G. Ginot, H. Pourcelot - Intro. à l'homotopie

GABRIEL-ZISMAN LOCALIZATION AND MODEL STRUCTURES

Exercice 1 (Model structures on the category of sets). Let Sets denote the category of sets. Show that (Sets, $\mathcal{W} =$ bijections, Fib = All, Cof = All) determines a model structure.

In fact, there are precisely nine model structures in the category of sets. See link.

Exercice 2 (Whitehead Theorem for model categories). See the assignment.

Exercice 3 (Gabriel-Zisman localization). Let \mathcal{C} be a small category and \mathcal{W} a subset of the set of morphisms in Fun (I, \mathcal{C}) where I is the category with two objects 0 and 1 and a unique non-trivial morphism $0 \to 1$. A localization of \mathcal{C} with respect to \mathcal{W} is a functor

$$l: \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$$

satisfying the following universal property: For any category \mathcal{D} , composition with *l*:

$$\operatorname{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D}) \to \operatorname{Fun}(\mathcal{C}, D)$$

is a fully faithful functor and its essential image consists of those functors $\mathcal{C} \to \mathcal{D}$ sending \mathcal{W} to isomorphisms. In other words, l, if it exists if the universal functor sending \mathcal{W} to isomorphisms.

- 1. Check that $\mathcal{C}[\mathcal{W}^{-1}]$, if it exists, is unique up to canonical equivalences of categories.
- 2. Show that when C is the category with a single object * and a monoid M of endomorphisms, and $\mathcal{W} = M$ then $\mathcal{C}[\mathcal{W}^{-1}]$ is equivalent to the category with one object * and M^+ as endomorphisms, with M^+ the group completion of M.

Solution. Note that a functor $l: \mathcal{C} \to \mathcal{D}$ is a localization if and only if it verifies the following two properties:

- for any functor $F: \mathcal{C} \to \mathcal{E}$ that sends \mathcal{W} to isomorphisms, there exists $G: \mathcal{D} \to \mathcal{E}$ such that $F \cong G \circ l$,
- the map $\circ l$: Nat $(G_1, G_2) \to$ Nat $(G_1 \circ l, G_2 \circ l)$ is a bijection for all functors $G_1, G_2: \mathcal{D} \to \mathcal{E}$).
- 1. Suppose $l: \mathcal{C} \to \mathcal{E}$ and $l': \mathcal{C} \to \mathcal{E}'$ are two localizations along \mathcal{W} . By the universal property of these localizations, both l and l' are isomorphic to functors that invert \mathcal{W} . In particular, l' belongs to the essential image of the functor $-\circ l$: $\operatorname{Fun}(\mathcal{E}, \mathcal{E}') \to \operatorname{Fun}^{\mathcal{W}}(\mathcal{C}, \mathcal{E}')$, so that there exists a functor $G: \mathcal{E} \to \mathcal{E}'$ such that $l' \cong G \circ l$.



Similarly, $l \cong H \circ l'$ for some functor $H: \mathcal{E}' \to \mathcal{E}$. Using the bijection $\operatorname{Nat}(\operatorname{Id}_{\mathcal{E}}, H \circ G) \cong$ $\operatorname{Nat}(l, H \circ G \circ l) \cong \operatorname{Nat}(l, l)$, one finds a natural transformation $\alpha: \operatorname{Id}_{\mathcal{E}} \Rightarrow H \circ G$ (given by the image of Id_l). Similarly, there is a natural transformation $\beta: H \circ G \Rightarrow \operatorname{Id}_{\mathcal{E}}$. Finally, using the bijection $\operatorname{Nat}(\operatorname{Id}_{\mathcal{E}}, \operatorname{Id}_{\mathcal{E}}) \cong \operatorname{Nat}(l, l)$, one gets $\beta \circ \alpha = \operatorname{Id}_{\operatorname{Id}_{\mathcal{E}}}$ and similarly for the other composition. Hence $H \circ G \cong \operatorname{Id}_{\mathcal{E}}$. Reasoning with \mathcal{E}' in a similar manner shows that G and H realize an equivalence of categories $\mathcal{E} \simeq \mathcal{E}'$. 2. Recall that the group completion of a monoid is a group M^+ together with a monoid map $\iota: M \to M^+$ such that for any monoid map $\phi: M \to G$, there is a unique group morphism $\tilde{\phi}: M^+ \to G$ factorizing ϕ , that is $\phi = \tilde{\phi} \circ \iota$. It is usual abstract nonsense to prove it is unique up to isomorphism (and there is a unique such isomorphism that commutes with the structure maps from M to the completion). To prove the existence of M^+ , it is enough to construct it, which can be done by defining it as a quotient of the free group on the generating set M by the obvious equivalence relation identifying $m \star m'$ with $m \cdot m'$, where \star is the product in the free group and \cdot is the product in M.

