Derived Functors, Homotopy Colimits and Model structures

Exercice 1 (Composition of Derived Functors)**.** 1. Let $F_1 : \mathcal{C}_1 \to \mathcal{C}_2$ and $F_2 : \mathcal{C}_2 \to \mathcal{C}_3$ be functors and let $\mathcal{W}_i$ be a class of morphisms in $\mathcal{C}_i$. Assuming all the relevant total left derived functors exist, use their universal properties to construct a natural transformation $\mathbb{L}L_2\circ\mathbb{L}F_1 \to \mathbb{L}(F_2\circ F_1)$.

2. Suppose now that $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_3$ are model categories and that F_1 and F_2 are left Quillen functors. Show that all derived functors exist and the natural transformation of the previous exercise is a natural isomorphism.

Solution 1. 1. Denote $\pi_i : \mathcal{C}_i \to \mathbf{Ho}(\mathcal{C}_i)$ the canonical functors. Let us recall that the total left derived functor $\mathbb{L}F_1 : \mathbf{Ho}(\mathcal{C}_1) \to \mathbf{Ho}(\mathcal{C}_2)$ come equipped with a natural transformation $\mathbb{L}F_1 \circ \pi_1 \twoheadrightarrow F_1$ which is universal among such (it is a right Kan extension). In particular we have a commutative diagram

$$\begin{tikzcd}
\mathcal{C}_1 \arrow[r, "F_1"] \arrow[dr, "\pi_1"'] & \mathcal{C}_2 \arrow[r, "F_2"] \arrow[dr, "\pi_2"] & \mathcal{C}_3 \arrow[dr, "\pi_3"'] & \\
& \mathbf{Ho}(\mathcal{C}_1) \arrow[r, "\mathbb{L}F_1"] & \mathbf{Ho}(\mathcal{C}_2) \arrow[r, "\mathbb{L}F_2"] & \mathbf{Ho}(\mathcal{C}_3)
\end{tikzcd} .$$

and natural transformations, given forn any $X \in \mathcal{C}_1$ and $Y \in \mathcal{C}_2$ by $\mathbb{L}F_1(\pi_1(X)) \to \pi_2(F_1(X))$ and $\mathbb{L}F_2(\pi_2(Y) \to \pi_3(F_2(Y))$. Taking $Y = F_1(X)$, the commutativity of the diagram gives a natural transformation

$$\mathbb{L}F_2 \circ \mathbb{L}F_1(\pi_1(X)) \to \pi_3(F_2 \circ F_1(X))$$

hence by universal property we get a unique natural transformation $\mathbb{L}L_2 \circ \mathbb{L}F_1 \to \mathbb{L}(F_2 \circ F_1)$.

2. Let us now address the second question: first we remark that the composition of left Quillen functors is again a left Quillen functor. Indeed, by definition if F_1 preserves both cofibrations and acyclic cofibrations and F_2 also, clearly so does the composition $F_2 \circ F_1$. Therefore, by the theorem given in class, the model structures garantee the existence of $\mathbb{L}F_1$, $\mathbb{L}F_2$ and $\mathbb{L}(F_2\circ F_1)$, given on objects, respectively by $\mathbb{L}F_1(X) = F_1(Q_1(X))$, $\mathbb{L}F_2(Y) = F_2(Q_2(Y))$ and $\mathbb{L}(F_2\circ F_1)(X) = F_2(F_1(Q_1(X))$ where Q_1 is a cofibrant replacement functor in $\mathcal{C}_1$ and Q_2 is a cofibrant replacement in $\mathcal{C}_2$. In this case the natural transformation $\mathbb{L}L_2 \circ \mathbb{L}F_1 \to \mathbb{L}(F_2 \circ F_1)$ is given on each object $X \in \mathcal{C}$ by a morphism

$$F_2(Q_2(F_1(Q_1(X))) \to F_2(F_1(Q_1(X))$$

We only have to notice that by construction (in fact, we have to unfold the proof given in class that the formula $\mathbb{L}F = F \circ Q$ has the universal property of total left derived functor) this morphism is the image under F_2 of the cofibrant-replacement

$$Q_2(F_1(Q_1(X)) \to F_1(Q_1(X)$$

which by definition is a weak-equivalence with both source and target cofibrant: the source is cofibrant by definition. The target is cofibrant because F_1 is a left Quillen functor so sends cofibrant objects to cofibrant objects. Therefore by Brown's lemma[1] its image under F_2 is a weak-equivalence and therefore an isomorphism in the homotopy category.

