Corrigé du Devoir Maison

Exercice 1 (Whitehead Theorem for model categories)**.** The goal is to prove that in a model category C, *if X, Y are both fibrant and cofibrant objects, then a map $f : X \to Y$ is a weak equivalence if and only if it is an homotopy equivalence.*

1. Let$f \overset{l}{\sim} g$ be left homotopic. Show that f is a weak equivalence if and only if g is a weak equivalence.
2. Let $i : X \overset{\sim}{\hookrightarrow} C$ be an acyclic cofibration where X is both fibrant and cofibrant and C is fibrant. Prove that there is a retraction r of i and then show that r is an homotopy inverse of i.
3. Deduce from the previous question that a weak equivalence between fibrant and cofibrant objects is an homotopy equivalence.
4. Let $f : X \to Y$ be an homotopy equivalence between fibrant and cofibrant objects, and let $f : X \underset{\sim}{\overset{i}{\hookrightarrow}} C \overset{p}{\twoheadrightarrow} Y$ be a factorization where the first map is an acyclic cofibration.
   (a) Prove that C is both fibrant and cofibrant and that if g is an homotopy inverse of f, with left homotopy $H : C' \to Y$ between id_Y and $f \circ g$, there is a lift $H' : C' \to C$ such that $p \circ H' = H$ and $H' \circ i_0 = i \circ g$.
   (b) Deduce that $H' \circ i_1 \circ p$ is homotopical to id_C (one can note that i has an homotopy inverse) and then that it is a weak equivalence.
   (c) Prove that p is a retract of a weak equivalence and then conclude.

Solution:

1. Soit $H : C \to Y$ une homotopie à gauche. On a $H \circ i_0 = f$ équivalence faible. De plus i_0 l'est aussi car $id = X \overset{i_0}{\to} C \overset{\sim}{\to} X$ l'est et que la dernière aussi par définition d'un cylindre. Ainsi H est une équivalence faibe et par suite $H \circ i_1 = g$ aussi.

2. On a le diagramme
$$\begin{tikzcd} X \arrow[r, "id"] \arrow[d, "i"'] & X \arrow[d] \\ C \arrow[r] & \{*\} \end{tikzcd}$$
qui induit un relèvement $r; C \to X$ puisque X est fibrant.

   Puisque C est fibrant et i cofibration acyclique, l'aplication $- \circ i : Hom(C,C)/\overset{r}{\sim} \to Hom(X,C)\overset{r}{\sim}$ est une bijection. Mais
$$i \circ r \circ i = i \circ id = i.$$
   Ainsi $i^*([i \circ r]) = i * ([id])$ et donc $i \circ r$ est homotope à droite à id. Comme les objets sont fibrants et cofibrants (C est cofibrant par i et car X l'est), c'est une homotopie tout court.

3. On factorise $X \hookrightarrow C \overset{\sim}{\twoheadrightarrow} Y$. La première flèche est une équivalence faible par CM2. Et C est cofibrant et fibrant car X et Y le sont. Ainsi on obtient un inverse homotopique (droite et gauche) de la première flèche par 1. En dualisant l'argument de 1 on obtient que la deuxième flèche aussi à un inverse homotopique.

4. (a) Comme tout à l'heure, C est cofibrant et cofibrant (par composition). On écrit le diagramme commutatif

$$\begin{tikzcd} Y \arrow[r, "i \circ g"] \arrow[d, "i_0"'] & C \arrow[d, "p"] \\ C' \arrow[r, "H"'] \arrow[ur, dotted, "H'"'] & Y \end{tikzcd}$$

qui nous donne H' par relèvement.

(b) Soit $s : C \to X$ un inverse homotopique de i. On a alors $f \circ r = p \circ i \circ r \sim p$ On a alors, puique par défintion H' définit une homotopie $H' \circ i_1 \sim H' \circ i_0 = i \circ g$, que

$$H' \circ i_1 \circ p \sim i \circ g \circ p \sim i \circ g \circ f \circ r \sim i \circ r \sim id_C.$$

(c) On déduit de la question précédente et de la question 1 que $H' \circ i_1 \circ p$ est une équivalence faible. Il suffit donc de montrer que p est un rétracte de la précédente pour obtenir que c'est une équivalence faible. Ce qui impliquera le résultat pour f par composition. Or on a le diagramme commutatif :

$$\begin{tikzcd} C \arrow[r, "id"] \arrow[d, "p"'] & C \arrow[r, "id"] \arrow[d, "H' \circ i_1 \circ p"] & C \arrow[d, "p"] \\ Y \arrow[r, "H' \circ i_1"'] & C \arrow[r, "p"'] & Y \end{tikzcd}$$

qui permet de conclure (car $p \circ H' \circ i_1 = H \circ i_1 = id_Y$).