We know come to the proof. The key fact to note is the following: the full subcategory of Cat spanned by the categories with a unique object is isomorphic to the category of monoids. Therefore, given a monoid A, we will also write A for the category with one object * and End(*) = A as morphisms. Let M be a monoid. We now show that $\iota: M \to M^+$, viewed as a functor, satisfies the universal property of the localization of M along all morphisms. Let \mathcal{D} be a category. Then any functor $F: M^+ \to \mathcal{D}$ factors through the full subcategory $\mathcal{D}_{F(*)}$ spanned by the image of *. Thus

$$\operatorname{Fun}(M^+, \mathcal{D}) \cong \bigsqcup_{x \in \operatorname{ob}(\mathcal{D})} \operatorname{Fun}(M^+, \mathcal{D}_x)$$

as categories. Observe that any morphism of monoid $f: M^+ \to \operatorname{End}_{\mathcal{D}}(x)$ factors throught the subgroup of units $\operatorname{Aut}_{\mathcal{D}}(x)$. Hence precomposing with ι invert all morphisms in M^+ . Now the universal property of the group completion gives that the functor

$$-\circ\iota\colon\operatorname{Fun}(M^+,\operatorname{Aut}_{\mathcal{D}}(x))\longrightarrow\operatorname{Fun}^{\mathcal{W}}(M,\operatorname{Aut}_{\mathcal{D}}(x))$$

is bijective on objects, hence essential surjective. The category $\operatorname{Fun}(M^+, \operatorname{Aut}_{\mathcal{D}}(x))$ has objects given by the group morphisms $M^+ \to \operatorname{Aut}_{\mathcal{D}}(x)$; the morphisms between f and g are the elements $\alpha \in \operatorname{Aut}_{\mathcal{D}}(x)$ such that $g = \alpha f \alpha^{-1}$. Using this description, one easily see that $-\circ \iota$ is fully faithful, hence an equivalence of categories.

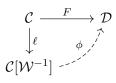
Exercice 4. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor having a right adjoint $G : \mathcal{D} \to \mathcal{C}^1$. Let \mathcal{W} denote the collection of morphisms f in \mathcal{C} such that F(f) is an isomorphism in \mathcal{D} . Show that the following are equivalent:

- 1. *G* is fully faithful;
- 2. The natural transformation $F \circ G \to \mathrm{Id}_{\mathcal{D}}$ is an isomorphism;
- 3. The natural functor $\mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{D}$ is an equivalence of categories.

Solution. To prove that 1 implies 2, one simply observe that

$$\mathcal{D}(x,y) \cong \mathcal{C}(G(x),G(y)) \cong \mathcal{D}(F \circ G(x),y).$$

Similarly, by reversing the argument one can show that 2 gives 1. Let us prove that 2 implies 3. By the universal property of the localization, the functor F factors as

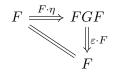


 $^{{}^{1}\}mathcal{D}$ is said to be a reflexive subcategory of \mathcal{C} .

We want to show that the ϕ is an equivalence of categories, with quasi-inverse ℓG . Note that the composite $\phi \ell G = FG$ is isomorphic to $\mathrm{Id}_{\mathcal{D}}$ by assumption. It remains to show that $\ell G \phi$ is isomorphic to $\mathrm{Id}_{\mathcal{C}[\mathcal{W}^{-1}]}$. Since precomposition with ℓ gives an equivalence of categories

$$\operatorname{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{C}[\mathcal{W}^{-1}]) \xrightarrow{\sim} \operatorname{Fun}^{\mathcal{W}}(\mathcal{C}, \mathcal{C}[\mathcal{W}^{-1}]),$$

it is enough to prove that $\ell GF \cong \ell$. Now we can use the triangle identity



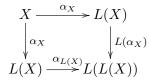
and the fact that ε is an isomorphism to deduce that $F \cdot \eta : \phi \ell \Rightarrow \phi \ell GF$ is also an isomorphism. But since \mathcal{W} is precisely $F^{-1}(iso_{\mathcal{D}})$, we deduce that each component $\eta_c : c \to GF(c)$ is in \mathcal{W} , for $c \in \mathcal{C}$. This implies that $\ell(\eta_c) : l(c) \cong \ell GF(c)$, as desired.

