Gabriel-Zisman localization and Model structures

Exercice 1 (Model structures on the category of sets)**.** Let Sets denote the category of sets. Show that $(\text{Sets}, \mathcal{W} = \text{bijections}, \text{Fib} = All, \text{Cof} = All)$ determines a model structure.

In fact, there are precisely nine model structures in the category of sets. See link.

Exercice 2 (Whitehead Theorem for model categories)**.** See the assignment.

Exercice 3 (Gabriel-Zisman localization)**.** Let $\mathcal{C}$ be a small category and $\mathcal{W}$ a subset of the set of morphisms in $\text{Fun}(I, \mathcal{C})$ where I is the category with two objects 0 and 1 and a unique non-trivial morphism $0 \to 1$. A localization of $\mathcal{C}$ with respect to $\mathcal{W}$ is a functor

$$l : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$$

satisfying the following universal property: For any category $\mathcal{D}$, composition with l:

$$\text{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D}) \to \text{Fun}(\mathcal{C}, D)$$

is a fully faithful functor and its essential image consists of those functors $\mathcal{C} \to \mathcal{D}$ sending $\mathcal{W}$ to isomorphisms. In other words, l, if it exists if the universal functor sending $\mathcal{W}$ to isomorphisms.

1. Check that $\mathcal{C}[\mathcal{W}^{-1}]$, if it exists, is unique up to canonical equivalences of categories.

2. Show that when $\mathcal{C}$ is the category with a single object $*$ and a monoid M of endomorphisms, and $\mathcal{W} = M$ then $\mathcal{C}[\mathcal{W}^{-1}]$ is equivalent to the category with one object $*$ and M^+ as endomorphisms, with M^+ the group completion of M.

Solution. Note that a functor $l \colon \mathcal{C} \to \mathcal{D}$ is a localization if and only if it verifies the following two properties:

- for any functor $F \colon \mathcal{C} \to \mathcal{E}$ that sends $\mathcal{W}$ to isomorphisms, there exists $G \colon \mathcal{D} \to \mathcal{E}$ such that $F \cong G \circ l$,

- the map $- \circ l \colon \text{Nat}(G_1, G_2) \to \text{Nat}(G_1 \circ l, G_2 \circ l)$ is a bijection for all functors $G_1, G_2 \colon \mathcal{D} \to \mathcal{E}$).

1. Suppose $l \colon \mathcal{C} \to \mathcal{E}$ and $l' \colon \mathcal{C} \to \mathcal{E}'$ are two localizations along $\mathcal{W}$. By the universal property of these localizations, both l and l' are isomorphic to functors that invert $\mathcal{W}$. In particular, l' belongs to the essential image of the functor $- \circ l \colon \text{Fun}(\mathcal{E}, \mathcal{E}') \to \text{Fun}^{\mathcal{W}}(\mathcal{C}, \mathcal{E}')$, so that there exists a functor $G \colon \mathcal{E} \to \mathcal{E}'$ such that $l' \cong G \circ l$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "l'"] \arrow[d, "l"'] & \mathcal{E}' \\ \mathcal{E} \arrow[ur, dashed, bend right, "G"] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, "l"] \arrow[d, "l'"'] & \mathcal{E} \\ \mathcal{E}' \arrow[ur, dashed, bend right, "H"] & \end{tikzcd}$$

Similarly, $l \cong H \circ l'$ for some functor $H \colon \mathcal{E}' \to \mathcal{E}$. Using the bijection $\text{Nat}(\text{Id}_{\mathcal{E}}, H \circ G) \cong \text{Nat}(l, H \circ G \circ l) \cong \text{Nat}(l, l)$, one finds a natural transformation $\alpha \colon \text{Id}_{\mathcal{E}} \Rightarrow H \circ G$ (given by the image of Id_l). Similarly, there is a natural transformation $\beta \colon H \circ G \Rightarrow \text{Id}_{\mathcal{E}}$. Finally, using the bijection $\text{Nat}(\text{Id}_{\mathcal{E}}, \text{Id}_{\mathcal{E}}) \cong \text{Nat}(l, l)$, one gets $\beta \circ \alpha = \text{Id}_{\text{Id}_{\mathcal{E}}}$ and similarly for the other composition. Hence $H \circ G \cong \text{Id}_{\mathcal{E}}$. Reasoning with $\mathcal{E}'$ in a similar manner shows that G and H realize an equivalence of categories $\mathcal{E} \simeq \mathcal{E}'$.