Exercice 2. Let $X, Y \in Ch_{\geq 0}(R)$ and $f : X \to Y$ be a chain complex map which is surjective in degree $n \geq 1$ and is a quasi-isomorphism.

1. Show that f is also surjective in degree 0.

2. Let $\psi : P \to Y$ be a map from a chain complex $P \in Ch_{\geq 0}(R)$ such that all P_n are projective.

   (a) Suppose a lift (i.e. $f \circ \tilde{\psi} = \psi$), denoted $\tilde{\psi}_{\leq n} : P_{\leq n} \to X_{\leq n}$, of ψ is given in degree $\leq n$, which is compatible with the differential. Prove that there exist a linear map $\tilde{\phi}_{n+1} : P_{n+1} \to X_{n+1}$ lifting ψ_{n+1} and that for any $p \in P_n$, one has $d \circ \tilde{\psi}_{n+1}(p) - \tilde{\psi}(dp)$ is a boundary in the subcomplex $\mathrm{Ker}(f : X \to Y)$ (hint: prove the latter is acyclic).

   (b) Deduce that there exist a lifting in the diagram of chain complex

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ P \arrow[r, "\psi"'] \arrow[ur, dotted, "h"'] & Y \end{tikzcd}$$

3. Let $A \hookrightarrow B$ be a chain map in $CH_{\geq 0}(R)$, injective in all degrees, whose cokernel P is made of projective modules in every degree. We consider a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d] & X \arrow[d, "f"] \\ B \arrow[r, "\psi"'] & Y \end{tikzcd}$$

   (a) Prove that there are splitting $B_n = A_n \oplus P_n$ such that the differential on the latter subcomplex is written as $d_B(a, p) = (d_A(a) + t(p), d_P(p))$ where d_A, d_B, d_P are the respective differentials of A, B, P.

(b) Let $\kappa : P_n \to X_n$ be a lift of $\psi_{n|P_n}$ and assume a lift of ψ has been constructed as a chain map in degree $\leq n-1$. Prove that $d \circ \kappa(p) - \kappa \circ d_p(p) - \phi \circ t(p)$ belongs to $\mathrm{Ker}(f : X \to Y)$.

(c) Deduce (using a proof similar to the first part of exercise) that one can build a lifting :

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d] & X \arrow[d, "f"] \\ B \arrow[r, "\psi"'] \arrow[ur, dotted, "h"] & Y \end{tikzcd} .$$

4. let P be a cofibrant chain complex with respect to the projective model structure and K be an acyclic chain complex. Prove that any chain map $f : P \to K$ is homotopic to the zero map.

Solution:

1. In degree 0, every element of a positively graded chain complex is a cycle. Since $H_0(f) : H_0(X) \to H_0(Y)$ is an isomoprhisme, there exists $x_0 \in X_0$ and $y_1 \in Y_1$ such that $y_0 = f(x_0) + d(y_1)$. Since f is surjective in degree ≥ 1, then we find x_1 such that $y_1 = f(x_1)$ hence $y_0 = f(x_0 + d(x_1))$ and f_0 is indeed surjective.

2. Let $\psi : P \to Y$ be a map from a chain complex $P \in Ch_{\geq 0}(R)$ such that all P_n are projective.

   (a) Since P_{n+1} is projective and f_{n+1} is surjective (for $n \geq -1$ by 1.) there exists a R-linear lift $\tilde{\phi}_{n+1} : P_{n+1} \to X_{n+1}$ lifting ψ_{n+1}. Since $f \circ \tilde{\psi} = \psi$ is a chain map and so is f, then

   $$f(\circ\tilde{\psi}_{n+1}(p) - \tilde{\psi}(dp)) = d(f \circ \tilde{\psi_{n+1}})(p) - f \circ \tilde{\psi}_n(dp)) = d\psi(p) - \psi(d(p) = 0.$$

