u\partial_r urdr = \int_0^\infty \partial_r(u^2)rdr = -\int_0^\infty u^2 dr.$$

Integrating the bound $M'(t) \leq -\frac{2}{3}\int_0^\infty u^6 rdr$ between two times a and b, with $T_- < a < b < T_+$, and letting $b \to T_+$ and $a \to T_-$ we obtain the desired conclusion. □

We next prove

LEMMA E.0.4 (Bound of the L^8 norm). *Let u be a radial solution of* (W5) *with $\sigma = -1$. Then*

$$\|u\|_{L^8(I_{\max}\times\mathbb{R}^3)} \lesssim E^{1/4}. \tag{E.0.2}$$

REMARK E.0.5. The pair $(8,8)$ is $\dot{\mathcal{H}}^1$-Strichartz admissible for the wave equation in space dimension 3.

PROOF.

STEP 1. We prove the bound when $\vec{u}_0$ is C^∞, compactly supported. For this we use the Morawetz estimate of Lemma E.0.2 and the radial Sobolev inequality (also known as Strauss' Lemma):

$$u^2(t,r) \leq \frac{1}{r}\int_r^\infty (\partial_\rho u(t,\rho))^2\rho^2 d\rho \lesssim \frac{1}{r}E. \tag{E.0.3}$$

This last inequality can be proved with the fundamental theorem of calculus and Cauchy-Schwarz inequality:

$$|u(r)| = \left|\int_r^\infty \partial_\rho u(\rho)d\rho\right| \leq \sqrt{\int_r^\infty (\partial_\rho u(\rho))^2\rho^2 d\rho}\sqrt{\int_r^\infty \rho^{-2}d\rho}.$$

Combining Lemma E.0.2 with (E.0.3), we obtain

$$\int_{I_{\max}} u^8(t,r)r^2 drdt \lesssim E\int_{I_{\max}}\int_0^\infty u^6(t,r)rdrdt \lesssim E^2,$$

which give (E.0.2) in this case.

STEP 2. To prove the bound for general solutions, we use a density argument. We consider a sequence of initial data $(\vec{u}_0^n)_n$ with $\vec{u}_0^n \in (C_0^\infty)^2$, radial, such that $\lim_n \vec{u}_0^n = \vec{u}_0$ in $\dot{\mathcal{H}}^1$. Let $K \subset I_{\max}(\vec{u}_0)$ compact. By continuity of the flow (Theorem IV.5.1), $K \subset I_{\max}(\vec{u}_0^n)$ for large n and

$$\lim_{n\to\infty} \|u^n - u\|_{L^\infty(K,L^6)} + \|u^n - u\|_{L^5(K,L^{10})} = 0.$$

Since Hölder inequality implies $L^5(K,L^{10}(\mathbb{R}^3)) \cap L^\infty(K,L^6(\mathbb{R}^3)) \subset L^8(K\times\mathbb{R}^3)$ with the bound

$$\|f\|^8_{L^8(K\times\mathbb{R}^3)} \leq \|f\|^3_{L^\infty(K,L^6)}\|f\|^5_{L^5(K,L^{10})},$$

we deduce

$$\lim_{n\to\infty} \left\|u^n - u^0\right\|_{L^8(K\times\mathbb{R}^3)} = 0. \tag{E.0.4}$$

By Step 1,

$$\|u^n\|^8_{L^8(K\times\mathbb{R}^3)} \lesssim (E(\vec{u}_0^n))^2 \underset{n\to\infty}{\longrightarrow} E^2,$$

which concludes the proof.

□

We are now ready to end the proof of Theorem V.2.5

PROOF OF THEOREM V.2.5. We fix $\vec{u}_0 \in \dot{\mathcal{H}}^1$. By Lemma E.0.4, we have $u \in L^8(I_{\max}\times\mathbb{R}^3)$. By Proposition V.2.2, and since $u \in L^5(K,L^{10})$ for all $K \in I_{\max}$ it is sufficient to prove $u \in L^5([\tau,T_+[,L^{10})$ for some $\tau \in I_{\max}$. We fix $\tau \in I_{\max}$ such that

$$\|u\|_{L^8([\tau,T_+[\times\mathbb{R}^3)} \leq \varepsilon, \tag{E.0.5}$$

where the small parameter $\varepsilon > 0$ is will be specified later. For $t \in [\tau, T_+[$, we have by Hölder's inequality

$$\|u\|_{L^5([\tau,t],L^{10}(\mathbb{R}^3))} \leq \|u\|^{2/5}_{L^8([\tau,t]\times\mathbb{R}^3)}\|u\|^{3/5}_{L^4([\tau,t],L^{12}(\mathbb{R}^3))}. \tag{E.0.6}$$