2. Recall that the group completion of a monoid is a group M^+ together with a monoid map $\iota : M \to M^+$ such that for any monoid map $\phi : M \to G$, there is a unique group morphism $\tilde{\phi} : M^+ \to G$ factorizing ϕ, that is $\phi = \tilde{\phi} \circ \iota$. It is usual abstract nonsense to prove it is unique up to isomorphism (and there is a unique such isomorphism that commutes with the structure maps from M to the completion). To prove the existence of M^+, it is enough to construct it, which can be done by defining it as a quotient of the free group on the generating set M by the obvious equivalence relation identifiying $m \star m'$ with $m \cdot m'$, where $\star$ is the product in the free group and $\cdot$ is the product in M.

   We know come to the proof. The key fact to note is the following: the full subcategory of Cat spanned by the categories with a unique object is isomorphic to the category of monoids. Therefore, given a monoid A, we will also write A for the category with one object $*$ and $\mathrm{End}(*) = A$ as morphisms. Let M be a monoid. We now show that $\iota\colon M \to M^+$, viewed as a functor, satisfies the universal property of the localization of M along all morphisms. Let $\mathcal{D}$ be a category. Then any functor $F\colon M^+ \to \mathcal{D}$ factors through the full subcategory $\mathcal{D}_{F(*)}$ spanned by the image of $*$. Thus
$$\mathrm{Fun}(M^+, \mathcal{D}) \cong \coprod_{x \in \mathrm{ob}(\mathcal{D})} \mathrm{Fun}(M^+, \mathcal{D}_x)$$
   as categories. Observe that any morphism of monoid $f\colon M^+ \to \mathrm{End}_{\mathcal{D}}(x)$ factors throught the subgroup of units $\mathrm{Aut}_{\mathcal{D}}(x)$. Hence precomposing with ι invert all morphisms in M^+. Now the universal property of the group completion gives that the functor
$$- \circ \iota\colon\ \mathrm{Fun}(M^+, \mathrm{Aut}_{\mathcal{D}}(x)) \longrightarrow \mathrm{Fun}^{\mathcal{W}}(M, \mathrm{Aut}_{\mathcal{D}}(x))$$
   is bijective on objects, hence essential surjective. The category $\mathrm{Fun}(M^+, \mathrm{Aut}_{\mathcal{D}}(x))$ has objects given by the group morphisms $M^+ \to \mathrm{Aut}_{\mathcal{D}}(x)$; the morphisms between f and g are the elements $\alpha \in \mathrm{Aut}_{\mathcal{D}}(x)$ such that $g = \alpha f \alpha^{-1}$. Using this description, one easily see that $- \circ \iota$ is fully faithful, hence an equivalence of categories.

Exercice 4. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor having a right adjoint $G : \mathcal{D} \to \mathcal{C}$[1]. Let $\mathcal{W}$ denote the collection of morphisms f in $\mathcal{C}$ such that $F(f)$ is an isomorphism in $\mathcal{D}$. Show that the following are equivalent:

1. G is fully faithful;
2. The natural transformation $F \circ G \to \mathrm{Id}_{\mathcal{D}}$ is an isomorphism;
3. The natural functor $\mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{D}$ is an equivalence of categories.

Solution. To prove that 1 implies 2, one simply observe that
$$\mathcal{D}(x, y) \cong \mathcal{C}(G(x), G(y)) \cong \mathcal{D}(F \circ G(x), y).$$
Similarly, by reversing the argument one can show that 2 gives 1.
Let us prove that 2 implies 3. By the universal property of the localization, the functor F factors as
$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\ell"'] & \mathcal{D} \\ \mathcal{C}[\mathcal{W}^{-1}] \arrow[ur, dashed, bend right, "\phi"'] & \end{tikzcd}$$

[1] $\mathcal{D}$ is said to be a reflexive subcategory of $\mathcal{C}$.