Now we show that 3 gives 1. Since $\operatorname{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D}) \simeq \operatorname{Fun}(\mathcal{C}, \mathcal{D})$ is fully faithful, so is $F^* \colon \operatorname{Fun}(\mathcal{D}, \mathcal{D}) \to \operatorname{Fun}(\mathcal{C}, \mathcal{D})$. Therefore, to show that $F \circ G \Rightarrow \operatorname{Id}$ is an isomorphism, we are reduced to showing that $FGF \Rightarrow F$ is an isomorphism. This can be obtained from the adjunction.

Exercice 5. Let $L : \mathcal{C} \to \mathcal{C}$ be a functor and denote by $L\mathcal{C} \subseteq \mathcal{C}$ its essential image. Show that the following are equivalent:

- 1. There exists a functor $F : \mathcal{C} \to \mathcal{D}$ with a fully faithful right adjoint $G : \mathcal{D} \to \mathcal{C}$ and a natural isomorphism between $G \circ F$ and L;
- 2. When regarded as a functor $\mathcal{C} \to L\mathcal{C}$, L is a left adjoint to the inclusion $L\mathcal{C} \subseteq \mathcal{C}$;
- 3. There exists a natural transformation $\alpha : \operatorname{Id}_{\mathcal{C}} \to L$ such that for each object $X \in \mathcal{C}$, the natural morphisms $L(\alpha_X)$ and $\alpha_{L(X)}$ are isomorphisms.

Solution: It is clear that (2) implies (1): just take \mathcal{D} to be the essential image of L. It is also clear that (1) implies (2): as G is fully faithful, we can replace \mathcal{D} by the essential image of G which by hypothesis is equal to the essential image of L. Let us show that (2) implies (3). Let $\alpha : \mathrm{Id}_{\mathbb{C}} \to L$ be the co-unit of the adjunction ensured by (2). Let us first remark that $\alpha_{L(X)}$ and $L(\alpha_X)$ are equal as maps. Indeed, because of the definition of natural transformation, we have the commutativity of the diagram



But we know that $\operatorname{Hom}_{L\mathcal{C}}(L(X), L(L(X)) \simeq \operatorname{Hom}_{\mathcal{C}}(X, L(L(X)))$ because L is a left adjoint to the inclusion. Through this isomorphism, both $\alpha_{L(X)}$ and $L(\alpha_X)$ correspond to the diagonal of the square, so, are equal. In this case it is enough to show that $\alpha_{L(X)}$ is an isomorphism.

Exercice 6. Let $\mathcal{C} = \operatorname{Mod}_{\mathbb{Z}}$ be the category of abelian groups.

- 1. (Localization at a single prime) Let p be a prime. Show that the base change functor $-\otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}]$: $\operatorname{Mod}_{\mathbb{Z}} \to \operatorname{Mod}_{\mathbb{Z}[\frac{1}{p}]}$ is a localization functor along the class \mathcal{W} of all maps of abelian groups $f: X \to Y$ such that both Ker f and coker f are p-torsion groups. (Hint: Use the flatness of $\mathbb{Z}[\frac{1}{p}]$ over \mathbb{Z} .)
- 2. Show that the map $\mathbb{Q} \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ sending $q \mapsto q \otimes 1$ is an isomorphism. Use this and the Exercice 3 to show that the category of \mathbb{Q} vector spaces is a localization of the category of abelian groups.

Solution. 1. Our strategy is the following: use the adjunction

$$F := (- \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]) : \operatorname{Mod}_{\mathbb{Z}} \xrightarrow{\longrightarrow} \operatorname{Mod}_{\mathbb{Z}[1/p]} : \operatorname{Hom}_{\mathbb{Z}[1/p]}(\mathbb{Z}[1/p], -) = U$$

and prove that the counit $\varepsilon \colon FU \Rightarrow \operatorname{Id}_{\operatorname{Mod}_{\mathbb{Z}[1/p]}}$ is an isomorphism. Then use exercise 4. (part 1 \implies 3) to identify $\operatorname{Mod}_{\mathbb{Z}[1/p]}$ with the localization of $\operatorname{Mod}_{\mathbb{Z}}$ along $F^{-1}(\operatorname{iso}_{\operatorname{Mod}_{\mathbb{Z}}})$. Observe that the counit evaluated at an abelian group A is the evaluation

