Devoir Maison

Exercice 1 (Whitehead Theorem for model categories)**.** The goal is to prove that in a model category C, *if* X, Y *are both fibrant and cofibrant objects, then a map* $f : X \to Y$ *is a weak equivalence if and only if it is an homotopy equivalence.*

1. Let$f \overset{l}{\sim} g$ be left homotopic. Show that f is a weak equivalence if and only if g is a weak equivalence.
2. Let $i : X \overset{\sim}{\hookrightarrow} C$ be an acyclic cofibration where X is both fibrant and cofibrant and C is fibrant. Prove that there is a retraction r of i and then show that r is an homotopy inverse of i.
3. Deduce from the previous question that a weak equivalence between fibrant and cofibrant objects is an homotopy equivalence.
4. Let $f : X \to Y$ be an homotopy equivalence between fibrant and cofibrant objects, and let $f : X \underset{\sim}{\overset{i}{\hookrightarrow}} C \overset{p}{\twoheadrightarrow} Y$ be a factorization where the first map is an acyclic cofibration.
   (a) Prove that C is both fibrant and cofibrant and that if g is an homotopy inverse of f, with left homotopy $H : C' \to Y$ between id_Y and $f \circ g$, there is a lift $H' : C' \to C$ such that $p \circ H' = H$ and $H' \circ i_0 = i \circ g$.
   (b) Deduce that $H' \circ i_1 \circ p$ is homotopical to id_C (one can note that i has an homotopy inverse) and then that it is a weak equivalence.
   (c) Prove that p is a retract of a weak equivalence and then conclude.

Exercice 2. Let $P, Q \in Ch_{\geq 0}(R)$ and $f : P \to Q$ be a chain complex map which is surjective in degree $n \geq 1$ and is a quasi-isomorphism.

1. Show that f is also surjective in degree 0.
2. Let $P \to Q$ be a map from a chain complex $\psi : P \in Ch_{\geq 0}(R)$ such that all P_n are projective.
   (a) Suppose a lift (i.e. $f \circ \tilde{\psi} = \phi$) $\tilde{\psi}_{\leq n} : P_{\leq n} \to X_{\leq n}$ of ψ is given in degree $\leq n$, which is compatible with the differential. Prove that there exist a linear map $\tilde{\phi}_{n+1} : P_{n+1} \to X_{n+1}$ lifting ϕ_n and that for any $p \in P_n$, one has $d \circ \tilde{\psi}_{n+1}(p) - \tilde{\psi}(dp)$ is a boundary in the subcomplex $\mathrm{Ker}(f : X \to Y)$ (hint: prove the latter is acyclic).
   (b) Deduce that there exist a lifting in the diagram of chain complex

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ P \arrow[r, "\psi"'] \arrow[ur, dotted, "h"'] & Y \end{tikzcd}$$

3. Let $A \hookrightarrow B$ be a chain map in $CH_{\geq 0}(R)$, injective in all degrees, whose cokernel P is made of projective modules in every degree. We consider a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d] & X \arrow[d, "f"] \\ B \arrow[r, "\psi"'] & Y \end{tikzcd}$$

(a) Prove that there are splitting $B_n = A_n \oplus P_n$ such that the differential on the latter subcomplex is written as $d_B(a,p) = (d_A(a) + t(p), d_P(p))$ where d_A, d_B, d_P are the respective differentials of A, B, P.

(b) Let $\kappa : P_n \to X_n$ be a lift of $\psi_{n|P_n}$. Prove that $d \circ \kappa(p) - \kappa \circ d_p(p) - \phi \circ t(p)$ belongs to $\mathrm{Ker}(f : X \to Y)$.

(c) Deduce (using a proof similar to the first part of exercise) that one can build a lifting :

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d] & X \arrow[d, "f"] \\ B \arrow[r, "\psi"'] \arrow[ur, dotted, "h"] & Y \end{tikzcd} .$$

4. let P be a cofibrant chain complex with respect to the projective model structure and K be an acyclic chain complex. Prove that any chain map $f : P \to K$ is homotpic to the zero map.

Exercice 3 (The canonical model structure in Cat)**.** Let Cat denote the category of small categories with morphisms given by functors between them. Let $\mathcal{W}$ be the collection of functors which are equivalences of categories.

1. Show that the Gabriel-Zisman localization $\mathrm{Cat}[\mathcal{W}^{-1}]$ is equivalent to the category whose objects are small categories and morphisms are isomorphism classes of functors.

2. A functor $F : \mathcal{C} \to \mathcal{D}$ between small categories is said to be an isofibration if for every object $c \in \mathcal{C}$ and every isomorphism $f : F(c) \to d$ in $\mathcal{D}$, there exists an object $c' \in \mathcal{C}$ and an isomorphism $u : c \to c'$ such that $d = F(c')$ and $f = F(u)$. Show that an isofibration that is an equivalence of categories is surjective on objects. Conversely, show that if a functor F is fully faithful and surjective on objects then it is an isofibration.

3. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be a cofibration if it is injective on objects. Let Fib denote the collection of all isofibrations and Cof the class of cofibrations. Show that $(\mathrm{Cat}, \mathcal{W}, \mathrm{Fib}, \mathrm{Cof})$ is a model structure and identify its fibrant-cofibrant objects.