Gabriel-Zisman localization and Model structures

Exercice 1 (Model structures on the category of sets)**.** Let Sets denote the category of sets. Show that $(\mathrm{Sets}, \mathcal{W} = \text{bijections}, \mathrm{Fib} = All, \mathrm{Cof} = All)$ determines a model structure.

In fact, there are precisely nine model structures in the category of sets. See link.

Exercice 2 (Whitehead Theorem for model categories)**.** The goal is to prove that in a model category C, *if* X, Y *are both fibrant and cofibrant objects, then a map* $f : X \to Y$ *is a weak equivalence if and only if it is an homotopy equivalence.*

1. Let$f \overset{l}{\sim} g$ be left homotopic. Show that f is a weak equivalence if and only if g is a weak equivalence.
2. Let $i : X \overset{\sim}{\hookrightarrow} C$ be an acyclic cofibration where X is both fibrant and cofibrant and C is fibrant. Prove that there is a retraction r of i and then show that r is an homotopy inverse of i.
3. Deduce from the previous question that a weak equivalence between fibrant and cofibrant objects is an homotopy equivalence.
4. Let $f : X \to Y$ be an homotopy equivalence between fibrant and cofibrant objects, and let $f : X \underset{\sim}{\overset{i}{\hookrightarrow}} C \overset{p}{\twoheadrightarrow} Y$ be a factorization where the first map is an acyclic cofibration.
   (a) Prove that C is both fibrant and cofibrant and that if g is an homotopy inverse of f, with left homotopy $H : C' \to Y$ between id_Y and $f \circ g$, there is a lift $H' : C' \to C$ such that $p \circ H' = H$ and $H' \circ i_0 = i \circ g$.
   (b) Deduce that $H' \circ i_1 \circ p$ is homotopical to id_C (one can note that i has an homotopy inverse) and then that it is a weak equivalence.
   (c) Prove that p is a retract of a weak equivalence and then conclude.

Solution:

1. Soit $H : C \to Y$ une homotopie à gauche. On a $H \circ i_0 = f$ équivalence faible. De plus i_0 l'est aussi car $id = X \overset{i_0}{\to} C \overset{\sim}{\to} X$ l'est et que la dernière aussi par définition d'un cylindre. Ainsi H est une équivalence faibe et par suite $H \circ i_1 = g$ aussi.

2. On a le diagramme
$$\begin{tikzcd} X \arrow[r, "id"] \arrow[d, "i"'] & X \arrow[d] \\ C \arrow[r] & \{*\} \end{tikzcd}$$
qui induit un relèvement $r; C \to X$ puisque X est fibrant.

   Puisque C est fibrant et i cofibration acyclique, l'aplication $-\circ i : Hom(C,C)/\overset{r}{\sim} \to Hom(X,C) \overset{r}{\sim}$ est une bijection. Mais
$$i \circ r \circ i = i \circ id = i.$$
   Ainsi $i^*([i \circ r]) = i * ([id])$ et donc $i \circ r$ est homotope à droite à id. Comme les objets sont fibrants et cofibrants (C est cofibrant par i et car X l'est), c'est une homotopie tout court.

3. On factorise $X \hookrightarrow C \overset{\sim}{\twoheadrightarrow} Y$. La première flèche est une équivalence faible par CM2. Et C est cofibrant et fibrant car X et Y le sont. Ainsi on obtient un inverse homotopique (droite et gauche) de la première flèche par 1. En dualisant l'argument de 1 on obtient que la deuxième flèche aussi à un inverse homotopique.

4. (a) Comme tout à l'heure, C est cofibrant et cofibrant (par composition). On écrit le diagramme commutatif

$$\begin{tikzcd} Y \arrow[r, "i\circ g"] \arrow[d, "i_0"'] & C \arrow[d, "p"] \\ C' \arrow[r, "H"'] \arrow[ur, dotted, "H'"'] & Y \end{tikzcd}$$

qui nous donne H' par relèvement.