 $\varepsilon_A \colon \operatorname{Hom}_{\mathbb{Z}[1/p]}(\mathbb{Z}[1/p], A) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \longrightarrow A.$

This map is clearly surjective. We now show its injectivity. Let $\sum_i \psi_i \otimes a_i$ be in the kernel of ε_A , i.e. $\sum_i \psi(a_i) = 0$. Without any loss of generality, we may assume that the ψ are linearly independent. Let's write $a_i = b_i/p^{n_i}$ and define $N = \max_i n_i$. Since multiplication by p is an isomorphism on A, we have $\sum_i p^{N-n_i} b_i \psi(1)$ in A. Since the ψ_i were linearly independent, all the b_i are zero.

We know that F is a localization; it only remains to identify \mathcal{W} with $F^{-1}(iso)$. Let $f: A \to B$ be a morphism of abelian groups. Since $\mathbb{Z}[1/p]$ is torsion-free and \mathbb{Z} is a principal ideal domain, $\mathbb{Z}[1/p]$ is a flat \mathbb{Z} -module. Therefore the sequence

$$0 \longrightarrow \ker(f) \otimes \mathbb{Z}[1/p] \longrightarrow A \otimes \mathbb{Z}[1/p] \longrightarrow B \otimes \mathbb{Z}[1/p] \longrightarrow \operatorname{coker}(f) \otimes \mathbb{Z}[1/p] \longrightarrow 0$$

is exact. From this, we observe that F(f) is an isomorphism if and only if $\ker(f) \otimes \mathbb{Z}[1/p]$ and $\operatorname{coker}(f) \otimes \mathbb{Z}[1/p]$ are zero. This is exactly the condition that $f \in \mathcal{W}$. Therefore $\operatorname{Mod}_{\mathbb{Z}[1/p]} \simeq \operatorname{Mod}_{\mathbb{Z}}[\mathcal{W}^{-1}]$.

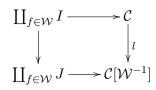
2. Let $\varphi \colon \mathbb{Q} \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ denote the map $q \mapsto q \otimes 1$. Every element of $\mathbb{Q} \otimes \mathbb{Q}$ can be written in the form

$$\sum_{i} \frac{a_i}{b_i} \otimes \frac{c_i}{d_i} = \left(\sum_{i} \frac{a_i c_i}{b_i d_i}\right) \otimes 1,$$

which proves the surjectivity of φ . Since every \mathbb{Z} -linear map between \mathbb{Q} -vector spaces is \mathbb{Q} -linear, for dimension reasons we get that φ is a bijection.

Now consider the endofunctor $L = (- \otimes_{\mathbb{Z}} \mathbb{Q})$: $\operatorname{Mod}_{\mathbb{Z}} \to \operatorname{Mod}_{\mathbb{Z}}$ and the natural transformation α : $\operatorname{Id}_{\operatorname{Mod}_{\mathbb{Z}}} \Rightarrow L$ given pointwise by $A \to A \otimes \mathbb{Q}, a \mapsto a \otimes 1$. Again, the strategy is to use the previous two exercises to ensure that the functor $L: \operatorname{Mod}_{\mathbb{Z}} \to L\operatorname{Mod}_{\mathbb{Z}}$ is a localization. To this end, we need to verify that $\alpha_{L(A)} = \alpha_{A \otimes \mathbb{Q}}$ and $L(\alpha_A)$ are isomorphisms for all abelian groups A. But this is an easy consequence of the fact that φ is an isomorphism. By construction, $L\operatorname{Mod}_{\mathbb{Z}}$ is the full subcategory of $\operatorname{Mod}_{\mathbb{Z}}$ spanned by abelian groups isomorphic to $\mathbb{Q}^{(I)}$ for some set I. We already noticed that this category is equivalent to that of \mathbb{Q} -vector spaces.