[1] it is always worth recalling that this lemma does imply that all left Quillen functors send all weak equivalences between cofibrant to weak equivalences and right Quillen functors send weak equivalences between fibrant to weak equivalences

Exercice 2 (Homotopy colimits)**.** In this exercise we first deal with generalities on homotopy pushouts and then specialized to chain complexes with the projective model structure. Let $\mathcal{C}$ be a model category and let I be the category given by the diagram-shape

$$\begin{tikzcd} b \arrow[r] \arrow[d] & c \\ a & \end{tikzcd}$$

1. Let $f : X \to Y$ be a natural transformation of diagrams $X, Y \in \mathrm{Fun}(I, \mathcal{C})$. Show that f has the left lifting property with respect to all projective acyclic fibrations if and only if the the natural maps
$$X(a) \coprod_{X(b)} Y(b) \to Y(a), \quad X(b) \to Y(b), \quad X(c) \coprod_{X(b)} Y(b) \to Y(c)$$
are cofibrations in $\mathcal{C}$ (Here we mean the usual pushouts in $\mathcal{C}$). Conclude that a diagram $Y : I \to \mathcal{C}$ is cofibrant if and only if $Y(b)$ is cofibrant in $\mathcal{C}$ and each map $Y(a) \to Y(b)$ and $Y(a) \to Y(c)$ is a cofibration. Moreover, show that $X \to Y$ has the left lifting property with respect to projective fibrations if and only the same three maps are acyclic cofibrations.

2. Show that the category of diagrams $\mathrm{Fun}(I, \mathcal{C})$ admits the projective model structure (without using the result seen in class that such structure exists since I is very small).

3. Show that the colimit functor $\mathrm{colim} : Fun(I, \mathcal{C}) \to \mathcal{C}$ is a left Quillen functor.

4. A model category $\mathcal{C}$ is said to be left proper if weak-equivalences are stable under pushouts along cofibrations. Show that $\mathcal{C}$ be left proper and
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ C \arrow[r] & C \coprod_A B \end{tikzcd}$$
is a pushout diagram with $A \to B$ a cofibration, then the diagram is also a homotopy pushout.

5. **Case of Topological spaces.** Assume now that $\mathcal{C} = \mathbf{Top}$.

   (a) Using that **Top** is proper (see exercise 6), deduce that that there is a canonical isomorphism
   $$\mathbb{L}\,\mathrm{colim}(X \leftarrow A \to Y) \cong X \coprod_A^h Y = X \coprod_{A \times \{0\}} \mathrm{Cyl}(A \to Y)$$
   in $\mathbf{Ho}(\mathbf{Top})$ between the homotopy pushout computed by the projective model structure and the formula given by the mapping cylinder.

   (b) Give a formula for computing the homotopy colimit of a tower $X_0 \overset{f_0}{\to} X_1 \overset{f_1}{\to} X_2 \overset{f_2}{\to} X_2 \to \dots$. as well as the homotopy limit of a tower $\dots Y_2 \to Y_1 \to Y_0$.

6. **Case of chain complexes.** Assume now that $\mathcal{C}$ is the model category of chain complexes over a ring R.

   (a) Show that $\mathcal{C}$ is left proper.

(b) Let $g : A \to B$ be a map of chain complexes. Recall that the *mapping cone* of g, denoted $C(g)$, is the chain complex given in level n by $B_n \oplus A_{n-1}$ and whose differential $B_{n+1} \oplus A_n \to B_n \oplus A_{n-1}$ is given $(b, a) \mapsto (\partial_B(b) + g(a), -\partial_A(a))$. Let I^1 denote the chain complex given by $R \oplus R$ in degree 0 and R in degree 1 with differential given by $\partial_R : R \to R \oplus R$ given by $r \mapsto (-r, r)$. We define the mapping cylinder of g, $Cyl(g)$ to be the pushout in chain complexes of