Thus it is sufficent to prove $u \in L^4([\tau,T_+[,L^{12})$. For this we use Strichartz estimate, (E.0.6) and (E.0.5):

$$\|u\|_{L^4([\tau,t,L^{12})} \leq C_S\|\vec{u}(\tau)\|_{\dot{\mathcal{H}}^1} + C_S\|u\|^5_{L^5([t_0,t],L^{10})} \leq 2C_S\sqrt{E} + C_S\varepsilon^2\|u\|^3_{L^4([t_0,t],L^{12})}.$$

We prove by a bootstrap argument:

$$\forall t \in [\tau, T_+[, \quad \|u\|_{L^4([\tau,t],L^{12})} \leq 3C_S\sqrt{E}. \tag{E.0.7}$$

Indeed if (E.0.7) holds for some t, we have

$$\|u\|_{L^4([\tau,t[,L^{12})} \leq 2C_S\sqrt{E} + C_S\varepsilon^2(3C_S\sqrt{E})^3 \leq \frac{5}{2}C_S\sqrt{E},$$

where we have chosen ε so small that $\varepsilon^2(3C_S)^3E \leq \frac{1}{2}$. This proves (E.0.7) by the intermediate value theorem.

By the same proof in a neighborhood of T_-, we obtain, $u \in L^5(I_{\max}, L^{10})$, which concludes the proof that u scatters in both time directions. □

EXERCISE E.2. In the setting of Theorem V.2.5, prove

$$\|u\|_{L^4(\mathbb{R},L^{12})} \leq C\big(E(u_0,u_1)\big)^2. \tag{E.0.8}$$

APPENDIX F

Complement: Nonradiative solutions

0.a. Definition and classification. In order to study the dynamics of nonlinear dispersive equation, it is common to classify solutions that are "completely nondispersive" in a certain sense. These solutions tend to play an important role on the dynamics, and their classification is crucial for its understanding. In this course, we considered a particular type of nondispersive solutions, the solutions that have a relatively compact trajectory (up to scaling and space translation). In this appendix we give a stronger notion of nondispersive solutions, and prove a rigidity theorem related to this notion for the focusing equation in the radial case. This rigidity theorem is crucial in the proof of the "resolution into stationary solutions" Theorem V.4.1.

We will give here the notion of "nonradiative solutions", that was introduced to prove the resolution into stationary solutions for radial solutions of (W5) (See Theorem V.4.1 below and [**13**]).

Definition F.0.1. Let u be a global solution of (W5) or of the linear wave equation (LW). Let $R \in \mathbb{R}$ and $t_0 \in \mathbb{R}$. The solution u is (R, t_0)-*nonradiative* when

$$\sum_{\pm} \lim_{t\to\pm\infty} \int_{|x|\geq R+|t-t_0|} e_u(t,x)dx = 0.$$

In the definition, we have used the notation $e_u(t,x) = \frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{2}(\partial_t u(t,x))^2$. To simplify notations, we will restrict without generality to the case $t_0 = 0$, and call the corresponding solutions R-nonradiative solutions. If $R = 0$, the solution with simply be called "nonradiative".