Exercice 7. Check that $\mathcal{C}[\mathcal{W}^{-1}]$ exists, given by the following pushout in Cat (the category of small categories):



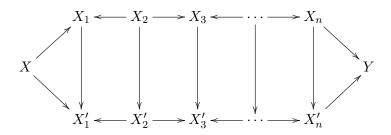
where J is the category with two objects 0 and 1 and unique morphism $0 \rightarrow 1$ which is an isomorphism.²

Exercice 8 (Explicit description). In this exercice we review an explicit model for the Gabriel-Zisman localization. Given the pair $(\mathcal{C}, \mathcal{W})$ we construct a new category \mathcal{D} as follows: the objects are the objects of \mathcal{C} , morphims from X to Y are given by strings of the form

$$X \to X_1 \leftarrow X_2 \to X_3 \leftarrow \dots \to X_n \to Y$$

²Why do pushouts in Cat exist?

where all arrows going to the left are in \mathcal{W} , submitted to the following equivalence relation: two strings are equivalent if there exists a commutative diagram



where the vertical arrows are in \mathcal{W} . Composition is given by concatenation of strings. Show that this equivalence relation is well-defined and that \mathcal{D} , together with the canonical functor $\mathcal{C} \to \mathcal{D}$ sending $X \mapsto X$ and $(f: X \to Y) \mapsto X \to Y = Y$ is a localization of \mathcal{C} along \mathcal{W} .

Exercice 9. In this exercice we check that the construction of the previous exercice can be simplified whenever \mathcal{W} satisfies some additional properties. Suppose that:

- 1. \mathcal{W} is stable under compositions;
- 2. For any diagram



with $s \in \mathcal{W}$, there exists a way to complete this diagram in a commutative diagram

$$\begin{array}{c} X' \xrightarrow{g} Y' \\ s & \uparrow \\ X \xrightarrow{f} Y \end{array}$$

with $t \in \mathcal{W}$.

3. Given $f, g: X \to Y$, if there exists $s \in W$ such that $f \circ s = g \circ s$ then there exists $t: Y \to Z$ such that $t \in W$ and $t \circ f = t \circ g$.

In this case \mathcal{W} is said to be a calculus of (right) fractions. Under these hypothesis we consider for each $X \in \mathcal{C}$ the category $\mathcal{W}_{X/.}$ whose objects are morphisms $s : X \to X'$ with $s \in \mathcal{W}$ and morphisms are commutative triangles over X. Assume that \mathcal{W} forms a calculus of fractions. Show that:

- 1. For each $X \in \mathcal{C}$, $\mathcal{W}_{X/.}$ is a filtered category.
- 2. The category $\mathcal{C}_{\mathcal{W}}$ whose objects are given by the objects of \mathcal{C} , hom-sets $\operatorname{Hom}_{\mathcal{C}_{\mathcal{W}}}(X,Y)$ are given by $\operatorname{colim}_{u:Y \to Y' \in \mathcal{W}_{Y'}}$. $\operatorname{Hom}_{\mathcal{C}}(X,Y')$ and compositions are induced from compositions in \mathcal{C} , is welldefined.

In other words, morphisms in $\mathcal{C}_{\mathcal{W}}$ between X and Y are given by equivalence classes of strings of lenght one

$$X \to Y' \leftarrow Y$$

where the left arrow belongs to \mathcal{W} . This simplifies the general explicit description given in (6).

3. Show that the canonical functor $Q : C \to C_W$ induced by the identity on objects and by the canonical map

$$\operatorname{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \operatorname{colim}_{u:Y \to Y' \in \mathcal{W}_{Y/.}} \operatorname{Hom}_{\mathcal{C}}(X,Y')$$

on morphisms, is well-defined.

4. Show that if $s: X \to X'$ is a map in \mathcal{W} and Y is an object in \mathcal{C} then the composition map $-\circ s$

$$\operatorname{Hom}_{\mathcal{C}_{\mathcal{W}}}(X',Y) \to \operatorname{Hom}_{\mathcal{C}_{\mathcal{W}}}(X,Y)$$

is a bijection. Conclude that Q sends \mathcal{W} to isomorphisms.

- 5. Show that Q is a localization of \mathcal{C} along \mathcal{W} .
- 6. Show that if C is an additive category and W is a calculus of fractions then the localization functor Q preserves finite colimits and $C[W^{-1}]$ is also additive.

Solution. It is worth recalling that a filtered category means a nonempty category such that³

- for any two objects, there is a thrid one linking them : that this they both have arrows to a common one,
- if $f, g: X \to Y$ are parallel maps, there is a map $Y \to Z$ which equalizes them after composition.