We want to show that the ϕ is an equivalence of categories, with quasi-inverse ℓG. Note that the composite $\phi\ell G = FG$ is isomorphic to $\mathrm{Id}_{\mathcal{D}}$ by assumption. It remains to show that $\ell G\phi$ is isomorphic to $\mathrm{Id}_{\mathcal{C}[\mathcal{W}^{-1}]}$. Since precomposition with ℓ gives an equivalence of categories

$$\mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{C}[\mathcal{W}^{-1}]) \xrightarrow{\sim} \mathrm{Fun}^{\mathcal{W}}(\mathcal{C}, \mathcal{C}[\mathcal{W}^{-1}]),$$

it is enough to prove that $\ell GF \cong \ell$. Now we can use the triangle identity

$$\begin{tikzcd} F \arrow[r, Rightarrow, "F\cdot\eta"] \arrow[dr, equal] & FGF \arrow[d, Rightarrow, "\varepsilon\cdot F"] \\ & F \end{tikzcd}$$

and the fact that ε is an isomorphism to deduce that $F \cdot \eta \colon \phi\ell \Rightarrow \phi\ell GF$ is also an isomorphism. But since $\mathcal{W}$ is precisely $F^{-1}(\mathrm{iso}_{\mathcal{D}})$, we deduce that each component $\eta_c \colon c \to GF(c)$ is in $\mathcal{W}$, for $c \in \mathcal{C}$. This implies that $\ell(\eta_c) \colon l(c) \cong \ell GF(c)$, as desired.

Now we show that 3 gives 1. Since $\mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is fully faithful, so is $F^* \colon \mathrm{Fun}(\mathcal{D}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$. Therefore, to show that $F \circ G \Rightarrow \mathrm{Id}$ is an isomorphism, we are reduced to showing that $FGF \Rightarrow F$ is an isomorphism. This can be obtained from the adjunction.

Exercice 5. Let $L : \mathcal{C} \to \mathcal{C}$ be a functor and denote by $L\mathcal{C} \subseteq \mathcal{C}$ its essential image. Show that the following are equivalent:

1. There exists a functor $F : \mathcal{C} \to \mathcal{D}$ with a fully faithful right adjoint $G : \mathcal{D} \to \mathcal{C}$ and a natural isomorphism between $G \circ F$ and L;

2. When regarded as a functor $\mathcal{C} \to L\mathcal{C}$, L is a left adjoint to the inclusion $L\mathcal{C} \subseteq \mathcal{C}$;

3. There exists a natural transformation $\alpha : \mathrm{Id}_{\mathcal{C}} \to L$ such that for each object $X \in \mathcal{C}$, the natural morphisms $L(\alpha_X)$ and $\alpha_{L(X)}$ are isomorphisms.

Solution: It is clear that (2) implies (1): just take $\mathcal{D}$ to be the essential image of L. It is also clear that (1) implies (2): as G is fully faithful, we can replace $\mathcal{D}$ by the essential image of G which by hypothesis is equal to the essential image of L. Let us show that (2) implies (3). Let $\alpha : \mathrm{Id}_{\mathbb{C}} \to L$ be the co-unit of the adjunction ensured by (2). Let us first remark that $\alpha_{L(X)}$ and $L(\alpha_X)$ are equal as maps. Indeed, because of the definition of natural transformation, we have the commutativity of the diagram

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[d, "\alpha_X"] & L(X) \arrow[d, "L(\alpha_X)"] \\ L(X) \arrow[r, "\alpha_{L(X)}"] & L(L(X)) \end{tikzcd}$$

But we know that $\mathrm{Hom}_{L\mathcal{C}}(L(X), L(L(X)) \simeq \mathrm{Hom}_{\mathcal{C}}(X, L(L(X))$ because L is a left adjoint to the inclusion. Through this isomorphism, both $\alpha_{L(X)}$ and $L(\alpha_X)$ correspond to the diagonal of the square, so, are equal. In this case it is enough to show that $\alpha_{L(X)}$ is an isomorphism.