It is possible, using the explicit formulas of Chapter 1, to prove that the only 0-nonradiative solution of the linear wave equation (LW) is the constant null solution. In the nonlinear case, using that the speed of travelling waves is always < 1, we see that travelling waves are also R-nonradiative solutions for all R. The rigidity conjecture for nonradiative solution says that this should be the only ones:

Conjecture F.0.2 (Rigidity conjecture for nonradiative solutions). *Let u be a nonradiative solution of* (W5f). *Then u is a travelling wave.*

We prove this conjecture in the radial case:

Theorem F.0.3 (Dynamical characterization of W). *Let $R_0 \geq 0$ and u be a radial, R_0-nonradiative solution of* (W5f). *Then one of the following occurs:*

- $u(t,x) = 0$ *for* $|x| > R_0 + |t|$.
- *there exists* $\lambda > 0$, $\iota \in \{\pm 1\}$,

$$\forall |x| > R_0 + |t|, \quad u(t,x) = \iota W_\lambda(x),$$

where

$$W_\lambda(x) = \frac{1}{\lambda^{1/2}} W\left(\frac{x}{\lambda}\right).$$

Remark F.0.4. In the case where $R_0 = 0$, we see that the theorem implies that $(u_0, u_1) = (\iota W_\lambda, 0)$, and thus that u is the stationary solution W_λ. This implies the uniqueness part in Theorem V.3.2, since any solution of the elliptic equation (Ell) is also a nonradiative solution of (W5f).

0.b. A lower bound of the exterior energy for the linear equation. The proof of Theorem F.0.3 is based on its (quantitative) analog for the linear equation (LW):

Proposition F.0.5. *Let $R \geq 0$. Let $(u_0, u_1) \in \dot{\mathcal{H}}^1$ and $u_L(t) = S_L(t)(u_0, u_1)$. Then*

$$\sum_{\pm} \lim_{t\to\pm\infty} \int_{R+|t|}^{+\infty} e_u(t,r) r^2\, dr = \frac{1}{2}\int_R^{+\infty} (\partial_r(ru_0))^2 + r^2 u_1^2\, dr. \tag{F.0.1}$$

The right-hand side of (F.0.1) can be compared to the $\dot{\mathcal{H}}^1(\{|x|>R\})$-norm by a simple integration by parts. Indeed, if $R>0$, we have, for any radial $\dot{H}^1$ function f on $\mathbb{R}^3$,

$$\int_R^\infty \Big(\partial_r(rf(r))\Big)^2 dr = \int_R^\infty \Big(\partial_r f(r)\Big)^2 r^2 dr - Rf(R)^2. \tag{F.0.2}$$

When $R=0$, the boundary term vanishes and we have

$$\int_0^\infty \Big(\partial_r(rf(r))\Big)^2 dr = \int_0^\infty \Big(\partial_r f(r)\Big)^2 r^2 dr.$$

The formula (F.0.1) reads

$$\sum_{\pm} \lim_{t\to\pm\infty} \int_{|x|\geq|t|}^{+\infty} e_u(t,x)dx = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u_0(x)|^2 + u_1^2(x)\right) dx. \tag{F.0.3}$$

Let us mention that (F.0.3) remains valid without the assumption that (u_0,u_1) is radial, and can be proved with the explicit formulas of Theorem I.5.3. It is still valid in any *odd* space dimension, as proved in [**12**], but not in *even* space dimension, even for radial solutions (see [**10**]).

Investigating (F.0.1), we see that the only radial R-nonradiative solutions of (W5) are the solutions that are equal to ℓ/r for $r>R+|t|$, (where $\ell\in\mathbb{R}$). Since ℓ/r is not in $\dot{H}^1(\mathbb{R}^3)$, we also obtain that 0 is the only 0-nonradiative solution.

Proof of Proposition F.0.5. This follows from the explicit formula for radial, 3D solutions (see (I.5.1)),

$$u(t,r) = \frac{1}{r}(\varphi(r+t) - \varphi(t-r)), \tag{F.0.4}$$

where

$$\varphi(\eta) = \frac{1}{2}\eta u_0(|\eta|) + \frac{1}{2}\int_0^\eta \sigma u_1(|\sigma|)d\sigma. \tag{F.0.5}$$

Using this formula, we see that

$$\Big(\partial_r(ru)\Big)^2 + \Big(\partial_t(ru)\Big)^2 = 2(\varphi'(r+t))^2 + 2(\varphi'(t-r))^2.$$

This gives

$$\sum_{\pm} \lim_{t\to\pm\infty} \int_{R+|t|}^{+\infty} |\partial_{t,r}(ru_L(t,r))|^2\, dx = 2\int_R^\infty \left(\varphi'(\eta)\right)^2 d\eta + 2\int_{-\infty}^{-R} \left(\varphi'(\eta)\right)^2 d\eta$$
$$= \int_R^{+\infty} \left(\partial_r(ru_0)\right)^2 + r^2u_1^2\, dr.$$

Using (F.0.2) we obtain (F.0.1). Indeed by by the formula (F.0.4), we have

$$\lim_{t\to\infty}(R+t)u^2(t,R+T) = 0,$$

since $|\varphi(\eta)|/\sqrt{|\eta|}$ goes to 0 as $|\eta|\to\infty$. □

0.c. Proof of the rigidity result. We next prove the rigidity Theorem F.0.3. The proof takes several steps.