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "i_0"'] & B \arrow[d] \\ I^1 \otimes A \arrow[r] & Cyl(g) \end{tikzcd}$$

where the vertical arrow $A \to I^1 \otimes A$ is the induced by the inclusion $i_0 : R \to I^1$ corresponding to the inclusion of the second factor $R \hookrightarrow R \oplus R$ in degree 0 and the differential on $I^1 \otimes A$ is given by $r \otimes a \mapsto \partial_R(x) \otimes a + (-1)^{\deg(r)} r \otimes \partial_A(a)$. Show that the mapping cone of g is the pushout of

$$\begin{tikzcd} I^1 \otimes A \arrow[r] \arrow[d] & Cyl(g) \arrow[d] \\ C(Id_A) \arrow[r] & C(g) \end{tikzcd}$$

(c) Let Δ^1 be the category with two objects and one non trivial morphism in between them. Show that the construction of the mapping cone defines a functor $C : Fun(\Delta^1, Ch(R)) \to Ch(R)$ sending natural transformations objectwise given by quasi-isomorphisms to quasi-isomorphisms.

(d) Let $Y := (0 \longleftarrow A \xrightarrow{g} B)$ be a diagram in $\mathcal{C}$. Show that there exists a diagram of the form $Y' := (0 \longleftarrow A' \xrightarrow{g'} B')$ with g' a cofibration and A' and B' cofibrant, together with a natural transformation $u : Y' \to Y$ which is objectwise a weak-equivalence. Notice that by the previous question the induced map $C(g') \to C(g)$ is a weak-equivalence.

(e) Let $Y := (0 \longleftarrow A \xrightarrow{g} B)$ be a diagram in $\mathcal{C}$ with A and B cofibrant and g a cofibration. Show that $A \to A \otimes \Delta^1$ is a weak-equivalence and show that we can construct a zigzag of diagrams $Y \leftarrow Y' \to Y''$

$$\begin{tikzcd} 0 & A \arrow[l] \arrow[r, "g"] & B \\ C(A) \arrow[u] \arrow[d] & A \arrow[l] \arrow[r, "g"] \arrow[u] \arrow[d] & B \arrow[u] \arrow[d] \\ C(A) & I^1 \otimes A \arrow[l] \arrow[r, "g"] & Cyl(g) \end{tikzcd}$$

where each vertical arrow is a weak-equivalence and the map $I^1 \otimes A \to Cyl(g)$ is a cofibration.

(f) Let $Y := (0 \longleftarrow A \xrightarrow{g} B)$. Conclude that the mapping cone $C(g)$ is a model for the homotopy colimit of the diagram Y.

Solution 2. First we advise the reader to write down a commutative square of functors in $Fun(I, \mathcal{C})$, which are given by glueing two commutative cubes on their common face, and in which each face is commutative, as well as to write down what a lifting mean (which is a family of three maps divding parallel faces into two commutative triangles). A key feature of the diagram we are considering is that the object b has only outgoing non-identity arrows and the other two objects have only incoming non-identity arrows. The object b and its image by a functor will play a specific role. **1.** Suppose a morphim $X \to Y$ in $Fun(I, \mathcal{C})$ has the left lifting property with respect to projective acyclic fibrations. We first show that the the map $X(b) \to Y(b)$ has the left lifting property. Thus, we need to see that for any $U \to V$ a acyclic fibration in $\mathcal{C}$ the dotted lifting arrow exists in the diagram

$$\begin{tikzcd} X(b) \arrow[r] \arrow[d] & U \arrow[d] \\ Y(b) \arrow[r] \arrow[ur, dashed] & V \end{tikzcd} .$$

For this, we notice that the data of such a diagram is equivalent to the data of a morphism of diagrams

$$\begin{tikzcd} X \arrow[r] \arrow[d] & (*, U, *) \arrow[d] \\ Y \arrow[r] \arrow[ur, dashed] & (*, V, *) \end{tikzcd}$$

where $(*, U, *)$ is a notation for the diagram $* \leftarrow U \to *$ (and $*$ is the terminal object). The lifting exists by the assumption that $X \rightarrowtail Y$. This shows that $X(b) \to Y(b)$ is a cofibration. Let us now use this to show that the map $X(a) \coprod_{X(b)} Y(b) \to Y(a)$ has the left lifting property

$$\begin{tikzcd} X(a) \coprod_{X(b)} Y(b) \arrow[r] \arrow[d] & U \arrow[d] \\ Y(a) \arrow[r] \arrow[ur, dashed] & V \end{tikzcd}$$

with respect to any acyclic fibration $U \to V$ in $\mathcal{C}$. We do this using the remark that the data of such a commutative square is equivalent to the data of a commutative square of diagrams

$$\begin{tikzcd} X \arrow[r] \arrow[d] & (U, U, *) \arrow[d] \\ Y \arrow[r] \arrow[ur, dashed] & (V, U, *) \end{tikzcd}$$