Exercice 6. Let $\mathcal{C} = \mathrm{Mod}_{\mathbb{Z}}$ be the category of abelian groups.

1. (Localization at a single prime) Let p be a prime. Show that the base change functor $- \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}] : \mathrm{Mod}_{\mathbb{Z}} \to \mathrm{Mod}_{\mathbb{Z}[\frac{1}{p}]}$ is a localization functor along the class $\mathcal{W}$ of all maps of abelian groups $f : X \to Y$ such that both $\mathrm{Ker} f$ and $\mathrm{coker} f$ are p-torsion groups. (Hint: Use the flatness of $\mathbb{Z}[\frac{1}{p}]$ over $\mathbb{Z}$.)

2. Show that the map $\mathbb{Q} \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ sending $q \mapsto q \otimes 1$ is an isomorphism. Use this and the Exercice 3 to show that the category of $\mathbb{Q}$ vector spaces is a localization of the category of abelian groups.

Solution. 1. Our strategy is the following: use the adjunction

$$F := (-\otimes_{\mathbb{Z}} \mathbb{Z}[1/p]) : \mathrm{Mod}_{\mathbb{Z}} \rightleftarrows \mathrm{Mod}_{\mathbb{Z}[1/p]} : \mathrm{Hom}_{\mathbb{Z}[1/p]}(\mathbb{Z}[1/p], -) = U$$

and prove that the counit $\varepsilon\colon FU \Rightarrow \mathrm{Id}_{\mathrm{Mod}_{\mathbb{Z}[1/p]}}$ is an isomorphism. Then use exercise 4. (part 1 $\implies$ 3) to identify $\mathrm{Mod}_{\mathbb{Z}[1/p]}$ with the localization of $\mathrm{Mod}_{\mathbb{Z}}$ along $F^{-1}(\mathrm{iso}_{\mathrm{Mod}_{\mathbb{Z}}})$.
Observe that the counit evaluated at an abelian group A is the evaluation

$$\varepsilon_A\colon \mathrm{Hom}_{\mathbb{Z}[1/p]}(\mathbb{Z}[1/p], A) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \longrightarrow A.$$

This map is clearly surjective. We now show its injectivity. Let $\sum_i \psi_i \otimes a_i$ be in the kernel of ε_A, i.e. $\sum_i \psi(a_i) = 0$. Without any loss of generality, we may assume that the ψ are linearly independent. Let's write $a_i = b_i/p^{n_i}$ and define $N = \max_i n_i$. Since multiplication by p is an isomorphism on A, we have $\sum_i p^{N-n_i} b_i \psi(1)$ in A. Since the ψ_i were linearly independent, all the b_i are zero.
We know that F is a localization; it only remains to identify $\mathcal{W}$ with $F^{-1}(\mathrm{iso})$. Let $f\colon A \to B$ be a morphism of abelian groups. Since $\mathbb{Z}[1/p]$ is torsion-free and $\mathbb{Z}$ is a principal ideal domain, $\mathbb{Z}[1/p]$ is a flat $\mathbb{Z}$-module. Therefore the sequence

$$0 \longrightarrow \ker(f) \otimes \mathbb{Z}[1/p] \longrightarrow A \otimes \mathbb{Z}[1/p] \longrightarrow B \otimes \mathbb{Z}[1/p] \longrightarrow \mathrm{coker}(f) \otimes \mathbb{Z}[1/p] \longrightarrow 0$$

is exact. From this, we observe that $F(f)$ is an isomorphism if and only if $\ker(f) \otimes \mathbb{Z}[1/p]$ and $\mathrm{coker}(f) \otimes \mathbb{Z}[1/p]$ are zero. This is exactly the condition that $f \in \mathcal{W}$. Therefore $\mathrm{Mod}_{\mathbb{Z}[1/p]} \simeq \mathrm{Mod}_{\mathbb{Z}}[\mathcal{W}^{-1}]$.