Step 1. Let $(u_0,u_1)\in\dot{\mathcal{H}}^1$ be as in Theorem F.0.3. Let $\varepsilon>0$ be a small parameter to be specified. In all the proof we fix $R_\varepsilon\geq R_0$ such that

$$\int_{R_\varepsilon}^{+\infty} \left((\partial_r u_0)^2 + u_1^2\right) r^2\, dr \leq \varepsilon^2. \tag{F.0.6}$$

In this step, we prove

$$\forall R\geq R_\varepsilon, \quad \int_R^{+\infty} \left(\partial_r(ru_0)\right)^2 + r^2u_1^2\, dr \leq CR^5u_0^{10}(R). \tag{F.0.7}$$

Let $R\geq R_\varepsilon$. We define the radial functions $v_0\in\dot{H}^1(\mathbb{R}^3)$, $v_1\in L^2(\mathbb{R}^3)$ as follows:

$$\begin{cases}(v_0,v_1)(r) = (u_0,u_1)(r) & \text{if } r>R\\ (v_0,v_1)(r) = (u_0(R),0) & \text{if } r\in(0,R).\end{cases} \tag{F.0.8}$$

We let $v(t)$ be the solution of (W5f) with initial data (v_0, v_1), and $v_L(t,r) = S_L(t)(v_0, v_1)$ be the corresponding solution to the free wave equation. We note that by final speed of propagation

$$v(t,r) = u(t,r), \quad r > R + |t|.$$

By the small data theory, since ε is small,

$$\sup_{t\in\mathbb{R}} \|\vec{v}(t) - \vec{v}_L(t)\|_{\dot{H}^1\times L^2} \leq C\, \|(v_0, v_1)\|^5_{\dot{H}^1\times L^2}\,. \tag{F.0.9}$$

By Proposition F.0.5,

$$\sum_{\pm} \lim_{t\to\pm\infty} \int_{R+|t|}^{+\infty} |\partial_{t,r}(v_L(t,r))|^2 r^2\, dr \geq \int_R^{+\infty} (\partial_r(ru_0))^2 + u_1^2\, dr. \tag{F.0.10}$$

By (F.0.9), and finite speed of propagation

$$\int_{R+|t|}^{+\infty} \left|\partial_{t,r}(v_L(t,r)) - \partial_{t,r}(u(t,r))\right|^2 r^2\, dr \leq C \left(\int_R^{+\infty} ((\partial_r u_0)^2 + u_1^2) r^2 dr \right)^5.$$

Combining with (F.0.10) and using that the solution is R-nonradiative, we obtain

$$\int_R^{+\infty} (\partial_r(ru_0))^2 + r^2 u_1^2\, dr \leq C \left(\int_R^{+\infty} \left((\partial_r u_0)^2 + u_1^2\right) r^2 dr \right)^5.$$

With the integration by parts formula (F.0.2) and the smallness of ε, we deduce (F.0.7).

Step 2. In this step we prove that there exists $\ell \in \mathbb{R}$ and $C > 0$ such that for large r,

$$\left| u_0(r) - \frac{\ell}{r} \right| \leq \frac{C}{r^3}, \qquad \int_r^{+\infty} \rho^2 u_1^2(\rho)\, d\rho \leq \frac{C}{r^5}. \tag{F.0.11}$$

First fix R and R' such that $R_\varepsilon \leq R \leq R' \leq 2R$. Letting $\zeta_0(r) = ru_0(r)$, we have, using Cauchy-Schwarz, then Step 1

$$|\zeta_0(R) - \zeta_0(R')| \leq \int_R^{R'} |\partial_r \zeta_0(r)|\, dr \leq \sqrt{R} \sqrt{\int_R^{R'} (\partial_r \zeta_0)^2 dr} \leq \frac{1}{R^2} \zeta_0^5(R). \tag{F.0.12}$$

