Homework, to be handed back the day of the exam

**Exercice 1** (Whitehead Theorem for model categories)**.** The goal is to prove that in a model category $C$, *if* $X$, $Y$ *are both fibrant and cofibrant objects, then a map* $f : X \to Y$ *is a weak equivalence if and only if it is an homotopy equivalence.*

1. Let$f \overset{l}{\sim} g$ be left homotopic. Show that $f$ is a weak equivalence if and only if $g$ is a weak equivalence.
2. Let $i : X \overset{\sim}{\hookrightarrow} C$ be an acyclic cofibration where $X$ is both fibrant and cofibrant and $C$ is fibrant. Prove that there is a retraction $r$ of $i$ and then show that $r$ is an homotopy inverse of $i$.
3. Deduce from the previous question that a weak equivalence between fibrant and cofibrant objects is an homotopy equivalence.
4. Let $f : X \to Y$ be an homotopy equivalence between fibrant and cofibrant objects, and let $f : X \underset{\sim}{\overset{i}{\hookrightarrow}} C \overset{p}{\twoheadrightarrow} Y$ be a factorization where the first map is an acyclic cofibration.
   (a) Prove that $C$ is both fibrant and cofibrant and that if $g$ is an homotopy inverse of $f$, with left homotopy $H : C' \to Y$ between $id_Y$ and $f \circ g$, there is a lift $H' : C' \to C$ such that $p \circ H' = H$ and $H' \circ i_0 = i \circ g$.
   (b) Deduce that $H' \circ i_1 \circ p$ is homotopical to $id_C$ (one can note that $i$ has an homotopy inverse) and then that it is a weak equivalence.
   (c) Prove that $p$ is a retract of a weak equivalence and then conclude.

**Exercice 2.** Let $G$ a finite group and $\mathbf{Top}^G$ be the category of topological spaces endowed with a continuous $G$-action with morphisms given by continuous $G$-equivariant map: for all $g \in G$, $x \in X$, $f(g \cdot x) = g \cdot f(x)$. Denote $BG$ the category with a single object and with arrows given by elements of $G$ with multiplication for composition.

1. Prove that the category of functors $\mathrm{Fun}(BG, \mathbf{Top})$ is equivalent to $\mathbf{Top}^G$ and deduce a model structure on $\mathbf{Top}^G$ (which will be called the projective model structure).
2. Prove that the forget functor $\mathbf{Top}^G \hookrightarrow \mathbf{Top}$ is right Quillen for the projective structure
3. Prove that the fixed points $(-)^G : \mathbf{Top}^G \to \mathbf{Top}$ and orbits fucntors $(-)_G : \mathbf{Top}^G \to \mathbf{Top}$, defined by $X^G = \{x \in X, \forall g \in G, g \cdot x = x\}$ and $X_G = X/x \sim g \cdot x$ have left and right adjoints respectively. Identify their adjoints.
4. Deduce that there exist right and left total derived functor, denoted $(-)^{hG}$ and $(-)_{hG}$ and interpret them as homotopys (co)limits.
5. Let $EG$ be cell complex with a free and cellular continuous $G$-action. Show that $E$ is cofibrant in $\mathrm{Fun}(BG, \mathbf{Top})$ for the projective structure.
6. Deduce a formula for $(-)_{hG}$ and then that $(\{*\}_{\mathbb{Z}/2\mathbb{Z}} \cong \mathbb{R}P^\infty$ (indic: consider $X \times_G Y = X \times Y/(g \cdot x, y) \sim (x, g \cdot y)$).
7. Find a formula for $(-)^{hG}$ in terms of $EG$ and compute $(\{*\})^{hG}$.
8. Prove that $E\mathbb{Z} \simeq \mathbb{R}$.

9. or $X$ be a compact metric space with trivial $\mathbb{Z}$-action, describe $(X)^{h\mathbb{Z}}$. Deduce that $(S^1)^{h\mathbb{Z}}$ (where the action is trivial) has infinitely many pathwise connected components.

**Exercice 3.** Let $Ch(R)$ be endowed with the projective model structure.

1. Prove that the functor $(A \xrightarrow{f} B) \mapsto \mathrm{coker}(f)$ has a total left derived functor $\mathbb{L}\mathrm{coker}$. Compute, for all module map $M \xrightarrow{g} N$ the homology groups of $\mathbb{L}\mathrm{coker}(g)$. (Indic: one can rewrite coker as a colimit).
2. Compute the homotopy pullback $\{0\} \times^h_{X_*} \{0\}$ and its homology groups for $X_*$ a chain complex

**Exercice 4** (Eilenberg Mac Lane spaces)**.** We equip **sSet** with its usual model structure and $Ch_{\geq 0}(R)$ with its projective model structure. For a simplicial set $X_*$, let $D_n(X_*)$ be the subset of degenerate[1] simplices of $X_n$. We define the normailzed chain complex $N_*(X_*)$ given in degree $n$ as the quotient

$$N_n(X_*) = \Big(\bigoplus_{X_n} \mathbb{R}\Big) / \bigoplus_{D_n(X_*)} R$$

with differential given by $d = \sum_{i=0}^{n} (-1)^i d_i$ where $d_i$ are the face maps.

1. Check that $N_*(X_*)$ is indeed a chain complex. **We admit[2] in what follows that this functor has a right adjoint** $B : Ch_{\geq 0}(R) \longrightarrow$ **sSet**.
2. Prove that the adjunction $N_* :$ **sSet** $\rightleftarrows Ch_{\geq 0}(R) \ : B$ is a Quillen adjunction.
3. For $C$ a chain complex, we call $K(C) := |B(C)|$ its (generalized) Eilenberg-Mac Lane space. Prove that there is a natural isomorphism of topological spaces
$$\mathrm{Hom}_{\mathbf{HoTop}}(X, K(S^n(R))) \ \cong \ H^n(X, R)$$
where $S^n(M)$ is the complex given by $M$ in degree $n$ and $0$ else.(Hint: use the relationship in between left homotopy and chain homotopy when appropriate cofibrancy and fibrancy are satisfied).
4. Prove that we have a natural isomorphism of chain complexes
$$\mathrm{hom}_{\mathbf{Ho}Ch_{\geq 0}(R)}(S^n(R), C) \ \cong \ H_n(C).$$
5. Prove that $N_*(\Delta^n/\partial\Delta^n) \cong S^n(R)$.
6. Deduce that for any chain complex $C$, the homotopy groups of the Eilenberg mac Lane spaces are given by
$$\pi_n(K(C)) \cong H_n(C).$$
7. **bonus : link with exercice 2**: Let $G$ be abelian group (seen as a discrete topological group) and take $R = \mathbb{Z}$.
   (a) Prove that $K(S^0(\mathbb{Z})) \cong G$ and $K(D^1(\mathbb{Z}))$ is contractible.
   (b) *we admit that there is are weak-equivalences* $B(C) \times B(D) \xrightarrow{\sim} B(C \otimes D)$. Deduce that $B(D^1(G)))$ is homotopy equivalent to $EG$ and that $K(S^1(G))$ is a model for the classifying space $BG$ of $G$ of exercice 2.

[1] that is those in the image of a degeneracy

[2] this follows from the Dold-Kan correspondance and forget-free adjunction in between simplical $R$-module and **sSet**