

DYNAMICS OF SEMI-LINEAR WAVE EQUATIONS. HOMEWORK, CHAPTER 5

Exercise 1. Let $s \geq 2$ be an integer, prove that $\dot{H}^s \cap \dot{H}^1$ is an algebra.

In the next exercises, we consider the equations:

$$(W5) \quad (\partial_t^2 - \Delta)u = \sigma u^5.$$

and

$$(LW) \quad (\partial_t^2 - \Delta)u_L = 0.$$

in space dimension 3. A “solution” will always mean a finite-energy solution which satisfies the equation in the usual Duhamel sense.

Exercise 2. Let $L \in \mathbb{R}$ and u and v be two global solutions of (W5) such that

$$\vec{u}(0, x) = \vec{v}(0, x) \text{ for all } x = (x_1, x_2, x_3) \text{ with } x_1 > L.$$

Prove that

$$\vec{u}(t, x) = \vec{v}(t, x) \text{ for all } x = (x_1, x_2, x_3) \text{ with } x_1 > L + |t|.$$

Exercise 3. (1) Let u_L be a solution of (LW). Prove that

$$\lim_{t \rightarrow \infty} \|u_L(t)\|_{L^6} = 0.$$

Hint: start with the case where $\vec{u}_L(0)$ is smooth and compactly supported.

- (2) Let f be a measurable function defined on $I \times \mathbb{R}^3$. Assume that the $L^\infty(I, L^6)$ and $L^4(I, L^{12})$ norms of f are finite. Prove that $f \in L^5(I, L^{10})$ and give a bound of the norm of f in this space in terms of the 2 preceding norms.
- (3) Prove that there exists $\varepsilon > 0$ with the following property. For any solution u of (W5) with $T_+(u) = +\infty$ and

$$\limsup_{t \rightarrow \infty} \|u(t)\|_{L^6} \leq \varepsilon.$$

the solution u scatters to a linear solution.

Exercise 4 (Lorentz transformation). (1) Let u be a solution of (LW) (respectively (W5)). Prove that the momentum of u ,

$$P(\vec{u})(t) = \int \nabla u(t, x) \partial_t u(t, x) dx,$$

is independent of t . What is the momentum of u when the initial data \vec{u}_0 of u is radial?

- (2) Assume that \vec{u}_0 is smooth and compactly supported. Let $c > 0$ and

$$u_c(t, x) = u \left(\frac{t - cx_1}{\sqrt{1 - c^2}}, \frac{x_1 - ct}{\sqrt{1 - c^2}}, x_2, x_3 \right).$$

Prove that u_c is a solution of (LW) (respectively (W5)). Compute the momentum and energy of u_c in terms of c , the momentum and the energy of u . The term “Energy” has to be interpreted as the conserved energy for the equation (LW) or (W5) satisfied by u .

Exercise 5. Let Q be a solution of $-\Delta Q = Q^5$ with $Q \in \dot{H}^1(\mathbb{R}^3)$. Let $F(x) := \mathcal{K}(Q)(x) = \frac{1}{|x|} Q(\frac{x}{|x|^2})$ (Kelvin transform of Q).

- (1) Prove that $F \in \dot{H}^1$ and that $-\Delta F = F^5$.
- (2) Compute $\mathcal{K}(\mathcal{K}Q)$.
- (3) Compute $\mathcal{K}W$, where $W = (1 + |x|^2/3)^{-1/2}$.
- (4) Assume that $Q(0) \neq 0$. Give the asymptotic behaviour of $F(x)$ as $|x| \rightarrow \infty$.