(b) Soit $s : C \to X$ un inverse homotopique de i. On a alors $f \circ r = p \circ i \circ r \sim p$ On a alors, puique par défintion H' définit une homotopie $H' \circ i_1 \sim H' \circ i_0 = i \circ g$, que

$$H' \circ i_1 \circ p \sim i \circ g \circ p \sim i \circ g \circ f \circ r \sim i \circ r \sim id_C.$$

(c) On déduit de la question précédente et de la question 1 que $H' \circ i_1 \circ p$ est une équivalence faible. Il suffit donc de montrer que p est un rétracte de la précédente pour obtenir que c'est une équivalence faible. Ce qui impliquera le résultat pour f par composition. Or on a le diagramme commutatif :

$$\begin{tikzcd} C \arrow[r, "id"] \arrow[d, "p"'] & C \arrow[r, "id"] \arrow[d, "H'\circ i_1\circ p"] & C \arrow[d, "p"] \\ Y \arrow[r, "H'\circ i_1"'] & C \arrow[r, "p"'] & Y \end{tikzcd}$$

qui permet de conclure (car $p \circ H' \circ i_1 = H \circ i_1 = id_Y$).

Exercice 3 (Gabriel-Zisman localization)**.** Let $\mathcal{C}$ be a small category and $\mathcal{W}$ a subset of the set of morphisms in $\mathrm{Fun}(I, \mathcal{C})$ where I is the category with two objects 0 and 1 and a unique non-trivial morphism $0 \to 1$. A localization of $\mathcal{C}$ with respect to $\mathcal{W}$ is a functor

$$l : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$$

satisfying the following universal property: For any category $\mathcal{D}$, composition with l:

$$\mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, D)$$

is a fully faithful functor and its essential image consists of those functors $\mathcal{C} \to \mathcal{D}$ sending $\mathcal{W}$ to isomorphisms. In other words, l, if it exists if the universal functor sending $\mathcal{W}$ to isomorphisms.

1. Check that $\mathcal{C}[\mathcal{W}^{-1}]$, if it exists, is unique up to canonical equivalences of categories.

2. Show that when $\mathcal{C}$ is the category with a single object $*$ and a monoid M of endomorphisms, and $\mathcal{W} = M$ then $\mathcal{C}[\mathcal{W}^{-1}]$ is equivalent to the category with one object $*$ and M^+ as endomorphisms, with M^+ the group completion of M.

Indication: we recall that the group completion M^+ of a monoid is a group M^+ together with a monoid map $can : M \to M^+$ such that for any monoid map $\phi : M \to G$, there is a unique group morphism $\tilde{\phi} : M^+ \to G$ factorizing ϕ, that is $\phi = \tilde{\phi} \circ can$. It is usual abstract nonsense to prove it is unique up (to unique if one requires that the isomorphisms commutes with the structure maps from M to the completion) isomorphism. To prove the existence of M^+, it is enough to construct it which can be obtained by defining as a quotient of the free group on the generating set M by the obvious equivalence relation identifiying $m \star m'$ with $m \cdot m'$ if $\star$ is the product in the free group and $\cdot$ is the product in M.

Exercice 4. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor having a right adjoint $G : \mathcal{D} \to \mathcal{C}$[1]. Let $\mathcal{W}$ denote the collection of morphisms f in $\mathcal{C}$ such that $F(f)$ is an isomorphism in $\mathcal{D}$. Show that the following are equivalent:

1. G is fully faithful;
2. The natural transformation $F \circ G \to Id_{\mathcal{D}}$ is an isomorphism;
3. The natural functor $\mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{D}$ is an equivalence of categories.

Solution: Que 1 implique 2 est standard pour toute adjonction:

$$\mathcal{D}(x,y) \cong \mathcal{C}(G(x), G(y)) \cong \mathcal{D}(F \circ G(x), y).$$