The case of the remaining map is completely analogous.

We now have to check the converse, that if $X \to Y$ is of the form given in the exercise then it has the left lifting property with respect to projective acyclic fibrations. The idea is again to use first the fact that $X(b) \to Y(b)$ is a cofibration in $\mathcal{C}$ to construct the lifting in the middle. This is possible since each arrow $U(a) \to V(a)$, $U(b) \to V(b)$ and $U(c) \to V(c)$ are acyclic fibrations if $U \to V$ is an acyclic fibration in $Fun(I, \mathcal{C})$.

This being done, we see that the lifing $Y(b) \to U(b)$ gives a commutative diagram

$$\begin{tikzcd} & X(b) \arrow[r] \arrow[d] \arrow[dl] & X(a) \arrow[d] \\ Y(b) \arrow[r] & U(b) \arrow[r] & U(a) \end{tikzcd}$$

from which we get a canonical map $X(a) \coprod_{X(b)} Y(b) \to U(a)$ which, by the commutativity of the diagram of squares fits into the commuative square $\begin{tikzcd} X(a) \coprod_{X(b)} Y(b) \arrow[r] \arrow[d] & U(a) \arrow[d, "\wr"] \\ Y(a) \arrow[r] \arrow[ur, dashed] & V(a) \end{tikzcd}$ for which the dotted arrow exists since the left hand vertical map is assumed to be a cofibration. The remaining lifts is the same. This proves the first equivalence.

The case of acyclic cofibrations is similar, using fibration on the right hand side instead of acyclic ones.

2.One has to check that all the axioms are satisfied. First one checks that $\mathrm{Fun}(I,\mathcal{C})$ admits all limits and colimits: this is true as long as they exist in $\mathcal{C}$ because colimits and limits in $\mathrm{Fun}(I,\mathcal{C})$ are computed objectwise in $\mathcal{C}$. Then one has to check the two-out-of-three property of weak-equivalences. But again this follows by definition of the weak-equivalences as objectwise weak-equivalences in $\mathcal{C}$ which verifies this property. Then we have to check that fibrations, cofibrations and weak-equivalences are stable under retracts. For fibrations and weak-equivalences this follows again from the definitions, so we only have to say something about cofibrations: but since cofibrations are maps defined by a left lifting property, and the latter are stable under retracts, this is also OK (see the proof of the closedness of a model category in Class).

The lifting properties were already checked in the previous question so all we have to check is the factorization property: we explain the case $X \to Y$ factored as acyclic cofibration + fibration. Here is the idea: again, first we factor the middle term $X(b) \to Y(b)$ as a acyclic cofibration followed by a fibration $X(b) \to A \to Y(b)$ in $\mathcal{C}$. Then we complete this into a diagram by taking pushouts $X(c) \to X(c)\coprod_{X(b)} A \to Y(b)$ and $X(a) \to X(a)\coprod_{X(b)} A \to Y(a)$. Now we factor the last two maps $X(c)\coprod_{X(b)} A \to H_c \to Y(b)$ and $X(a)\coprod_{X(b)} A \to H_a \to Y(a)$ again in $\mathcal{C}$. The resulting factorization $X \to H \to Y$ has the required properties.

3.This follows because by definition its right adjoint is the constant diagram functor which is right Quillen as by definition it preserves fibrations and acyclic fibrations.