The first question is an easy consequence of the axioms. We now define composition in the category $\mathcal{C}_{\mathcal{W}}$. Let $X \longrightarrow Y' \xleftarrow{\sim} Y$ and $Y \longrightarrow Z' \xleftarrow{\sim} Z$ represent two morphisms in that category (denoting morphisms in \mathcal{W} with the symbol $\xleftarrow{\sim}$). Using axiom 2, choose an object W' and morphisms to make the following diagram commute

$$\begin{array}{ccc} Y & \longrightarrow & Z' \\ \sim & & \downarrow \sim \\ Y' & \longrightarrow & W' \end{array}$$

We want to define the composite as represented by $X \longrightarrow Y' \longrightarrow W' \xleftarrow{\sim} Z' \xleftarrow{\sim} Z$. We show that this definition does not depend on the choice of the object W' and of the maps $Y' \longrightarrow W' \xleftarrow{\sim} Z'$. Suppose $Y' \longrightarrow W'' \xleftarrow{\sim} Z'$ is another choice. Then again using axiom 2, we can construct a diagram

$$\begin{array}{ccc} Z' & \stackrel{\sim}{\longrightarrow} & W'' \\ \sim & & \downarrow \\ W' & \stackrel{\sim}{\longrightarrow} & W. \end{array}$$

Note that at that point, there is no reason for $Y' \to W' \to W$ and $Y' \to W'' \to W$ to coincide. But since these composite become equal after precomposition with $Y \xrightarrow{\sim} Y'$, there exists a morphism $W \xrightarrow{\sim} \widetilde{W}$ in \mathcal{W} that makes the diagram

$$\begin{array}{ccc} Y' \longrightarrow W'' \\ \downarrow & \downarrow \\ W' \longrightarrow \widetilde{W} \end{array}$$

commute. Then $X \longrightarrow \widetilde{W} \xleftarrow{\sim} W \xleftarrow{\sim} W' \xleftarrow{\sim} Z' \xleftarrow{\sim} Z$ is equal to both $X \longrightarrow W' \xleftarrow{\sim} Z$ and $X \longrightarrow W' \xleftarrow{\sim} Z$ in $\operatorname{Hom}_{\mathcal{C}_{W}}(X, Z)$. One shows similarly that the composition does not depend either on the choice of a representative such as $X \longrightarrow Y' \xleftarrow{\sim} Y$ of the morphisms to compose.

Similar arguments permit to solve question 4.: one verifies that the injectivity of the map follows from condition 3 and surjectivity from condition 2 in the definition of right calculus of fractions.

Since questions 3 and 5 are easily verified, let us now focus on question 6. We first show that the localization functor Q commutes with finite colimits. Let $d: I \to C$ be a diagram in C and assume the

³Equivalently, a filtered category is one in which every finite diagram admits a cocone. More generally, if κ is a regular cardinal, a category is said to be κ -filtered if every diagram with less than κ arrows has a cocone.

colimit colim_I d exists in \mathcal{C} . Then we want to show that $Q(\operatorname{colim}_I d)$ is a colimit of $Q \circ d$ in $\mathcal{C}_{\mathcal{W}}$. To see this notice that

$$\operatorname{Hom}_{\mathcal{C}_{\mathcal{W}}}(Q(\operatorname{colim}_{I} d), Z) := \operatorname{Hom}_{\mathcal{C}_{\mathcal{W}}}(\operatorname{colim}_{I} d, Z) := \operatorname{colim}_{Z \to Z' \in \mathcal{W}} \operatorname{Hom}_{\mathcal{C}}(\operatorname{colim}_{I} d, Z')$$
$$\simeq \operatorname{colim}_{Z \to Z' \in \mathcal{W}} \lim_{i \in I} \operatorname{Hom}_{\mathcal{C}}(d(i), Z') \simeq \lim_{i \in I} \operatorname{colim}_{Z \to Z' \in \mathcal{W}} \operatorname{Hom}_{\mathcal{C}}(d(i), Z')$$

because filtered colimits commute with finite limits. By definition, the last becomes

$$\lim_{i \in I} \operatorname{Hom}_{\mathcal{C}_{\mathcal{W}}}(d(i), Z),$$

which concludes the argument. Let us now show that if C is additive then C_W is additive. Recall that an additive category is a category such that the hom-sets are abelian groups and the composition is bilinear. Moreover, the category has a zero object. Clearly, if C is additive then the localization is also additive because the hom-sets in the localization are filtered colimits of abelian groups which are again abelian groups. We know now that if 0 is a zero object in C then Q(0) is an initial object in the localization. We only have to show that Q(0) is also a final object. To see this we show that every diagram $c \to d \leftarrow 0$ is equivalent to the diagram $c \to 0 = 0$ using condition 3 in the definition of right calculus of fractions together with the fact that 0 is terminal in C.