2. Let $\varphi\colon \mathbb{Q} \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ denote the map $q \mapsto q \otimes 1$. Every element of $\mathbb{Q} \otimes \mathbb{Q}$ can be written in the form

$$\sum_i \frac{a_i}{b_i} \otimes \frac{c_i}{d_i} = \left(\sum_i \frac{a_i c_i}{b_i d_i} \right) \otimes 1,$$

which proves the surjectivity of φ. Since every $\mathbb{Z}$-linear map between $\mathbb{Q}$-vector spaces is $\mathbb{Q}$-linear, for dimension reasons we get that φ is a bijection.
Now consider the endofunctor $L = (-\otimes_{\mathbb{Z}} \mathbb{Q})\colon \mathrm{Mod}_{\mathbb{Z}} \to \mathrm{Mod}_{\mathbb{Z}}$ and the natural transformation $\alpha\colon \mathrm{Id}_{\mathrm{Mod}_{\mathbb{Z}}} \Rightarrow L$ given pointwise by $A \to A \otimes \mathbb{Q}, a \mapsto a \otimes 1$. Again, the strategy is to use the previous two exercises to ensure that the functor $L\colon \mathrm{Mod}_{\mathbb{Z}} \longrightarrow L\mathrm{Mod}_{\mathbb{Z}}$ is a localization. To this end, we need to verify that $\alpha_{L(A)} = \alpha_{A\otimes\mathbb{Q}}$ and $L(\alpha_A)$ are isomorphisms for all abelian groups A. But this is an easy consequence of the fact that φ is an isomorphism. By construction, $L\mathrm{Mod}_{\mathbb{Z}}$ is the full subcategory of $\mathrm{Mod}_{\mathbb{Z}}$ spanned by abelian groups isomorphic to $\mathbb{Q}^{(I)}$ for some set I. We already noticed that this category is equivalent to that of $\mathbb{Q}$-vector spaces.

Exercice 7. Check that $\mathcal{C}[\mathcal{W}^{-1}]$ exists, given by the following pushout in Cat (the category of small categories):

$$\begin{tikzcd} \coprod_{f\in\mathcal{W}} I \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "l"] \\ \coprod_{f\in\mathcal{W}} J \arrow[r] & \mathcal{C}[\mathcal{W}^{-1}] \end{tikzcd}$$

where J is the category with two objects 0 and 1 and unique morphism $0 \to 1$ which is an isomorphism.[2]

Exercice 8 (Explicit descripition)**.** In this exercice we review an explicit model for the Gabriel-Zisman localization. Given the pair $(\mathcal{C}, \mathcal{W})$ we construct a new category $\mathcal{D}$ as follows: the objects are the objects of $\mathcal{C}$, morphims from X to Y are given by strings of the form

$$X \to X_1 \leftarrow X_2 \to X_3 \leftarrow \ldots \to X_n \to Y$$

[2] Why do pushouts in Cat exist?

where all arrows going to the left are in $\mathcal{W}$, submitted to the following equivalence relation: two strings are equivalent if there exists a commutative diagram

$$\begin{tikzcd}
 & X_1 \arrow[d] & X_2 \arrow[l] \arrow[r] \arrow[d] & X_3 \arrow[d] & \cdots \arrow[l] \arrow[r] \arrow[d] & X_n \arrow[d] \arrow[dr] & \\
X \arrow[ur] \arrow[dr] & & & & & & Y \\
 & X_1' & X_2' \arrow[l] \arrow[r] & X_3' & \cdots \arrow[l] \arrow[r] & X_n' \arrow[ur] &
\end{tikzcd}$$

where the vertical arrows are in $\mathcal{W}$. Composition is given by concatenation of strings. Show that this equivalence relation is well-defined and that $\mathcal{D}$, together with the canonical functor $\mathcal{C} \to \mathcal{D}$ sending $X \mapsto X$ and $(f : X \to Y) \mapsto X \to Y = Y$ is a localization of $\mathcal{C}$ along $\mathcal{W}$.