On voit de même que 2 implique 1 en allant dans l'autre sens.
Montrons 2 implique 3: On a que la composée $\mathcal{D} \overset{F}{\to} \mathcal{C} \to \mathcal{C}[W^{-1}] \to \mathcal{D}$ est $G \circ F$ donc iso à l'identité. Il suffit de montrer que $\mathcal{C}[W^{-1}] \to \mathcal{D} \overset{G}{\to} \mathcal{C} \to \mathcal{C}[W^{-1}]$ est iso à Id. Par universalité de $\mathcal{C} \to \mathcal{C}[W^{-1}]$, il suffit de montrer que $\mathcal{C} \to \mathcal{C}[W^{-1}] \to \mathcal{D} \overset{G}{\to} \mathcal{C} \to \mathcal{C}[W^{-1}]$ est isomorphe à $\mathcal{C} \to \mathcal{C}[W^{-1}]$. De la composition $F \to FGF \to F = id_F$ donnée par l'unité et la counité, on déduit que la transformation $F \to FGF$ est un isomorphisme puisque la deuxième partie l'est. Cela nous dit que $\mathcal{C} \to \mathcal{C}[W^{-1}] \to \mathcal{D} \overset{G}{\to} \mathcal{C} \to \mathcal{C}[W^{-1}]$ est bien ce que l'on veut.
Montrons que 3 donne 1. On a que $Hom(\mathcal{C}[W^{-1}], \mathcal{D}) \cong Hom(\mathcal{C}, \mathcal{D})$ est pleinement fidèle, ainsi on a par composition que $F^* : Hom(\mathcal{D}, \mathcal{D}) \cong Hom(\mathcal{C}, \mathcal{D})$ est pleinement fidèle. On veut montrer que $F \circ G \to Id$ est un iso, ce qui se ramène à $FGF \to F$ est un isomorphisme ce qui est une conséquence de l'adjonction.

Exercice 5. Let $L : \mathcal{C} \to \mathcal{C}$ be a functor and denote by $L\mathcal{C} \subseteq \mathcal{C}$ its essential image. Show that the following are equivalent:

1. There exists a functor $F : \mathcal{C} \to \mathcal{D}$ with a fully faithful right adjoint $G : \mathcal{D} \to \mathcal{C}$ and a natural isomorphism between $G \circ F$ and L;
2. When regarded as a functor $\mathcal{C} \to L\mathcal{C}$, L is a left adjoint to the inclusion $L\mathcal{C} \subseteq \mathcal{C}$;
3. There exists a natural transformation $\alpha : Id_{\mathcal{C}} \to L$ such that for each object $X \in \mathcal{C}$, the natural morphisms $L(\alpha_X)$ and $\alpha_{L(X)}$ are isomorphisms.

Solution: It is clear that (2) implies (1): just take $\mathcal{D}$ to be the essential image of L. It is also clear that (1) implies (2): as G is fully faithful, we can replace $\mathcal{D}$ by the essential image of G which by hypothesis is equal to the essential image of L. Let us show that (2) implies (3). Let $\alpha : Id_{\mathbb{C}} \to L$ be the co-unit of the adjunction ensured by (2). Let us first remark that $\alpha_{L(X)}$ and $L(\alpha_X)$ are equal as maps. Indeed, because of the definition of natural transformation, we have the commutativity of the diagram

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[d, "\alpha_X"] & L(X) \arrow[d, "L(\alpha_X)"] \\ L(X) \arrow[r, "\alpha_{L(X)}"] & L(L(X)) \end{tikzcd}$$

But we know that $Hom_{L\mathcal{C}}(L(X), L(L(X)) \simeq Hom_{\mathcal{C}}(X, L(L(X))$ because L is a left adjoint to the inclusion. Through this isomorphism, both $\alpha_{L(X)}$ and $L(\alpha_X)$ correspond to the diagonal of the square, so, are equal. In this case it is enough to show that $\alpha_{L(X)}$ is an isomorphism.

Exercice 6. Let $\mathcal{C} = \mathrm{Mod}_{\mathbb{Z}}$ be the category of abelian groups.

[1] $\mathcal{D}$ is said to be a reflexive subcategory of $\mathcal{C}$.

1. (Localization at a single prime) Let p be a prime. Show that the base change functor $-\otimes_{\mathbb{Z}}\mathbb{Z}[\frac{1}{p}]$: $\mathrm{Mod}_{\mathbb{Z}} \to \mathrm{Mod}_{\mathbb{Z}[\frac{1}{p}]}$ is a localization functor along the class $\mathcal{W}$ of all maps of abelian groups $f : X \to Y$ such that both $\mathrm{Ker} f$ and $\mathrm{coker} f$ are p-torsion groups. (Hint: Use the flatness of $\mathbb{Z}[\frac{1}{p}]$ over $\mathbb{Z}$.)

2. Show that the map $\mathbb{Q} \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ sending $q \mapsto q \otimes 1$ is an isomorphism. Use this and the Exercice 3 to show that the category of $\mathbb{Q}$ vector spaces is a localization of the category of abelian groups.