Remark on computations of homotopy pushouts. As we have seen in class, the last point implies in particular that homotopy pushouts exists for any model category $\mathcal{C}$ and are computed as the left total derived functor of the pushout functor $Fun(I,\mathcal{C}) \to \mathbf{Ho}(\mathcal{C})$ where the diagram category is given the projective model structure. This means that it is computed by taking the pushout of a cofibrant replacement of $X(a) \leftarrow X(b) \to X(c)$ in $Fun(I,\mathcal{C})$, that is

$$\mathbb{L}\,\mathrm{colim}\,\big(X(a) \leftarrow X(b) \to X(c)\big) = \mathrm{colim}\,\big(L_X(a) \leftarrow L_X(b) \to L_X(c)\big) = L_X(a) \coprod_{L_X(b)} L_X(c)$$

where $L_X \overset{\sim}{\twoheadrightarrow} X$ is the cofibrant replacement.

Note that by question 2., we have that a diagram Z is cofibrant if $Z(b)$ is cofibrant and (since $0\coprod_{O} Z(b) = Z(b)$) the maps $Z(b) \to Z(c)$ and $Z(b) \to Z(a)$ are cofibrations. Thus:

a cofibrant replacement of a diagram X is a diagram $L_X(a) \leftarrowtail L_X(b) \rightarrowtail L_X(c)$, with $L_X(b)$ cofibrant, and a commutative diagram:

$$\begin{tikzcd}
L_X(a) \arrow[d, two heads, "\wr"] & L_X(b) \arrow[l, tail] \arrow[r, tail] \arrow[d, two heads, "\wr"] & L_X(c) \arrow[d, two heads, "\wr"] \\
X(a) & X(b) \arrow[l] \arrow[r] & X(c).
\end{tikzcd}$$

The next question and the proposition below shows that in model categories where weak equivalences are preserved by pushouts, there is an easier formula to compute it.

4. Indeed, let

$$\begin{tikzcd}
C' \arrow[d, two heads, "\wr"] & A' \arrow[l, tail] \arrow[r, tail] \arrow[d, two heads, "\wr"] & B' \arrow[d, two heads, "\wr"] \\
C & A \arrow[l] \arrow[r, tail, "f"] & B
\end{tikzcd}$$

be a cofibrant resolution of the diagram $C \leftarrow A \to B$ (as explained in the remark above). We have to show that the natural map

$$C' \coprod_{A'} B' \to C \coprod_{A} B$$

is a weak-equivalence. But this map can be obtained as a composition of two maps : $C' \coprod_{A'} B' \to C' \coprod_{A'} B$ followed by $C' \coprod_{A'} B \to C \coprod_A B$. The first map can be obtained as a pushout

$$\begin{tikzcd} B' \arrow[r] \arrow[d] & C' \coprod_{A'} B' \arrow[d] \\ B \arrow[r] & C' \coprod_{A'} B \end{tikzcd}$$

The top horizontal arrow is a cofibration (because cofibrations are stable under pushout and $A' \rightarrowtail C'$ is a cofibration) and as $B' \to B$ is a weak-equivalence , left properness implies that the left vertical arrow is a weak-equivalence. The second map can be obtained as a composition of pushout diagrams.

$$\begin{tikzcd} A' \arrow[r, "cof"] \arrow[d, "\sim"] & C' \arrow[d, "\sim"] \arrow[dr, "\sim"] & \\ A \arrow[r] \arrow[d, tail, "f"] & C' \coprod_{A'} A \arrow[r, "\sim(2of3)"] \arrow[d, "cof"] & C \arrow[d] \\ B \arrow[r] & B \coprod_A (C' \coprod_{A'} A) \simeq B \coprod_{A'} C' \arrow[r, "\sim(proper)"] & B \coprod_A C \end{tikzcd}$$

where the middle vertical arrow is a cofibration as a pushout of the cofibration f and the lower right horizontal arrow is a weak-equivalence thanks to the properness assumption.

5.

a. Noticing that the factorisation $A \hookrightarrow A \times [0,1] \coprod\limits_{A \times \{1\}} Y \xrightarrow{\sim} Y$ given by the mapping cylinder is a relative cell complex followed by a weak equivalence, we wee that the result will follow from the following general fact:

Proposition (Homotopy pushout in left proper model categories). *If $\mathcal{C}$ is a left proper model category, and $A \rightarrowtail B' \xrightarrow{\sim} B$ is a replacement of a morphism $i : A \to B$ by a cofibration, then there is a natural isomorphism* $\mathbb{L}\operatorname{colim}(X \leftarrow A \to B) \cong X \coprod\limits_A B'$.