Since by the definition (F.0.6) of R_ε and the integration by parts formula (F.0.2) one has

$$\frac{1}{R} \zeta_0^2(R) \leq \varepsilon^2, \tag{F.0.13}$$

we deduce from (F.0.12):

$$|\zeta_0(R) - \zeta_0(R')| \leq C\varepsilon^4 \zeta_0(R), \qquad R_\varepsilon \leq R \leq R' \leq 2R. \tag{F.0.14}$$

Let $\alpha = \log_2\left(1 + C\varepsilon^4\right)$, so that $2^\alpha = (1 + C\varepsilon^4)$. By (F.0.14), for all k, $|\zeta_0(2^{k+1}R_\varepsilon)| \leq 2^\alpha |\zeta_0(2^k R_\varepsilon)|$. Thus the sequence $\left(\frac{|\zeta_0(2^k R_\varepsilon)|}{(2^k)^\alpha}\right)_{k\geq 0}$ is nonincreasing. This implies that $|\zeta_0(2^k R_\varepsilon)| \lesssim \left(2^k R_\varepsilon\right)^\alpha$, for $k \geq 0$ and thus, using (F.0.14) again,

$$|\zeta_0(R)| \lesssim R^\alpha.$$

We can take ε small enough, so that $\alpha \leq 1/5$. The inequality (F.0.12) yields

$$|\zeta_0(R) - \zeta_0(R')| \lesssim \frac{1}{R}, \quad R_\varepsilon \leq R \leq R' \leq 2R. \tag{F.0.15}$$

This shows that $\sum_{k\geq 0} \left|\zeta_0(2^k R) - \zeta_0(2^{k+1}R_\varepsilon)\right| < \infty$, and thus that $\left(\zeta_0(2^k R_\varepsilon)\right)_k$ has a limit ℓ as $k \to \infty$. By (F.0.15)

$$\lim_{R\to\infty} \zeta_0(R) = \ell.$$

This implies that ζ_0 is bounded. The inequality (F.0.12) then yields

$$\forall k \geq 0, \quad \forall R \geq R_\varepsilon, \quad \left|\zeta_0\left(2^k R\right) - \zeta_0\left(2^{k+1} R\right)\right| \lesssim \frac{1}{2^{2k} R^2}.$$

Summing over $k \geq 0$ and using the triangle inequality, we obtain,

$$|\zeta_0(R) - \ell| \lesssim \frac{1}{R^2},$$

which is the first inequality in (F.0.11). Combining with Step 1, we obtain the second inequality in (F.0.11).

STEP 3. In this step, we assume $\ell = 0$ and prove that $(u_0, u_1) \equiv (0,0)$. Indeed by (F.0.14), if $R \geq R_\varepsilon$ and $k \in \mathbb{N}$,

$$\left|\zeta_0\left(2^{k+1}R\right)\right| \geq (1 - C\varepsilon^4)\left|\zeta_0\left(2^k R\right)\right|.$$

Hence by induction on k,

$$\left|\zeta_0\left(2^k R\right)\right| \geq (1 - C\varepsilon^4)^k |\zeta_0(R)|.$$

Since by the preceding step and the assumption $\ell = 0$, $|\zeta_0(2^k R)| \lesssim 1/(2^k R)^2$, we deduce, choosing ε small enough and letting $k \to \infty$ that $\zeta_0(R) = 0$. Combining with (F.0.7) we deduce

$$R \geq R_\varepsilon \Longrightarrow \int_R^{+\infty} (\partial_r \zeta_0)^2 + u_1^2(r)\, dr = 0,$$

that is $u_0(r)$ and $u_1(r)$ are 0 for almost every $r \geq R_\varepsilon$. Going back to the definition of R_ε we see that we can choose any $R_\varepsilon > R_0$, which concludes this step.