Solution: 1. first note that if $F : \mathrm{Mod}_{\mathbb{Z}} \to \mathcal{D}$ is a functor sending maps of $\mathcal{W}$ onto isoorphisms then $F(M) \cong F(\{0\})$ for all p-torsion abelian groups M and further, since the map $N \xrightarrow{p\cdot} N$ is in $\mathcal{W}$, then $F(p\cdot)$ is an isomorphism. Then we define a factorsation $\tilde{F} : \mathrm{Mod}_{\mathbb{Z}[\frac{1}{p}]} \to \mathcal{D}$ by the formula $\tilde{F}(M \xrightarrow{f} N) = F(\overline{f})$ where $\overline{f}$ is the underlying map of abelian groups. The only thing ot check is that the knowledge of $F(\overline{f})$ completely determines $\tilde{F}(f)$. This is the case since $f(x/p) = (p^{-1})\overline{f}(x)$ and that $F(p\cdot)$ is an isomorphism hence $F(p^{-1}) = F(p)^{-1}$ exists.
2. Apply the last question of Exercise 3.

Exercice 7. Check that $\mathcal{C}[\mathcal{W}^{-1}]$ exists, given by the following pushout in Cat (the category of small categories):

$$\begin{tikzcd}
\coprod_{f\in\mathcal{W}} I \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "l"] \\
\coprod_{f\in\mathcal{W}} J \arrow[r] & \mathcal{C}[\mathcal{W}^{-1}]
\end{tikzcd}$$

where J is the category with two objects 0 and 1 and unique morphism $0 \to 1$ which is an isomorphism.[2]

Exercice 8 (Explicit descripition)**.** In this exercice we review an explicit model for the Gabriel-Zisman localization. Given the pair $(\mathcal{C}, \mathcal{W})$ we construct a new category $\mathcal{D}$ as follows: the objects are the objects of $\mathcal{C}$, morphims from X to Y are given by strings of the form

$$X \to X_1 \leftarrow X_2 \to X_3 \leftarrow ... \to X_n \to Y$$

where all arrows going to the left are in $\mathcal{W}$, submitted to the following equivalence relation: two strings are equivalent if there exists a commutative diagram

$$\begin{tikzcd}
 & X_1 \arrow[d] & X_2 \arrow[l] \arrow[r] & X_3 \arrow[d] & \cdots \arrow[l] \arrow[r] \arrow[d] & X_n \arrow[d] \arrow[dr] & \\
X \arrow[ur] \arrow[dr] & & & & & & Y \\
 & X'_1 & X'_2 \arrow[l] \arrow[r] & X'_3 & \cdots \arrow[l] \arrow[r] & X'_n \arrow[ur] &
\end{tikzcd}$$

where the vertical arrows are in $\mathcal{W}$. Composition is given by concatenation of strings. Show that this equivalence relation is well-defined and that $\mathcal{D}$, together with the canonical functor $\mathcal{C} \to \mathcal{D}$ sending $X \mapsto X$ and $(f : X \to Y) \mapsto X \to Y = Y$ is a localization of $\mathcal{C}$ along $\mathcal{W}$.

Exercice 9. In this exercice we check that the construction of the the previous exercice can be simplified whenever $\mathcal{W}$ satisfies some additional properties. Suppose that:

1. $\mathcal{W}$ is stable under compositions;

[2] Why do pushouts in Cat exist?

2. For any diagram

$$\begin{tikzcd} X' \\ X \arrow[u, "s"] \arrow[r, "f"] & Y \end{tikzcd}$$

with $s \in \mathcal{W}$, there exists a way to complete this diagram in a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "g"] & Y' \\ X \arrow[u, "s"] \arrow[r, "f"] & Y \arrow[u, "t"'] \end{tikzcd}$$

with $t \in \mathcal{W}$.

3. Given $f, g : X \to Y$, if there exists $s \in \mathcal{W}$ such that $f \circ s = g \circ s$ then there exists $t : Y \to Z$ such that $t \in \mathcal{W}$ and $t \circ f = t \circ g$.

In this case $\mathcal{W}$ is said to be a calculus of (right) fractions. Under these hypothesis we consider for each $X \in \mathcal{C}$ the category $\mathcal{W}_{X/.}$ whose objects are morphisms $s : X \to X'$ with $s \in \mathcal{W}$ and morphisms are commutative triangles over X. Assume that $\mathcal{W}$ forms a calculus of fractions. Show that:

1. For each $X \in \mathcal{C}$, $\mathcal{W}_{X/.}$ is a filtered category.

2. The category $\mathcal{C}_{\mathcal{W}}$ whose objects are given by the objects of $\mathcal{C}$, hom-sets $Hom_{\mathcal{C}_{\mathcal{W}}}(X, Y)$ are given by $\mathrm{colim}_{u:Y \to Y' \in \mathcal{W}_{Y/.}} Hom_{\mathcal{C}}(X, Y')$ and compositions are induced from compositions in $\mathcal{C}$, is well-defined.