   Hence $d \circ \tilde{\psi}_{n+1}(p) - \tilde{\psi}(dp)$ is a cycle in the subcomplex $\mathrm{Ker}(f : X \to Y)$. But since f is surjective in every degree, the sequence $\mathrm{Ker}(f) \to X \to Y$ is a short exact sequence of complexes. Hence the long exact sequences in homology and the fact that $f_* : H_*(X) \to H_*(Y)$ is an isomorphism imply that $H_*(\mathrm{Ker}(f)) = 0$. Hence any cycle is a boundary. Thus there exists $z \in X_{n+1}$ such that $f(z) = 0$ and $d(z) = d \circ \tilde{\psi}_{n+1}(p) - \tilde{\psi}(dp)$. We can choose z linearly since P is projective and the boundary of $Ker(f)$ surjects onto its cycles by what we have just seen.

   (b) We work by induction to define h_n in degree n. We set $h = 0$ in degree ≤ -1 which is obviously a chain map and a lift. We now assume h has been defined in degree $\leq n$ for $n \geq -1$. By the previous question (with $\tilde{\psi}_n = h_n$) we can find $\tilde{\psi}_{n+1}$ as a linear map such that for any p, $f \circ \tilde{\psi}_{n+1}(p) = \psi(p)$ and we can find linearly $z \in X_{n+1}$ such that $f(z) = 0$ and $d(z) = d \circ \tilde{\psi}_{n+1}(p) - \tilde{\psi}(dp)$. Then we set $h_{n+1}(p) = \tilde{\psi}_{n+1}(p) - z$. Then $f \circ h_{n+1} = \psi_{n+1}$ and $d(h_{n+1}(p) = h_n(d(p))$ completing the induction hypothesis.

3. (a) Since $P_n = B_n/A_n$ is projective, there is a section $s : P_n \to B_n$ of the projection $p : B_n \twoheadrightarrow P_n$ given by the diagram $\begin{tikzcd} & B_n \arrow[d, "p"] \\ P_n \arrow[r, equal] \arrow[ur, dotted, "s"] & P_n \end{tikzcd}$. It follows that the map $b \mapsto \big(b - s(p(b)), p(b)\big)$ provides the splitting $B_n = A_n \oplus P_n$. Since A is a subcomplex, the restriction of the differential on the summand A is the differential of A. The differential restricted to p decomposes as $d_B(0, p) = (t(p), d_P(p)$ since $p : B \to P$ is a chain, map.

   (b) By assumption, for any $p \in P_n$, one has

   $$\begin{aligned} f(d \circ \kappa(p) - \kappa \circ d_p(p) - \phi \circ t(p)) &= d(f \circ \kappa(p)) - f \circ \kappa(d_P(p)) - f \circ \phi(t(p)) \\ &= d\psi(0, p) - \psi(d_P(p)) - \psi(t(p)) \\ &= d\psi(p) - \psi(d_B(0, p)) = 0 \end{aligned}$$

   since ψ is a chain map and by the previous question.

(c) Again we work by induction starting with $h = 0$ in negative degrees. We assume h has been constructed in degre $n-1 \geq -1$ and we choose a lift $\kappa : P_n \to X_n$ as in the previous question. We can do it by projectivity of P_n. As we have seen before, the chain complex $\mathrm{Ker}(f)$ has zero homology. Thus, as in question (2), we can find a linear map $z : P_n \to \mathrm{Ker}(f_n : X_n \to Y_n$ such that for any $p \in P_n$, one has

$$d(z(p)) = d \circ \kappa(p) - \kappa \circ d_p(p) - \phi \circ t(p).$$

Then we set $h_n(a,p) = \psi_n(a) + \kappa(p) - z(p)$. Then $f \circ h(a,p) = f \circ \phi(a) + f \circ \kappa(p) = \psi(a) + \psi(p) = \psi(a,p)$ hence, h_n is a linear lift. It is also compatible with the differential

$$d(h_n(a,p)) = d \circ \psi(a) + d \circ \kappa(p) - dz(p) = \psi(d_A(a)) + \kappa \circ (d_P(p)) + \phi \circ t(p) = h_{n-1}((d_A(a) + t(p), d_P(p))$$

and the latter is precisely $h_{n-1}(d(a,p))$.

