DYNAMICS OF SEMI-LINEAR WAVE EQUATIONS. 2026. HOMEWORK OF CHAPTER III

Exercise 1. Let $u_0 \in (\dot{H}^{1/2} \times \dot{H}^{-1/2})(\mathbb{R}^3)$, $f \in L^{4/3}(\mathbb{R} \times \mathbb{R}^3)$. Let

$$u(t) = \cos(t|D|)u_0 + \frac{\sin(t|D|)}{|D|}u_1 + \int_0^t \frac{\sin((t-s)|D|)}{|D|} f(s)ds.$$

Using the dispersion inequality

$$\left\| \frac{e^{it|D|}}{|D|}\varphi \right\|_{L^4(\mathbb{R}^3)} \lesssim \frac{1}{t^{1/2}} \|\varphi\|_{L^{4/3}(\mathbb{R}^3)},$$

and the general strategy of the proof of Strichartz estimates of Chapter III prove:

1. $u \in L^4(\mathbb{R} \times \mathbb{R}^3)$ and
$$\|u\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|f\|_{L^{4/3}(\mathbb{R}\times\mathbb{R}^3)} + \|(u_0,u_1)\|_{\dot{H}^{1/2}\times\dot{H}^{-1/2}}.$$
2. $(u, \partial_t u) \in C^0(\mathbb{R}, \dot{H}^{1/2} \times \dot{H}^{-1/2})$ and
$$\sup_t \|(u,\partial_t u)\|_{\dot{H}^{1/2}\times\dot{H}^{-1/2}} \lesssim \|f\|_{L^{4/3}(\mathbb{R}\times\mathbb{R}^3)} + \|(u_0,u_1)\|_{\dot{H}^{1/2}\times\dot{H}^{-1/2}}.$$

Exercise 2. For $u_0 \in \mathcal{S}'(\mathbb{R}^N)$, $N \geq 1$ we denote $u(t) = e^{it\Delta}u_0$ the element of $C^0(\mathbb{R}, \mathcal{S}'(\mathbb{R}))$ defined by

$$u(t) = \overline{\mathcal{F}}\left(e^{-it|\xi|^2}\widehat{u_0}(\xi)\right),$$

which is (formally at least) the solution of the linear Schrödinger equation

$$i\partial_t u + \Delta u = 0.$$

One can show, by explicit calculation, the dispersion inequality:

$$\left\|e^{it\Delta}u_0\right\|_{L^\infty(\mathbb{R}^N)} \lesssim \frac{1}{|t|^{N/2}}\|u_0\|_{L^1(\mathbb{R}^N)}.$$

Let $(p,q) \in [2,\infty]^2$, with $p > 2$ and $\frac{2}{p} + \frac{N}{q} = \frac{N}{2}$. Show

$$\left\|e^{it\Delta}u_0\right\|_{L^p(\mathbb{R},L^q(\mathbb{R}^N))} \lesssim \|u_0\|_{L^2(\mathbb{R}^N)}.$$

Exercise 3. We fix $N \geq 2$. An element x of $\mathbb{R}^N$ is denoted by (x_1, x') with $x_1 \in \mathbb{R}$ and $x' = (x_2, \ldots, x_N) \in \mathbb{R}^{N-1}$. Let $\varepsilon > 0$ be a small parameter and denote by

$$S_\varepsilon = \{\xi \in \mathbb{R}^N \;:\; 1 \leq \xi_1 \leq 2,\ |\xi'| \leq \varepsilon\}, \quad u_\varepsilon(t,x) = \int_{S_\varepsilon} e^{ix\cdot\xi + it|\xi|}d\xi.$$

1. Prove that u_ε is a solution to the wave equation, and that for all σ,
$$\|u_\varepsilon(0)\|_{\dot{H}^\sigma} \approx \|\partial_t u_\varepsilon(0)\|_{\dot{H}^\sigma} \approx \varepsilon^{\frac{d-1}{2}},$$
uniformly with respect to small $\varepsilon > 0$.
2. Prove that on S_ε, $|\xi| = \xi_1 + O(\varepsilon^2)$.
3. Prove that if $c > 0$ is small enough, then for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$ with
$$-c \leq t + x_1 \leq c, \quad |t| \leq c\varepsilon^{-2}, \quad |x'| \leq c\varepsilon^{-1},$$
then $\operatorname{Re} e^{ix\dot{\xi}+it|\xi'|} \geq \frac{1}{2}$. Deduce a lower bound of $\|u_\varepsilon\|_{L^p(\mathbb{R},L^q(\mathbb{R}^N))}$ by a power of ε.
4. Prove that if the Strichartz inequality
$$\|u\|_{L^p(\mathbb{R},L^q(\mathbb{R}^N))} \lesssim \|(u_0,u_1)\|_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}}$$
holds for all solution of the wave equation $\partial_t^2 u - \Delta u = 0$ with initial data (u_0,u_1), then
$$\frac{2}{p} + \frac{d-1}{q} \leq \frac{d-1}{2}.$$
This counterexample to a potential Strichartz inequality is known as Knapp's example.