   In other words, morphisms in $\mathcal{C}_{\mathcal{W}}$ between X and Y are given by equivalence classes of strings of lenght one
$$X \to Y' \leftarrow Y$$
   where the left arrow belongs to $\mathcal{W}$. This simplifies the general explicit description given in (6).

3. Show that the canonical functor $Q : \mathcal{C} \to \mathcal{C}_{\mathcal{W}}$ induced by the identity on objects and by the canonical map
$$Hom_{\mathcal{C}}(X, Y) \to \mathrm{colim}_{u:Y \to Y' \in \mathcal{W}_{Y/.}} Hom_{\mathcal{C}}(X, Y')$$
   on morphisms, is well-defined;

4. Show that if $s : X \to X'$ is a map in $\mathcal{W}$ and Y is an object in $\mathcal{C}$ then the composition map $- \circ s$
$$Hom_{\mathcal{C}_{\mathcal{W}}}(X', Y) \to Hom_{\mathcal{C}_{\mathcal{W}}}(X, Y)$$
   is a bijection. Conclude that Q sends $\mathcal{W}$ to isomorphisms.

5. Show that Q is a localization of $\mathcal{C}$ along $\mathcal{W}$.

6. Show that if $\mathcal{C}$ is an additive category and $\mathcal{W}$ is a calculus of fractions then the localization functor Q preserves finite colimits and $\mathcal{C}[\mathcal{W}^{-1}]$ is also additive.

Solution: It is worth recalling that a filtered category means a nonempty category such that for any two objects, there is a thrid one linking them : that this they both have arrows to a common one. And further if $f, g; X \to Y$ are two maps, there is a map $Y \to Z$ which equalizes them after composition.
The only non-immediate questions are 4. and 6.. To solve 4. we see that the injectivity of the map follows from condition 3 and surjectivity from condition 2 in the definition of right calculus of fractions. The solution to Question 6 takes some extra work:Let us first show that the localization functor Q

commutes with finite colimits. Let $d : I \to \mathcal{C}$ be a diagram in $\mathcal{C}$ and assume the colimit $\mathrm{colim}_I d$ exists in $\mathcal{C}$. Then we want to show that $Q(\mathrm{colim}_I d)$ is a colimit of $Q \circ d$ in $\mathcal{C}_{\mathcal{W}}$. To see this notice that

$$Hom_{\mathcal{C}_{\mathcal{W}}}(Q(\mathrm{colim}_I d), Z) := Hom_{\mathcal{C}_{\mathcal{W}}}(\mathrm{colim}_I d, Z) := \mathrm{colim}_{Z \to Z' \in \mathcal{W}} Hom_{\mathcal{C}}(\mathrm{colim}_I d, Z')$$

$$\simeq \mathrm{colim}_{Z \to Z' \in \mathcal{W}} \mathrm{lim}_{i \in I} Hom_{\mathcal{C}}(d(i), Z') \simeq \mathrm{lim}_{i \in I} \mathrm{colim}_{Z \to Z' \in \mathcal{W}} Hom_{\mathcal{C}}(d(i), Z')$$

because filtered colimits commute with finite limits. By definition, the last becomes

$$\mathrm{lim}_{i \in I} Hom_{\mathcal{C}_{\mathcal{W}}}(d(i), Z)$$

Concluding the argument. Let us now show that if $\mathcal{C}$ is additive then $\mathcal{C}_{\mathcal{W}}$ is additive. Recall that an additive category is a category such that the hom-sets are abelian groups and the composition is bilinear. Moreover, the category has a zero object. Clearly, if $\mathcal{C}$ is additive then the localization is also additive because the hom-sets in the localization are filtered colimits of abelian groups which are again abelian groups. We know now that if 0 is a zero object in $\mathcal{C}$ then $Q(0)$ is an initial object in the localization. We only have to show that $Q(0)$ is also a final object. To see this we show that every diagram $c \to d \leftarrow 0$ is equivalent to the diagram $c \to 0 = 0$ using condition 3 in the definition of right calculus of fractions together with the fact that 0 is terminal in $\mathcal{C}$.