Exercice 9. In this exercice we check that the construction of the the previous exercice can be simplified whenever $\mathcal{W}$ satisfies some additional properties. Suppose that:

1. $\mathcal{W}$ is stable under compositions;
2. For any diagram

$$\begin{tikzcd}
X' & \\
X \arrow[u, "s"] \arrow[r, "f"] & Y
\end{tikzcd}$$

   with $s \in \mathcal{W}$, there exists a way to complete this diagram in a commutative diagram

$$\begin{tikzcd}
X' \arrow[r, "g"] & Y' \\
X \arrow[u, "s"] \arrow[r, "f"] & Y \arrow[u, "t"']
\end{tikzcd}$$

   with $t \in \mathcal{W}$.
3. Given $f, g : X \to Y$, if there exists $s \in \mathcal{W}$ such that $f \circ s = g \circ s$ then there exists $t : Y \to Z$ such that $t \in \mathcal{W}$ and $t \circ f = t \circ g$.

In this case $\mathcal{W}$ is said to be a calculus of (right) fractions. Under these hypothesis we consider for each $X \in \mathcal{C}$ the category $\mathcal{W}_{X/.}$ whose objects are morphisms $s : X \to X'$ with $s \in \mathcal{W}$ and morphisms are commutative triangles over X. Assume that $\mathcal{W}$ forms a calculus of fractions. Show that:

1. For each $X \in \mathcal{C}$, $\mathcal{W}_{X/.}$ is a filtered category.
2. The category $\mathcal{C}_{\mathcal{W}}$ whose objects are given by the objects of $\mathcal{C}$, hom-sets $\mathrm{Hom}_{\mathcal{C}_{\mathcal{W}}}(X, Y)$ are given by $\mathrm{colim}_{u:Y\to Y'\in\mathcal{W}_{Y/.}} \mathrm{Hom}_{\mathcal{C}}(X, Y')$ and compositions are induced from compositions in $\mathcal{C}$, is well-defined.

   In other words, morphisms in $\mathcal{C}_{\mathcal{W}}$ between X and Y are given by equivalence classes of strings of lenght one

$$X \to Y' \leftarrow Y$$

   where the left arrow belongs to $\mathcal{W}$. This simplifies the general explicit description given in (6).
3. Show that the canonical functor $Q : \mathcal{C} \to \mathcal{C}_{\mathcal{W}}$ induced by the identity on objects and by the canonical map

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \operatorname*{colim}_{u:Y\to Y'\in\mathcal{W}_{Y/.}} \mathrm{Hom}_{\mathcal{C}}(X, Y')$$

   on morphisms, is well-defined.

4. Show that if $s : X \to X'$ is a map in $\mathcal{W}$ and Y is an object in $\mathcal{C}$ then the composition map $- \circ s$

$$\mathrm{Hom}_{\mathcal{C}_{\mathcal{W}}}(X', Y) \to \mathrm{Hom}_{\mathcal{C}_{\mathcal{W}}}(X, Y)$$

   is a bijection. Conclude that Q sends $\mathcal{W}$ to isomorphisms.

5. Show that Q is a localization of $\mathcal{C}$ along $\mathcal{W}$.

6. Show that if $\mathcal{C}$ is an additive category and $\mathcal{W}$ is a calculus of fractions then the localization functor Q preserves finite colimits and $\mathcal{C}[\mathcal{W}^{-1}]$ is also additive.

Solution. It is worth recalling that a filtered category means a nonempty category such that[3]

- for any two objects, there is a thrid one linking them : that this they both have arrows to a common one,
- if $f, g \colon X \to Y$ are parallel maps, there is a map $Y \to Z$ which equalizes them after composition.

The first question is an easy consequence of the axioms. We now define composition in the category $\mathcal{C}_{\mathcal{W}}$. Let $X \longrightarrow Y' \xleftarrow{\sim} Y$ and $Y \longrightarrow Z' \xleftarrow{\sim} Z$ represent two morphisms in that category (denoting morphisms in $\mathcal{W}$ with the symbol $\xleftarrow{\sim}$). Using axiom 2, choose an object W' and morphisms to make the following diagram commute

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\sim"'] & Z' \arrow[d, "\sim"] \\ Y' \arrow[r] & W' \end{tikzcd}$$

