DYNAMICS OF SEMI-LINEAR WAVE EQUATIONS 2026. HOMEWORKS, CHAPTER I

In this exercise sheet, we consider the linear wave equation:

$$\partial_t^2 u - \Delta u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{LW}$$

with initial data:

$$\vec{u}_{\restriction t=0} = (u_0, u_1). \tag{ID}$$

Exercise 1. Let $f : \mathbb{R}^N \to \mathbb{R}$ $(N \geq 1)$. Suppose f is radial (i.e. That it depends only on the variable $r = |x| = \sqrt{x_1^2 + x_2^2 + \ldots + x_N^2}$). Denote $f(x) = g(|x|)$, where $g : [0, \infty[\to \mathbb{R}$.

(1) Show that f is continuous on $\mathbb{R}^N$ if and only if g is continuous on $[0, \infty[$.
(2) Show that f is C^1 on $\mathbb{R}^N$ if and only if g is C^1 on $[0, \infty[$ and $g'(0) = 0$.
(3) Show that for any $k \geq 2$, f is C^k on $\mathbb{R}^N$ if and only if g is C^k on $\mathbb{R}^N$ and $g^{(j)}(0) = 0$ for all odd integers $j \leq k$.
(4) Assuming f is C^1, compute $\frac{\partial f}{\partial x_j}$ in terms of g', $j = 1, \ldots, N$. Compute $g'(r)$ in terms of ∇f.
(5) Assuming f is C^2 on $\mathbb{R}^N$, prove the formula

$$\Delta f(x) = g''(|x|) + \frac{N-1}{|x|} g'(|x|).$$

To lighten notation, we use the same notation (f) for functions f and g, and denote $g' = \frac{df}{dr}$, etc...

Exercise 2 (Loss of regularity for the radial wave equation in dimension $1+3$). Let $k \geq 0$ and $f \in C^k(\mathbb{R}^3)$ be a *radial* function. Define a function u on $\mathbb{R} \times (\mathbb{R}^3 \setminus \{0\})$, radial with respect to the space variable, by

$$u(t,x) = \frac{1}{2r}\Big((r+t) f(|r+t|) + (r-t) f(|r-t|)\Big),$$

where $r = |x|$. Note that this defines a function of class C^k on $\mathbb{R} \times \big(\mathbb{R}^3 \setminus \{0\}\big)$.

(1) Suppose that f is supported in the annulus $\{\frac{1}{2} \leq |x| \leq 2\}$ and is such that for $|\eta - 1| \leq 1/10$,

$$f(\eta) = \begin{cases} 2 - \eta & \text{if } \eta > 1 \\ \eta & \text{if } \eta < 1 \end{cases}.$$

Calculate $\lim_{r \to 0} u(t,r)$ when $t = 1$, $t > 1$, and $t < 1$ (close to 1). Conclude that u cannot be extended to a continuous function on $\mathbb{R} \times \mathbb{R}^3$.
(2) Similarly, give an example of a C^2 function f such that u cannot be extended to a C^2 function on $\mathbb{R} \times \mathbb{R}^3$.
(3) Assume f is C^3. Show that u defines a C^2 function on $\mathbb{R} \times \mathbb{R}^3$.
(4) Let g be a C^2 radial function on $\mathbb{R}^3$. Show that

$$u(t,r) = \frac{1}{2r} \int_{r-t}^{r+t} \sigma g(|\sigma|) d\sigma,$$

extends to a C^2 function on $\mathbb{R}^3$.

Exercise 3 (Explicit solutions of the radial wave equation in odd space dimension). Let $N \geq 3$ be an odd integer, written as $N = 2k+1$. Let T_k be the operator defined by

$$T_k\phi = \left(r^{-1}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\phi(r)\right).$$

(1) Show that

$$T_k\varphi = \sum_{j=0}^{k-1} c_j r^{j+1}\phi^{(j)}r,$$

for some $c_j \in \mathbb{R}$. Determine c_0 and c_{k-1}.

(2) Show that for any function $\varphi \in C^{k+1}([0,+\infty[)$,

$$\frac{d^2}{dr^2}(T_k\varphi) = \left(r^{-1}\frac{d}{dr}\right)^k (r^{2k}\varphi'(r)).$$

Hint: You can start by verifying that the formula is true when $\varphi(r) = r^m$ for any integer m.

(3) Consider a solution $u(t,x)$ of the linear wave equation in space dimension N, radial with respect to the space variable. Suppose u is C^{k+1} on $\mathbb{R}^{1+N}$. Show prove

$$(\partial_t^2 - \partial_r^2)(T_k u) = 0.$$

Deduce an expression of $T_k u$ in terms of u_0 and u_1.

(4) Express $u(t,r)$ in terms of u_0 and u_1 when $N = 5$. What regularity of u_0 and u_1 is required for u to be C^2 on $\mathbb{R}^{1+5}$?

Exercise 4. Let u be a solution of the wave equation (LW) in spce dimension $N \geq 3$, radial with respect to the space variable. Recall that $\Delta u = \frac{d^2}{dr^2} + \frac{N-1}{r}\frac{d}{dr}$. Suppose $u \in C^2(\mathbb{R}^{1+N})$, with compactly supported initial data. Let

$$v(t,r) = \int_r^\infty \rho\partial_t u(t,\rho)d\rho.$$

Show that v defines a radial solution, of class C^2, to the wave equation in space dimension $N-2$.