Exercice 10. The following generalizes the exercice (4): Let $\mathcal{C}$ be an abelian category and let $\mathcal{D} \subseteq \mathcal{C}$ be a thick subcategory, ie, a full subcategory such that for each exact sequence

$$0 \to X_1 \to X_2 \to X_3 \to 0$$

in $\mathcal{C}$, X_2 is in $\mathcal{D}$ if and only if X_1 and X_3 are in $\mathcal{D}$. Let $\mathcal{W}$ denote the colllection of morphisms f in $\mathcal{C}$ such that $\mathrm{Ker} f$ and $\mathrm{coker} f$ are in $\mathcal{D}$. Show that $\mathcal{W}$ admits a calculus of fractions and that $\mathcal{C}[\mathcal{W}^{-1}]$ is equivalent to the pushout $\mathcal{C}/\mathcal{D}$ in Cat given by

$$\begin{tikzcd} \mathcal{D} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ 0 \arrow[r] & \mathcal{C}/\mathcal{D} \end{tikzcd}$$

Exercice 11 (The canonical model structure in Cat)**.** Let Cat denote the 1-category of small categories and morphisms given by functors between them. Let $\mathcal{W}$ be the collection of functors which are equivalences of categories.

1. Show that the Gabriel-Zisman localization $\mathrm{Cat}[\mathcal{W}^{-1}]$ is equivalent to the category whose objects are small categories and morphisms are isomorphism classes of functors.

2. A functor $F : \mathcal{C} \to \mathcal{D}$ between small categories is said to be an isofibration if for every object $c \in \mathcal{C}$ and every isomorphism $f : F(c) \to d$ in $\mathcal{D}$, there exists an object $c' \in \mathcal{C}$ and an isomorphism $u : c \to c'$ such that $d = F(c')$ and $f = F(u)$. Show that an isofibration that is an equivalence of categories is surjective on objects. Conversely, show that if a functor F is fully faithful and surjective on objects then it is an isofibration.

3. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be a cofibration if it is injective on objects. Let Fib denote the collection of all isofibrations and Cof the class of cofibrations. Show that $(\mathrm{Cat}, \mathcal{W}, \mathrm{Fib}, \mathrm{Cof})$ is a model structure and identify its fibrant-cofibrant objects.

Solution: 1. Denote $\pi_1(\mathrm{Cat})$ the category with objects small categories and morphisms given by isomorphism classes of functors. There is an obvious functor $\pi : \mathrm{Cat} \to \pi_1(\mathrm{Cat})$ which is the identity on objects and maps a functor to its isomorphism class. We wish to prove that the category $\pi_1(\mathrm{Cat})$

and the functor π satisfy the universal property of $\mathrm{Cat}[\mathcal{W}^{-1}]$. If $\psi : \mathrm{Cat} \to \mathbf{E}$ is a functor such that $\psi(W)$ is an siomorphism when W is an equivalence of categories, then we wish to define a functor $\tilde{\psi} : \pi_1(\mathrm{Cat}) \to \mathbf{E}$ splitting ψ by the obvious formula $\tilde{\psi}(\mathcal{D}) = \psi(\mathcal{D})$ for any samll category $\mathcal{D}$ and, given any isomorphism class $[F] : \mathcal{C} \to \mathcal{D}$ of functor , by $\tilde{\psi}([F]) = \psi(F)$. The only difficulty now is to prove that the latter formula does not depend on the choice of the representative. In other words to prove that if F and G are naturally isomorphic, then $\psi(F) = \psi(G)$

