DYNAMICS OF SEMI-LINEAR WAVE EQUATIONS 2026. HOMEWORK, CHAPTER II

Exercise 1. Let $\sigma \in \mathbb{R}$. Let $N \geq 1$, $\chi \in C^\infty(\mathbb{R}^N)$ such that $0 \leq \chi \leq 1$ and

$$\chi(x) = \begin{cases} 0 & \text{if } |x| \leq 1/2 \text{ or } |x| \geq 3 \\ 1 & \text{if } 1 \leq |x| \leq 2 \end{cases}.$$

Let $\varphi_\varepsilon \in \mathcal{S}(\mathbb{R}^N)$ such that $\widehat{\varphi_\varepsilon}(\xi) = |\xi|^{-\sigma-N}\chi\left(\frac{\xi}{\varepsilon}\right)$. Let $\|\cdot\|_{\dot{B}^\sigma}$ be the Besov-type norm defined in Chapter II of the course to state the "improved Sobolev inequality".

1. Prove that $\lim_{\varepsilon\to 0} \|\varphi_\varepsilon\|_{\dot{H}^\sigma} = +\infty$ and $\|\varphi_\varepsilon\|_{\dot{B}^\sigma} \leq C$, where C is a constant which is independent of ε.
2. Assume $\sigma < \frac{N}{2}$ and let p be defined by $\frac{N}{2} - \sigma = \frac{N}{p}$. Construct a family of function $(\psi_\varepsilon)_{\varepsilon>0}$ with
$$\|\psi_\varepsilon\|_{\dot{H}^\sigma} = 1, \quad \lim_{\varepsilon\to 0} \|\psi_\varepsilon\|_{L^p} = 0,$$

Exercise 2. Let $\sigma \in \mathbb{R}$. We define the (inhomogeneous) Sobolev space $H^\sigma(\mathbb{R}^N)$ by the space of $u \in \mathcal{S}'(\mathbb{R}^N)$ such that $\hat{u}$ is in $L^1(K)$ for all compact K and that the following quantity is finite

$$\|f\|_{H^\sigma}^2 = \int_{\mathbb{R}^N} (1+|\xi|^2)^\sigma |\widehat{f}(\xi)|^2 d\xi.$$

1. Prove that H^σ is an Hilbert space.
2. Assume that σ is a positive integer. Prove that f is in H^σ is and only if for all $\alpha \in \mathbb{N}^N$ with $|\alpha| \leq \sigma$, $\partial_x^\alpha f \in L^2(\mathbb{R}^N)$. Prove that the norm on H^σ is equivalent to the norm defined by
$$\|f\|^2 = \sum_{|\alpha|\leq\sigma} \|\partial_x^\alpha f\|_{L^2}^2.$$
In the case where σ is even, prove that this norm is also equivalent to the norm defined by
$$\|f\|^2 = \|f\|_{L^2}^2 + \|\Delta^{\sigma/2} f\|_{L^2}^2.$$
3. Prove that for all $(u_0, u_1) \in H^\sigma \times H^{\sigma-1}$ the formula
$$u(t) = \cos(t|D|)u_0 + \frac{\sin(t|D|)}{|D|}u_1 \tag{1}$$
defines $u \in C^0(\mathbb{R}, H^\sigma)$ such that $u \in C^k(\mathbb{R}, H^{\sigma-k})$ for all $k \in \mathbb{N}$, and that u satisfies the linear wave equation in $H^{\sigma-k}$ and in the sense of distributions on $\mathbb{R}^{N+1}$.

Exercise 3. Let $u_1 \in L^2(\mathbb{R}^N)$ and consider $u(t) = \frac{\sin(t|D|)}{|D|}u_1$ the solution of the wave equation with initial data $(0, u_1)$.

1. Prove that for all t, $u(t) \in L^2(\mathbb{R}^N)$, and $\|u(t)\|_{L^2} \leq t\|u_1\|_{L^2}$.
2. Give a sufficient condition on u_1 so that $\sup_{t\in\mathbb{R}} \|u(t)\|_{L^2}$ is finite.
3. Give examples of functions $u_1 \in L^2$ such that this quantity is not finite.

Exercise 4 (Conservation laws). Let $(u_0, u_1) \in \mathcal{S}'(\mathbb{R}^N) \times \mathcal{S}'(\mathbb{R}^N)$. Assume that $\widehat{u_0}$ and $\widehat{u_1}$ are locally integrable functions. Let

$$u(t,x) = \cos(t|D|)u_0 + \frac{\sin(t|D|)}{|D|}u_1.$$

Let $\varphi \in C^\infty(\mathbb{R}^N)$ such that $\left(|\widehat{u_0}(\xi)|^2|\xi|^2 + |\widehat{u_1}(\xi)|^2\right)\varphi(\xi)$ is integrable on $\mathbb{R}^N$. Prove that

$$\int \left[|\xi|^2 \left|\widehat{u}(t,\xi)\right|^2 + \left|\widehat{\partial_t u}(\xi)\right|^2\right] \varphi(\xi) d\xi$$

is well-defined and independent of t.

Exercise 5. Consider the fonction defined on $\mathbb{R}^3$ by

$$S(x) = \frac{\sin(|x|)}{|x|}. \tag{2}$$

Compute the Fourier transform of S. Hint: compare the two formulas for the wave equation in space dimension 3 obtained in Chapter I and Chapter II.