Strictly speaking the proposition asserts that there is an isomorphism in $\mathbf{Ho}(\mathcal{C})$ between the homotopy pushout and the pushout induced by the cofibrant replacement of $A \to B$ and that in fact, this isomorphism is inducec by a *natural* zigzag of weak equivalence

$$L_X \coprod_{L_A} L_B \xleftarrow{\sim} ? \xrightarrow{\sim} X \coprod_A B'$$

where the $L_X \leftarrow L_A \to L_B$ is a cofibrant replacement of $X \leftarrow A \to B$ (and thus the source of the weak equivalence is precisely the homotopy pushout) and the question mark ? depends functorially on the diagram.

We now prove the proposition. By question (3.), the target $X \coprod\limits_A B'$ is the homotopy pushout $\mathbb{L}\operatorname{colim}(X \leftarrow A \rightarrowtail B')$. The map $B' \to B$ induces a map of diagrams

$$\begin{tikzcd} X \arrow[d, equal] & A \arrow[l] \arrow[r, tail] \arrow[d, equal] & B' \arrow[d, "\wr"] \\ X & A \arrow[l] \arrow[r] & B \end{tikzcd}$$

for which all vertical maps are weak equivalences. Hence this is a weak equivalence of diagrams, and thus, the induced map on homotopy colimits is an isomorphism in $\mathbf{Ho}(\mathcal{C})$.

We thus have a natural isomorphism $\mathbb{L}\operatorname{colim}(X \leftarrow A \to B) \overset{\simeq}{\leftarrow} \mathbb{L}\operatorname{colim}(X \leftarrow A \rightarrowtail B') \overset{\simeq}{\to} X \coprod\limits_{A} B'$ in $\mathbf{Ho}(\mathcal{C})$ as claimed and the question mark ? above is just the pushout $L_X \coprod\limits_{L_A} L_{B'}$ where $L_X \leftarrow L_A \to L_{B'}$ is the cofibrant replacement of $X \leftarrow A \to B'$.

b. The category N depicting the colimit of tower is simply $0 \to 1 \to 2 \to 3 \to \cdots$ (that is the category associated to the ordinal $\mathbb{N}$, or said otherwise to the ordered set $\mathbb{N}$) the category with exactly one arrow in between two consecutive non-negative integers. It is not a very small category so that the theorem seen in class does not guarantee the existence of homotopy colimit.

However, we can apply the same ideas as in the study of the homotopy pushout. Proceeding exactly as in question 1. we see that, for any model category $\mathcal{C}$, a morphism $X \to Y$ in $Fun(N, \mathcal{C})$ is a projective cofibration (resp. acyclic cofibration) if and only if $X(0) \to Y(0)$ is a cofibration (resp. acyclic cofibration) and for every $i > 0$, the natural map $X_i \coprod\limits_{Y_{i-1}} X_{i-1} \to Y_i$ is a cofibration (resp. acyclic cofibration). Then one can prove as in 2. that the projective structure on $Fun(N, \mathcal{C})$ makes the category of towers a model category so that the homotopy colimit of the tower exists. Further a cofibrant replacement of a diagram $X : N \to \mathcal{C}$ is thus given by a cofibrant object $L_X(0)$ and cofibrations $L_X(\rightarrowtail L_X(i+1)$ (for any $i \in \mathbb{N}$) together with acyclic fibrations $L_X(i) \overset{\sim}{\twoheadrightarrow} X_i$ making the obvious squares commutative. In the specific case where $X(0)$ is cofibrant and all the maps $X(i) \to X(i+1)$ are cofibrations, we thus have that X is cofibrant and therefore as seen in class, the canonical map from the homotopy pushout of the tower X to its pushout $\operatorname{colim} X(i)$ is a weak equivalence. It follows that if we have a commutative diagram

$$\begin{tikzcd}
Y(0) \arrow[r, tail] \arrow[d, "\wr"] & Y(1) \arrow[r, tail] \arrow[d, "\wr"] & Y(2) \arrow[r, tail] \arrow[d, "\wr"] & \cdots \\
X(0) \arrow[r] & Y(1) \arrow[r] & Y(2) \arrow[r] & \cdots
\end{tikzcd}$$

with $Y(0)$ cofibrant, then the diagram is a weak equivalence of diagram and by above we thus have a zigzag of weak equivalences

$$\operatorname*{colim}_{\mathbb{N}} Y(i) \overset{\sim}{\leftarrow} \operatorname*{colim}_{\mathbb{N}} L_Y(i) \overset{\sim}{\to} \operatorname*{colim}_{\mathbb{N}} L_X(i).$$

This proves that to compute the homotopy colimit of a tower it is enough to replace it by a weakly equivalent tower consisting of cofibrations whose first object is cofibrant.