Exercise 5 (Conservation of momentum). (1) Let u be a C^2 solution of (LW) on $\mathbb{R} \times \mathbb{R}^N$, and $j \in 1, \ldots N$. Let $p_{j,u}(t,x) = \partial_{x_j} u(t,x)\partial_t u(t,x)$. Show

$$\frac{\partial p_{j,u}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x_j}\left((\partial_t u)^2 - |\nabla u|^2\right) + \nabla \cdot V,$$

where V is a C^1 vector field to be specified.

(2) Assume that (u_0, u_1) has finite energy. Justify that

$$P_j(\vec{u}(t)) = \int_{\mathbb{R}^N} p_{j,u}(t,x)dx$$

is defined for all times. Show that this quantity is independent of time. You can start by considering a local version of the momentum

$$\int_{[-R,R]^N} p_{j,u}(t,x)dx \text{ or } \int_{\mathbb{R}^N} p_{j,u}(t,x)\varphi\left(\frac{x}{R}\right)dx$$

then let R tend to $+\infty$. Here φ denotes a C^2 function with compact support equal to 1 in a neighborhood of the origin.

Exercise 6 (Positivity properties in low dimensions).

(1) Let $u_1 \in C^2(\mathbb{R}^3)$ such that
$$\forall t \geq 0,\ \forall x \in \mathbb{R}^3, \quad u_1(x) \geq 0.$$
Assume $u_0 = 0$. Let u be the corresponding solution of (LW). Prove
$$\forall t \geq 0,\ \forall x \in \mathbb{R}^3, \quad u(t,x) \geq 0.$$

(2) Suppose now $N = 1$ or $N = 2$. Let u be the solution of (LW), (ID), with $(u_0, u_1) \in C^3 \times C^2$ (if $N = 2$) or $C^2 \times C^1$ (if $N = 1$).
Show that if $u_1 \geq 0$ and $u_0 = 0$ then $u(t,x)$ has the sign of t for all x and $t \neq 0$.
When $N = 1$, give a weaker sufficient condition on (u_0, u_1) such that:
$$\forall t \geq 0,\ \forall x \in \mathbb{R}, \quad u(t,x) \geq 0.$$

Exercise 7. Assume $N = 1$ or $N = 2$. Let u be a solution of
$$\partial_t^2 u - \Delta u = f,$$
with $(u, \partial_t u) = 0$ at $t = 0$, and f of class C^1 (if $N = 1$) or C^2 (if $N = 2$). Express u in terms of f.

Exercise 8. The *Minkowski spacetime* of dimension N is the space $\mathbb{R}^{1+N}$, equipped with the quadratic form of signature $(1, N)$:
$$g(X) = x_0^2 - \sum_{j=1}^{N} x_j^2 = t^2 - |x|^2 = {}^tXJX,$$
where ${}^tX$ is the transpose of X,
$$X = (x_0, x_1, \ldots, x_N),\ t = x_0,\ x = (x_1, \ldots, x_N),$$
and $J = [J_{\mu,\nu}]_{0 \leq \mu\ \nu \leq N}$ is the matrix such that $J_{0,0} = 1$, $J_{\ell,\ell} = -1$ if $\ell \in 1, \ldots, N$, and $J_{\mu,\nu} = 0$ if $\mu \neq \nu$.

The Lorentz group $\mathrm{O}(1, N)$ is the group of real square matrices P of size $1 + N$ which leave the quadratic form g invariant, i.e., such that $g(PX) = g(X)$ for all X in $\mathbb{R}^{1+N}$. In other words, if P is a $(1 + N) \times (1 + N)$ matrix,
$$P \in \mathrm{O}(1, N) \iff {}^tPJP = J.$$

(1) Prove that a function v of class C^2 on $\mathbb{R}^{1+N}$ satisfies the wave equation (LW) if and only if $\mathrm{Tr}(Jv'') = 0$, where v'' is the Hessian matrix $\left[\partial_{x_\mu} \partial_{x_\nu} v\right]_{0 \leq {\mu \atop \nu} \leq N}$.

(2) Let $P \in \mathrm{O}(1, N)$, $v \in C^2(\mathbb{R}^{1+N})$, and $w(X) = v(PX)$. Then
$$(\partial_t^2 - \Delta)v = 0 \iff (\partial_t^2 - \Delta)w = 0.$$

(3) Prove that the space rotations:
$$\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & R \end{bmatrix}, \quad R \in \mathrm{O}(N)$$
and the Lorentz boosts
$$\mathcal{R}\sigma = \begin{bmatrix} R_\sigma & \mathbf{0} \\ \mathbf{0} & I_{N-1}. \end{bmatrix}, \quad R_\sigma = \begin{bmatrix} \cosh(\sigma) & \sinh(\sigma) \\ \sinh(\sigma) & \cosh(\sigma) \end{bmatrix},$$
where I_{N-1} denotes the identity matrix $(N-1) \times (N-1)$ and $\sigma \in \mathbb{R}$ are Lorentz transformations. In these formulas, $\mathbf{0}$ always denotes the zero matrix of appropriate size.