We want to define the composite as represented by $X \longrightarrow Y' \longrightarrow W' \xleftarrow{\sim} Z' \xleftarrow{\sim} Z$. We show that this definition does not depend on the choice of the object W' and of the maps $Y' \longrightarrow W' \xleftarrow{\sim} Z'$. Suppose $Y' \longrightarrow W'' \xleftarrow{\sim} Z'$ is another choice. Then again using axiom 2, we can construct a diagram

$$\begin{tikzcd} Z' \arrow[r, "\sim"] \arrow[d, "\sim"'] & W'' \arrow[d] \\ W' \arrow[r, "\sim"] & W. \end{tikzcd}$$

Note that at that point, there is no reason for $Y' \to W' \to W$ and $Y' \to W'' \to W$ to coincide. But since these composite become equal after precomposition with $Y \xrightarrow{\sim} Y'$, there exists a morphism $W \xrightarrow{\sim} \widetilde{W}$ in $\mathcal{W}$ that makes the diagram

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & W'' \arrow[d] \\ W' \arrow[r] & \widetilde{W} \end{tikzcd}$$

commute. Then $X \longrightarrow \widetilde{W} \xleftarrow{\sim} W \xleftarrow{\sim} W' \xleftarrow{\sim} Z' \xleftarrow{\sim} Z$ is equal to both $X \longrightarrow W' \xleftarrow{\sim} Z$ and $X \longrightarrow W'' \xleftarrow{\sim} Z$ in $\mathrm{Hom}_{\mathcal{C}_{\mathcal{W}}}(X, Z)$. One shows similarly that the composition does not depend either on the choice of a representative such as $X \longrightarrow Y' \xleftarrow{\sim} Y$ of the morphisms to compose.
Similar arguments permit to solve question 4.: one verifies that the injectivity of the map follows from condition 3 and surjectivity from condition 2 in the definition of right calculus of fractions.
Since questions 3 and 5 are easily verified, let us now focus on question 6. We first show that the localization functor Q commutes with finite colimits. Let $d : I \to \mathcal{C}$ be a diagram in $\mathcal{C}$ and assume the

[3]Equivalently, a filtered category is one in which every finite diagram admits a cocone. More generally, if κ is a regular cardinal, a category is said to be κ-filtered if every diagram with less than κ arrows has a cocone.

colimit $\mathrm{colim}_I d$ exists in $\mathcal{C}$. Then we want to show that $Q(\mathrm{colim}_I d)$ is a colimit of $Q \circ d$ in $\mathcal{C}_{\mathcal{W}}$. To see this notice that

$$\mathrm{Hom}_{\mathcal{C}_{\mathcal{W}}}(Q(\operatorname*{colim}_I d), Z) := \mathrm{Hom}_{\mathcal{C}_{\mathcal{W}}}(\operatorname*{colim}_I d, Z) := \operatorname*{colim}_{Z \to Z' \in \mathcal{W}} \mathrm{Hom}_{\mathcal{C}}(\operatorname*{colim}_I d, Z')$$

$$\simeq \operatorname*{colim}_{Z \to Z' \in \mathcal{W}} \lim_{i \in I} \mathrm{Hom}_{\mathcal{C}}(d(i), Z') \simeq \lim_{i \in I} \operatorname*{colim}_{Z \to Z' \in \mathcal{W}} \mathrm{Hom}_{\mathcal{C}}(d(i), Z')$$

because filtered colimits commute with finite limits. By definition, the last becomes

$$\lim_{i \in I} \mathrm{Hom}_{\mathcal{C}_{\mathcal{W}}}(d(i), Z),$$

which concludes the argument. Let us now show that if $\mathcal{C}$ is additive then $\mathcal{C}_{\mathcal{W}}$ is additive. Recall that an additive category is a category such that the hom-sets are abelian groups and the composition is bilinear. Moreover, the category has a zero object. Clearly, if $\mathcal{C}$ is additive then the localization is also additive because the hom-sets in the localization are filtered colimits of abelian groups which are again abelian groups. We know now that if 0 is a zero object in $\mathcal{C}$ then $Q(0)$ is an initial object in the localization. We only have to show that $Q(0)$ is also a final object. To see this we show that every diagram $c \to d \leftarrow 0$ is equivalent to the diagram $c \to 0 = 0$ using condition 3 in the definition of right calculus of fractions together with the fact that 0 is terminal in $\mathcal{C}$.