A completely dual analysis shows that the injective model structure is also a model category for $Fun(N, \mathcal{C})$ and thus that homotopy limit of tower exists and can be computed by replacing a tower by a weakly equivalent tower such that all maps are fibrations and the last object Y_0 is fibrant.

Now, recall from class that in **Top**, every object is fibrant and that for every object X_0 there is a CW-complex $\tilde{X}_0$ weakly equivalent to it: $\tilde{X}_0 \overset{\sim}{\to} X_0$ (and by composition we have an induced map $\tilde{X}_0 \to X_1$).

Hence in **Top** , the homotopy colimit of a tower is given by the "telescope"

$$\mathbb{L}colim X_i \cong \tilde{X}_0 \times [0,1] \coprod_{\tilde{X}_0 \times \{1\}} X_1 \times [1,2] \coprod_{X_1 \times \{2\}} X_2 \times [2,3] \coprod_{X_2 \times \{3\}} X_3 \times [3,4] \coprod \cdots$$

that is a tower of glued cylinders. Now consider the colimit of almost the same telescope but for which we start at X_0. Then we have a pushout diagram

$$\begin{tikzcd} \tilde{X}_0 \times [0,1[\arrow[r, tail] \arrow[d, "\wr"] & \tilde{X}_0 \times [0,1] \coprod_{\tilde{X}_0} \big(\coprod_{X_{i-1}} X_i \times [i, i+1] \big) \arrow[d] \\ X_0 \times [0,1[\arrow[r, tail] & X_0 \times [0,1] \coprod_{\tilde{X}_0} \big(\coprod_{X_{i-1}} X_i \times [i, i+1] \big) \end{tikzcd}$$

in which the right vertical arrow is a weak equivalence by left properness. Hence the homotopy colimit of a tower $X_0 \to X_1 \to \cdots$ is given by the telescope

$$\mathbb{L}colim X_i \cong X_0 \times [0,1] \coprod_{X_0 \times \{1\}} X_1 \times [1,2] \coprod_{X_1 \times \{2\}} X_2 \times [2,3] \coprod_{X_2 \times \{3\}} X_3 \times [3,4] \coprod \cdots .$$

By a similar argument and induction one can prove that if all the maps in the sequence $X_0 \to X_1 \to \cdots$ are cofibration then the colimit of the sequence $\mathrm{colim}(X_i)$ is weakly equivalent to its homotopy colimit $\mathbb{L}colim X_i$.

Similarly a homotopy limit of $\ldots Y_2 \to Y_1 \to Y_0$ by replacing each map by a fibration and taking the limit hence as a limit of path spaces.

6.

a. Let

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "f"] & M' \arrow[d, "f'"] \\ N \arrow[r, "g'"] & N' \end{tikzcd}$$

be a pushout diagram in $Ch(A)$ where g is assumed to be a cofibration and f is weak-equivalence. We must show that f' is a weak-equivalence. But notice that as g is a cofibration and therefore injective, we have a short exact sequence of chain complexes and therefore long exact sequence of homology groups, and finally we have maps of exact sequences

$$\begin{tikzcd} H_{n+1}(M'/M) \arrow[r] \arrow[d] & H_n(M) \arrow[r] \arrow[d] & H_n(M') \arrow[r] \arrow[d] & H_n(M'/M) \arrow[r] \arrow[d] & H_{n-1}(M) \arrow[d] \\ H_{n+1}(N'/N) \arrow[r] & H_n(N) \arrow[r] & H_n(N') \arrow[r] & H_n(N'/N) \arrow[r] & H_{n-1}(N) \end{tikzcd}$$

where the first and fourth vertical maps are isomorphisms because the diagram is a pushout and the second and last vertical maps are isomorphisms because f is a weak-equivalence. So f' is also a weak-equivalence.

