DYNAMICS OF SEMI-LINEAR WAVE EQUATIONS 2025
EXERCISES, CHAPTER 4

In this worksheet, we consider the equations:

$$(\partial_t^2 - \Delta)u = \sigma u^5. \tag{W5}$$

Exercise 1. Assume $\sigma = 1$. Let u be a solution of (W5) such that $u_0 = 0$ and $u_1 \in L^2 \cap C^2$ is nonnegative. Prove that $u(t,x) \geq 0$ for all x and for all $t \geq 0$ in the domain of existence of u. *Hints: Write u as the limit of a sequence given by Picard iteration. Use exercise 6 of Chapter 1, where it is shown that $\frac{\sin(t|D|)}{|D|} f \geq 0$ whenever $f \in C^2(\mathbb{R}^3)$ is such that $f \geq 0$.*

Exercise 2. (1) Let $R > 0$ and u, v be two solutions of (W5) defined on $[0,\infty[\times\mathbb{R}^3$. Assume that $\vec{u}(0,x) = \vec{v}(0,x)$ for $|x| > R$. Prove that $\vec{u}(t,x) = \vec{v}(t,x)$ for $t \geq 0$, $|x| > R + |t|$.

(2) Prove that there exists a small $\delta_1 > 0$ with the following property: if u is a solution of (W5) with initial data $\vec{u}_0 = (u_0, u_1) \in \dot{\mathcal{H}}^1$ at $t = 0$ with maximal time of existence $T_+ = +\infty$ and such that

$$\int_{|x|>R} |\nabla u_0|^2 + (u_1)^2 dx = \delta^2 \leq \delta_1^2,$$

Then

$$\int_{|x|>R+|t|} |\nabla u(t)|^2 + (\partial_t u(t))^2 dx \leq 2\delta^2 \tag{1}$$

Exercise 3. Check that the definition of finite energy solutions in the course does not depend on the choice of the initial time. In other words, if u is a solution of (W5) on an interval I such that $t_0 \in I$, with initial data $\vec{u}(t_0) = (u_0, u_1) \in \dot{\mathcal{H}}^1$, and $t_1 \in I$, then

$$\forall t \in I, \quad u(t) = \cos\big((t-t_1)|D|\big)u(t_1) + \frac{\sin\big((t-t_1)|D|\big)}{|D|}\partial_t u(t_1) + \int_{t_1}^{t} \frac{\sin\big((t-s)|D|\big)}{|D|} u^5(s) ds.$$

Exercise 4. Let $R > 0$, and $\Gamma = \Big\{(t,x) \in \mathbb{R} \times \mathbb{R}^N \ : \ t_0 \leq t \leq t_1,\ |x| \leq R - |t|\Big\}$ Let $u, v \in C^2(\Gamma)$ be two classical solutions of (W5) on Γ. We suppose $(u, \partial_t u)(0,x) = (v, \partial_t v)(0,x)$ for all x with $|x| < R$. We want to prove $u(t,x) = v(t,x)$ for all $(t,x) \in \Gamma$. Let

$$V(t) = \frac{1}{2}\int_{|x|<R-t} (u(t,x) - v(t,x))^2 dx + \frac{1}{2}\int_{|x|<R-t} (\partial_t u(t,x) - \partial_t v(t,x))^2 dx + \frac{1}{2}\sum_{j=1}^{3} \int_{|x|<R-t} (\partial_{x_j} u(t,x) - \partial_{x_j} v(t,x))^2 dx.$$

(1) Prove that $V'(t) \leq CV(t)$ for $t \in [0, t_1]$.
(2) Prove that $V(t) = 0$ for all $t \in [0, t_1]$

Exercise 5. Assume $\sigma = 1$.

(1) Prove that there exists a solution of the ODE: $Y'' = Y^5$ defined on $]-1,+1[$ and such that

$$\lim_{t \to \pm 1} |Y(t)| = +\infty$$

(2) Prove that for all a, b with $a < b$, there exists a solution of (W5) with maximal interval of existence (a, b). *Remark: this is an extension from the construction of the course where we constructed a solution that blows up in finite time in the future (but not necessarily in the past).*