STEP 4. We next assume $\ell \neq 0$. To fix ideas, we assume that ℓ is positive. By the definition (V.3.2) of W and the definition of W_λ we have, for $\lambda > 0$

$$W_\lambda(r) = \frac{\sqrt{3\lambda}}{r} + \mathcal{O}\left(\frac{1}{r^3}\right), \quad r \to \infty.$$

We choose $\lambda > 0$ such that $\sqrt{3\lambda} = \ell$ so that

$$\left|W_\lambda(r) - \frac{\ell}{r}\right| \lesssim \frac{1}{r^3} \tag{F.0.16}$$

for large r. In this step we prove that $(u_0 - W_\lambda, u_1)$ has compact support. Let $f = u - W_\lambda$. Then

$$\begin{cases} \partial_t^2 f - \Delta f = P_\lambda(f) := \displaystyle\sum_{k=1}^{5} \binom{5}{k} W_\lambda^{5-k} f^k. \\ \vec{f}_{\restriction t=0} = (f_0, f_1) := (u_0 - W_\lambda, u_1), \end{cases} \tag{F.0.17}$$

For $\varepsilon > 0$ small, we fix $R'_\varepsilon \gg 1$ such that

$$\int_{R'_\varepsilon}^{+\infty} \left(|\partial_r f_0(r)|^2 + |f_1(r)|^2\right) r^2 dr \leq \varepsilon^2 \tag{F.0.18}$$

$$\int_{\mathbb{R}} \left(\int_{R'_\varepsilon + |t|}^{+\infty} W_\lambda^{10}(r) r^2\, dr\right)^{\frac{1}{2}} dt \leq \varepsilon^5. \tag{F.0.19}$$

Let f_L be the solution of $\partial_t^2 f_L = \Delta f_L$ with

$$\vec{f}_{L\restriction t=0} = (\tilde{f}_0, \tilde{f}_1),$$

where $(\tilde{f}_0, \tilde{f}_1)$ coincides with (f_0, f_1) for $r > R'_\varepsilon$ and is defined as in (F.0.8). Using (F.0.17) and the assumptions (F.0.18) and (F.0.19) on R'_ε, we obtain

$$\sup_{t\in\mathbb{R}} \left\| 1\!\!1_{\{|x|>|t|+R'_\varepsilon\}} \left|\nabla_{t,x}(\tilde{f}(t) - \tilde{f}_L(t))\right| \right\|_{L^2} \lesssim \varepsilon^4 \left\|(\tilde{f}_0, \tilde{f}_1)\right\|_{\dot{H}^1 \times L^2}. \tag{F.0.20}$$

Let $R \geq R'_\varepsilon$. Using that by Proposition F.0.5,

$$\sum_{\pm} \lim_{t\to\pm\infty} \int_{R+|t|}^{+\infty} \left(\partial_{t,r}\left(\tilde{f}_L(t,r)\right)\right)^2 r^2 dr \gtrsim \int_R^{+\infty} \left(\left(\partial_r(r\tilde{f})\right)^2 + r^2 \tilde{f}_1^2\right) dr,$$

and since

$$\sum_{\pm} \lim_{t\to\pm\infty} \int_{R+|t|}^{+\infty} \left(\partial_{t,r}\left(\tilde{f}(t,r)\right)\right)^2 r^2 dr = 0,$$

we deduce from (F.0.20)

$$\varepsilon^8 \int_R^{+\infty} \left((\partial_r f_0)^2 + f_1^2\right) r^2 dr \gtrsim \int_R^{+\infty} \left(\left(\partial_r(rf_0)\right)^2 + r^2 f_1^2\right) dr,$$

and thus

$$\varepsilon^8 R f_0^2(R) \gtrsim \int_R^{\infty} \left(\left(\partial_r(rf_0)\right)^2 + r^2 f_1^2\right) dr. \tag{F.0.21}$$

Letting $g_0 = rf_0$, we deduce by Cauchy-Schwarz that for $R \geq R'_\varepsilon$, $k \in \mathbb{N}$,

$$\left|g_0\left(2^{k+1}R\right) - g_0\left(2^kR\right)\right| \lesssim \int_{2^kR}^{2^{k+1}R} |\partial_r g_0| dr \lesssim \varepsilon^4 \left|g_0(2^kR)\right|.$$

This yields by an easy induction $|g_0(2^kR)| \geq \left(1 - C\varepsilon^4\right)^k |g_0(R)|$, where $C > 0$ is a constant which is independent of ε. Since by Step 2,

$$\frac{C}{\left(2^kR\right)^2} \geq \left|g_0\left(2^kR\right)\right|,$$

we obtain choosing ε small enough that $g_0(R) = 0$ for large R. Combining with (F.0.21), we deduce that $(f_0(r), f_1(r)) = 0$ a.e. for large R, concluding this step.