4. In that question we no longer assume P to be concentrated in non negative degrees. Let $C_*(K) = K_* + 1 \oplus K_*$ be the cocone of K. We recall that the differential of $C_*(K)$ is given by $d(x,y) = \big(-d(x) + y, d(y)\big)$ and in particular $H_*(C_*(K)) = 0$ in every degree. The projection $(x,y) \mapsto y$ on the second summand is a chain map $\pi : C_*(K) \to K$ which is surjective in every degree and a quasi-isomorphisms since K has no homology. It is thus an acyclic fibration and since P is cofibrant we have a lift H in the diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & K_{*+1} \oplus K_* \arrow[d, "\sim"] \\ P \arrow[r, "f"'] \arrow[ur, dotted, "H"] & K. \end{tikzcd}$$

The map H decomposes as $H(p) = (h(p), f(p))$ where $h = (h_n : P_n \to K_{n+1})_{n \in \mathbb{Z}}$ since $\pi \circ H = f$. Since H is a chain map we get also $d \circ H(p) = H(d(p))$ which, on the first summand, implies $-d \circ h(p) + f(p) = h \circ d(p)$ showing that $f - 0 = d \circ h + h \circ d$ hence $f : P \to K$ is homotopic to the zero map.

Exercice 3 (The canonical model structure in Cat)**.** Let Cat denote the category of small categories with morphisms given by functors between them. Let $\mathcal{W}$ be the collection of functors which are equivalences of categories.

1. Show that the Gabriel-Zisman localization $\mathrm{Cat}[\mathcal{W}^{-1}]$ is equivalent to the category whose objects are small categories and morphisms are isomorphism classes of functors.

2. A functor $F : \mathcal{C} \to \mathcal{D}$ between small categories is said to be an isofibration if for every object $c \in \mathcal{C}$ and every isomorphism $f : F(c) \to d$ in $\mathcal{D}$, there exists an object $c' \in \mathcal{C}$ and an isomorphism $u : c \to c'$ such that $d = F(c')$ and $f = F(u)$. Show that an isofibration that is an equivalence of categories is surjective on objects. Conversely, show that if a functor F is fully faithful and surjective on objects then it is an isofibration.

3. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be a cofibration if it is injective on objects. Let Fib denote the collection of all isofibrations and Cof the class of cofibrations. Show that $(\mathrm{Cat}, \mathcal{W}, \mathrm{Fib}, \mathrm{Cof})$ is a model structure and identify its fibrant-cofibrant objects.

Solution: The only non-trivial part is in (3) to show that we have the factorization and lifting properties. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We have to exhibit a factorization of F as $u : \mathcal{C} \to \mathcal{A}$ followed by $v : \mathcal{A} \to \mathcal{D}$ where u is an acyclic cofibration and v is a fibration. Here is the strategy: we define a new category $\mathcal{A}$ as follows as the full subcategory of the comma category $F/\mathcal{D}$, consisting of all those triples $(c \in \mathcal{C}, d \in \mathcal{D}, F(c) \to d)$ where the map $F(c) \to d$ is an isomorphism. Clearly the forgetful functor $(c, d, \phi : F(c) \simeq d) \mapsto d$ defines an isofibration $\mathcal{A} \to \mathcal{D}$ and the inclusion $\mathcal{C} \to \mathcal{A}$ sending

$c \mapsto (c, F(c), Id_{F(c)})$ is injective on objects and is an equivalence as clearly it is essentially surjective and fully faithful by definition of morphisms in this comma category (all morphisms are uniquely determined by morphisms in $\mathcal{C}$). Let us now construct the factorization cofibration + acylic fibration. In this case we can take $\mathcal{A}$ as follows: its objects are the objects of $\mathcal{C}$ disjoint union with the objects of $\mathcal{D}$. Morphisms are defined as follows:

$$Hom_{\mathcal{A}}(x,y) := \begin{cases} Hom_{\mathcal{D}}(F(x), F(y)) & \text{if } x, y \in \mathcal{C} \\ Hom_{\mathcal{D}}(x, F(y)) & \text{if } y \in \mathcal{C}, x \in \mathcal{D} \\ Hom_{\mathcal{D}}(F(x), y) & \text{if } y \in \mathcal{D}, x \in \mathcal{C} \\ Hom_{\mathcal{D}}(x, y) & \text{if } y \in \mathcal{D}, x \in \mathcal{D} \end{cases}$$

And composition is the composition in $\mathcal{D}$. This way the functor $\mathcal{A} \to \mathcal{D}$ is surjective on objects and fully faithful, so that it is a fibration. Moreover, the inclusion $\mathcal{C} \to \mathcal{A}$ is injective on objects, so a cofibration. We are left to show the lifting properties which are routine checks.