The idea we use is that natural isomorphisms are like (a simple version of) homotopy equivalences in Cat (the notation π_1 was also reminiscent of that and that idea can actually be made very precise using question (3); this is a good exercise). So one can use a proof similar to the proof that the morphisms in the homotopy category are computed as the quotient of morphisms by homotopy equivalences. That is the idea is to construct a natural "path object" for $\mathcal{D}$ and a "(right) homotopy" in between F and G. Concretely let $Iso(\mathcal{D})$ be the category of isomorphisms in $\mathcal{D}$. That is an object of $Iso(\mathcal{D})$ is a morphism $x_0 \stackrel{f}{\to} x_1$ in $\mathcal{D}$ (where x_0, x_1 are any two objects), which is further an *isomorphism*. An arrow α in $Iso(\mathcal{D})$ between $f : x_0 \to x_1$ and $f' : y_0 \to y_1$ is a commutative diagram
$$\begin{tikzcd} x_0 \arrow[r, "f", "\cong"'] \arrow[d, "\alpha_0"'] & x_1 \arrow[d, "\alpha_1"] \\ y_0 \arrow[r, "f'", "\cong"'] & y_1 \end{tikzcd}$$
in $\mathcal{D}$. The composition $\alpha \circ \beta$ is given by the diagram obatined by taking $\alpha_0 \circ \beta_0$, $\alpha_1 \circ \beta_1$ as vertical arrows (that is by stacking the two diagrams). In other words $Iso(\mathcal{D}) = Fun(\{0 \stackrel{\cong}{\to} 1\}, \mathcal{D})$ is the category of functors from the small category[3] with two objects and only two non-identity morphisms which are inverse of each other.

The restriction to 0, 1 induces two functors $proj_i : Iso(\mathcal{D}) \to \mathcal{D}$ which are explicitly given by $proj_i(x_0 \stackrel{f}{\to} x_1) = x_i$ and $proj_i(\alpha) = \alpha_i$. On the other hand we have a canonical functor $I : \mathcal{D} \to Iso(\mathcal{D})$ which maps x to the identity morphism $x \stackrel{id}{\to} x$ and sends f to the diagram in which $\alpha_i = f$.

Now we claim that the last map is an natural equivalence. For this we first note that $proj_i \circ I = Id_{\mathcal{D}}$. Now the following commutative diagram:

$$\begin{tikzcd} x_0 \arrow[r, "id"] \arrow[d, "id"'] & x_0 \arrow[d, "f"] \\ x_0 \arrow[r, "f", "\cong"'] & x_1 \end{tikzcd}$$

defines an natural equivalence from $I \circ proj_0$ to $Id_{Iso(\mathcal{D})}$. This proves that I is an natural equivalence and thus $\psi(I)$ is an isomorphism and further is the inverse of $\psi(proj_0)$. And since $proj_1 \circ I = Id$ as well, we have $\psi(proj_1) = \psi(proj_0)$.

Finally the data of the natural isomorphism τ between F and G yields a functor $H : \mathcal{D} \to Iso(\mathcal{D})$ define for any $g : c \to c'$ by the commutative diagram

$$H(g) = \quad \begin{tikzcd} F(c) \arrow[r, "\tau_c", "\cong"'] \arrow[d, "F(g)"'] & G(c) \arrow[d, "G(g)"] \\ F(c') \arrow[r, "\tau_{c'}", "\cong"'] & G(c') \end{tikzcd} .$$

We thus have $proj0 \circ H = F$ and $proj_1 \circ H = G$. Applying ψ and since $\psi(proj_1) = \psi(proj_0)$ we get that $\psi(F) = \psi(G)$ which concludes the proof of 1.

2. Suppose F is surjective on objects. Then for every object $c \in \mathcal{C}$ and every isomorphism $f : F(c) \to d$ in $\mathcal{D}$, there exists an object $c' \in \mathcal{C}$ with $F(c') = d$. If F is fully faithfull, then $\mathrm{Hom}_{\mathcal{C}}(c, y) \cong \mathrm{Hom}_{\mathcal{D}}(F(c), F(y)$ hence $f : c \to c'$ is an isomorphism if and only if $F(f)$ is an isomorphism (since those are characterized by existence of an inverse). Thus if F is both, it is in particular an isofibration.

[3] denoted J in exercise 7

Conversely, if F is an equivalence, then for any object $d \in \mathcal{D}$, there is $c \in \mathcal{C}$ and an isomorphism $f : F(c) \to d$. If F is an isofibration we get that d is in the image of F.