STEP 5. In this step we still assume $\ell \neq 0$ and conclude the proof. We let

$$\rho = \inf\left\{R > R_0 \ : \ \int_R^{+\infty} \left((\partial_r f_0)^2 + f_1^2\right) r^2 dr = 0\right\}$$

and prove that $\rho = R_0$ i.e.that $u_0(r) = W_\lambda(r)$ for $r > R_0$.

We argue by contradiction, assuming $\rho > R_0$. By the preceding step and finite speed of propagation, the essential support of f is included in $\{r \leq \rho + |t|\}$. Thus f is solution of

$$\begin{cases} \partial_t^2 f - \Delta f = \mathbb{1}_{\{|x| \leq \rho + |t|\}} P_\lambda(f). \\ \vec{f}_{\restriction t=0} = (f_0, f_1) := (u_0 - W_\lambda, u_1), \end{cases}$$

Fix $R''_\varepsilon \in (R_0, \rho)$ such that,

$$\int_{R''_\varepsilon}^{+\infty} \left(|\partial_r f_0(r)|^2 + |f_1(r)|^2\right) r^2 dr \leq \varepsilon^2$$

$$\int_{\mathbb{R}} \left(\int_{R''_\varepsilon + |t|}^{\rho + |t|} W_\lambda^{10}(r) r^2 \, dr\right)^{\frac{1}{2}} dt \leq \varepsilon^5.$$

The same argument as in the preceding step, replacing R'_ε by R''_ε, yields that $(f_0, f_1) = 0$ for almost every $r > R''_\varepsilon$, which contradicts the definition of ρ. The proof is complete.

Bibliography

[1] Aubin, T. Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire. *J. Math. Pures Appl. (9) 55*, 3 (1976), 269–296.

[2] Bahouri, H., Chemin, J.-Y., and Danchin, R. *Fourier analysis and nonlinear partial differential equations*, vol. 343 of *Grundlehren Math. Wiss.* Berlin: Springer, 2011.

[3] Bahouri, H., and Gérard, P. High frequency approximation of solutions to critical nonlinear wave equations. *Amer. J. Math. 121*, 1 (1999), 131–175.

[4] Bahouri, H., and Shatah, J. Decay estimates for the critical semilinear wave equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire 15*, 6 (1998), 783–789.

[5] Bony, J.-M. *Cours d'analyse. Théorie des distributions et analyse de Fourier.* Palaiseau: Les Éditions de l'École Polytechnique, 2001.

[6] Bourgain, J. Global wellposedness of defocusing critical nonlinear Schrödinger equation in the radial case. *J. Amer. Math. Soc. 12*, 1 (1999), 145–171.

[7] Cazenave, T. *Semilinear Schrödinger equations*, vol. 10 of *Courant Lecture Notes in Mathematics.* New York University Courant Institute of Mathematical Sciences, New York, 2003.

[8] Chemin, J.-Y., and Xu, C. Sobolev imbeddings, Weyl-Hörmander calculus and subelliptic vector fields. *Ann. Sci. Éc. Norm. Supér. (4) 30*, 6 (1997), 719–751.

[9] Christ, M., and Kiselev, A. Maximal functions associated to filtrations. *J. Funct. Anal. 179*, 2 (2001), 409–425.

[10] Côte, R., Kenig, C. E., and Schlag, W. Energy partition for the linear radial wave equation. *Math. Ann. 358*, 3-4 (2014), 573–607.

[11] Ding, W. Y. On a conformally invariant elliptic equation on $\mathbf{R}^n$. *Comm. Math. Phys. 107*, 2 (1986), 331–335.

[12] Duyckaerts, T., Kenig, C., and Merle, F. Universality of the blow-up profile for small type II blow-up solutions of the energy-critical wave equation: the nonradial case. *J. Eur. Math. Soc. (JEMS) 14*, 5 (2012), 1389–1454.

[13] Duyckaerts, T., Kenig, C., and Merle, F. Classification of radial solutions of the focusing, energy-critical wave equation. *Camb. J. Math. 1*, 1 (2013), 75–144.

[14] Duyckaerts, T., and Negro, G. Global solutions with asymptotic self-similar behaviour for the cubic wave equation. *Commun. Math. Phys. 405*, 84 (2024).

[15] Folland, G. B. *Introduction to partial differential equations.*, 2nd ed. ed. Princeton, NJ: Princeton University Press, 1995.

[16] Gérard, P. Description du défaut de compacité de l'injection de sobolev. *ESAIM Control Optim. Calc. Var. 3* (1998), 213–233.

[17] Ginibre, J., Soffer, A., and Velo, G. The global Cauchy problem for the critical nonlinear wave equation. *J. Funct. Anal. 110*, 1 (1992), 96–130.

[18] Ginibre, J., and Velo, G. Generalized Strichartz inequalities for the wave equation. *J. Funct. Anal. 133*, 1 (1995), 50–68.

[19] Grafakos, L. *Modern Fourier analysis*, 2nd ed. ed., vol. 250 of *Grad. Texts Math.* New York, NY: Springer, 2009.

[20] Hörmander, L. *The analysis of linear partial differential operators. I: Distribution theory and Fourier analysis.*, reprint of the 2nd edition 1990 ed. Class. Math. Berlin: Springer, 2003.

[21] John, F. Blow-up of solutions of nonlinear wave equations in three space dimensions. *Proc. Nat. Acad. Sci. U.S.A. 76*, 4 (1979), 1559–1560.

[22] Keel, M., and Tao, T. Endpoint Strichartz estimates. *Amer. J. Math. 120*, 5 (1998), 955–980.

[23] Kenig, C. E., and Merle, F. Global well-posedness, scattering and blow-up for the energy-critical focusing non-linear wave equation. *Acta Math. 201*, 2 (2008), 147–212.

[24] Keraani, S. On the blow up phenomenon of the critical nonlinear Schrödinger equation. *J. Funct. Anal. 235*, 1 (2006), 171–192.

[25] Krieger, J., Schlag, W., and Tataru, D. Slow blow-up solutions for the $H^1(\mathbb{R}^3)$ critical focusing semilinear wave equation. *Duke Math. J. 147*, 1 (2009), 1–53.

[26] Levine, H. A. Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = -Au + \mathcal{F}(u)$. *Trans. Amer. Math. Soc. 192* (1974), 1–21.

[27] Martel, Y., and Merle, F. A Liouville theorem for the critical generalized Korteweg-de Vries equation. *J. Math. Pures Appl. (9) 79*, 4 (2000), 339–425.

[28] Payne, L. E., and Sattinger, D. H. Saddle points and instability of nonlinear hyperbolic equations. *Israel J. of Math 22*, 3-4 (1975), 273–303.

[29] Schwartz, L. Théorie des distributions. Nouv. éd., entièrement corr., ref. + augm. (Nouv. tirage). Paris: Hermann. 436 p. (1984)., 1984.

[30] Sogge, C. D. *Lectures on nonlinear wave equations.* Monographs in Analysis, II. International Press, Boston, MA, 1995.

[31] Stein, E. M., and Shakarchi, R. *Functional analysis. Introduction to further topics in analysis*, vol. 4 of *Princeton Lect. Anal.* Princeton, NJ: Princeton University Press, 2011.

[32] Strichartz, R. S. Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations. *Duke Math. J. 44*, 3 (1977), 705–714.

[33] Talenti, G. Best constant in Sobolev inequality. *Ann. Mat. Pura Appl. (4) 110* (1976), 353–372.

[34] Tao, T. *Nonlinear dispersive equations*, vol. 106 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2006. Local and global analysis.

[35] Tao, T., Visan, M., and Zhang, X. Minimal-mass blowup solutions of the mass-critical NLS. *Forum Math. 20*, 5 (2008), 881–919.

[36] Trudinger, N. S. Remarks concerning the conformal deformation of Riemannian structures on compact manifolds. *Ann. Scuola Norm. Sup. Pisa (3) 22* (1968), 265–274.