3. (Co)limits in Cat are inherited from those of sets. The 2 out of 3 property is easy since it boils down to check it on natural isomorphisms. The same applies to retract of equivalences of categories. The injectivity and surjectivity are preserved by retracts and thus cofibrations are preserved by retracts and it follows from (2) that fibrations are stable by retracts as well. The only non-trivial part in (3) is thus to show that we have the factorization and lifting properties. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We have to exhibit a factorization of F as $u : \mathcal{C} \to \mathcal{A}$ followed by $v : \mathcal{A} \to \mathcal{D}$ where u is an acyclic cofibration and v is a fibration. Here is the strategy: we define a new category $\mathcal{A}$ as follows. It is the full subcategory of the comma category $F/\mathcal{D}$, consisting of all those triples $(c \in \mathcal{C}, d \in \mathcal{D}, F(c) \to d)$ where the map $F(c) \to d$ is an isomorphism. Clearly the forgetful functor $(c, d, \phi : F(c) \simeq d) \mapsto d$ defines an isofibration $\mathcal{A} \to \mathcal{D}$ and the inclusion $\mathcal{C} \to \mathcal{A}$ sending $c \mapsto (c, F(c), Id_{F(c)})$ is injective on objects and is an equivalence as clearly it is essentially surjective and fully faithful by definition of morphisms in this comma category (all morphisms are uniquely determined by morphisms in $\mathcal{C}$). Let us now construct the factorization cofibration + acylic fibration. In this case we can take $\mathcal{A}$ as follows: its objects are the objects of $\mathcal{C}$ disjoint union with the objects of $\mathcal{D}$. Morphisms are defined as follows:

$$Hom_{\mathcal{A}}(x,y) := \begin{cases} Hom_{\mathcal{D}}(F(x), F(y)) & \text{if } x, y \in \mathcal{C} \\ Hom_{\mathcal{D}}(x, F(y)) & \text{if } y \in \mathcal{C}, x \in \mathcal{D} \\ Hom_{\mathcal{D}}(F(x), y) & \text{if } y \in \mathcal{D}, x \in \mathcal{C} \\ Hom_{\mathcal{D}}(x, y) & \text{if } y \in \mathcal{D}, x \in \mathcal{D} \end{cases}$$

And composition is the composition in $\mathcal{D}$. This way the functor $\mathcal{A} \to \mathcal{D}$ is surjective on objects and fully faithful, so that it is an isofibration, hence a fibration in the model structure. Moreover, the inclusion $\mathcal{C} \to \mathcal{A}$ is injective on objects, so is a cofibration. We are left to show the lifting properties which are routine checks. For instance if we have a commutative square
$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, tail, "J"'] & \mathcal{X} \arrow[d, "P", "\wr"'] \\ \mathcal{B} \arrow[r, "G"'] \arrow[ur, dotted, "\tilde{G}"] & \mathcal{Y} \end{tikzcd}$$
we construct a lift $\tilde{G}$ as follows. Since by (2) the functor P is surjective on objects, for any object $B \in \mathcal{B}$ we choose an object $X_B \in \mathcal{X}$ such that $P(X_B) = B$ in such a way that if $B = J(A)$, then $X_{J(A)} = F(A)$. There is no choice issues since J is injective on objects. This defines $\tilde{G}(B) = X_B$ on objects. If $f : B \to B'$ is a morphism, then since P is fully faithful we get a (unique) map $\alpha_f : X_B \to X_{B'}$ such that $P(\alpha_f) = G(f)$. By commutativity of the diagram (and uniqueness) we obtain that if $f = J(A \to A'))$, then $\alpha_f = F(A \to A')$. This proves half of the lifting axiom.

For the lifting of
$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, tail, "J"', "\wr"] & \mathcal{X} \arrow[d, "P"] \\ \mathcal{B} \arrow[r, "G"'] \arrow[ur, dotted, "\tilde{G}"] & \mathcal{Y} \end{tikzcd}$$
on objects $B \in \mathcal{B}$, we note that since J is an equivalence, there exist $A \in \mathcal{A}$ and an isomorphism $f : J(A) \to B$ and hence an isomorphism $G(f) : P(F(A)) = G(J(A) \to G(B)$. The isofibration property gives then an object X_B such that $P(X_B) = G(B)$ as above (such that if $B = J(A)$, then $X_{J(A)} = F(A)$). In particular we have that P is surjective on the objects of $G(B)$ and as above we can use the isofibration property to deduce the existence of a lift of $G(B \to B')$. If the objects come from $\mathcal{A}$ then we can use the fully faithfulness of J to ensure the compatibility of the lift $\tilde{G}$ on morphisms with the upper triangle.

Finally we note that every category is cofibrant (the initial object of Cat being the empty category) and fibrant as well (the terminal object being the point with only identity isomorphism).