b. Note that in degree n, one has $(I^1 \otimes A)_n = A_n \oplus A_n \oplus A_{n-1}$. The formula given for the differential gives $d(x, y, w) = \big(\partial_A(x) - z, \partial_A(y) + z, -\partial_A(z)\big)$. Hence the pushout $Cyl(g) := B \coprod_A I^1 \otimes A$ is given in degree n by $B_n \oplus A_n \oplus A_{n-1}$ and the map $I^1 \otimes A \to B \coprod_A I^1 \otimes A$ is given in degree n by $(x, y, w) \mapsto (g(y), x, w)$. Thus the differential on the pushout $Cyl(g)$ is given by

$$(b, x, w) \mapsto \big(\partial_B(b) + g(w), \partial_A(x) - w, -\partial_A(z)\big).$$

The formula for the differential of $I^1 \otimes A$ above shows that their linear maps $(I^1 \otimes A)_n = A_n \oplus A_n \oplus A_{n-1} \to A_n \oplus A_{n-1}$ given by $(x, y, z) \mapsto (y, z)$ defines a chain map $t : I^1 \otimes A \to C(Id_A)$.

Now we compute the pushout $Cyl(g) \coprod_{I^1 \otimes A} C(Id_A)$. In degree n, we have

$$\big(Cyl(g) \coprod_{I^1 \otimes A} C(Id_A)\big)_n = (B_n \oplus A_n \oplus A_{n-1}) \oplus (A_n \oplus A_{n-1})/(g(y), x, w, 0, 0) \sim (0, 0, 0, y, w)$$

and hence it is isomorphic to $B_n \oplus A_{n-1}$ (the terms corresponding to x being killed off in the quotient). The differential then reads $(b, w) \mapsto (\partial_B(b) + g(w), -\partial_A(w))$ which proves that the pushout is indeed the cone $C(g)$.

c. A functor from Δ^1 to any category is simply the data of two objects and one morphism between them, that is the data of an arrow $A \overset{g}{\to} B$. A map between functors is simply a natural transformation thus a commutative diagram
$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ A' \arrow[r, "g'"] & B' \end{tikzcd}$$
. Now if we are in chain complexes, the linear maps $\beta \oplus \alpha : B_n \oplus A_{n-1} \to B'_n \oplus A'_{n-1}$ are a map $C(g) \to C(g')$ of chain complexes (because α and β commutes with differential and the diagram is commutative). And it is easy to check that this assignement does make $g \mapsto C(g)$ into a functor $Fun(\Delta^1, Ch(R)) \to Ch(R)$. It reamains to prove it send objectwise weak equivalences to weak equivalence. To see this, we note that given $f : A \to B$ one has an exact sequence of complexes $0 \to B \to C(f) \to A[1] \to 0$. Hence a map of morphisms produces a map of exact sequences and if the maps are quasi-isomorphisms, by the five-lemma, the middle terms will also be.

d. Take $u_a : A' \overset{\sim}{\twoheadrightarrow} A$ a cofibrant replacement of A. Then choose a factorization of $A' \to A \to B$ as a cofibration $g' : A' \rightarrowtail B'$ followed by a acyclic fibration $u_b : B' \overset{\sim}{\twoheadrightarrow} B$ and set $u_c : 0 \to 0$ as the identity. This gives us the required natural transformation with g' a cofibration as we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[d, equal] & A' \arrow[l] \arrow[r, tail, "g'"] \arrow[d, "u_a"', "\wr"] & B' \arrow[d, "\wr"', "u_b"] \\ 0 & A \arrow[l, "g"'] \arrow[r] & B \end{tikzcd}$$

e. First note that the composition $A \overset{i_0}{I}^{1} \otimes A \to C(Id_A)$ is given $y \mapsto (0, y, 0) \mapsto$ Since g is assumed to be a cofibration and cofibrations are stable under pushouts, by definition of the mapping cylinder, the map $I^1 \otimes A \to Cyl(g)$ is a cofibration as well. Now we are only left to prove the vertical arrows in the diagram
$$\begin{tikzcd} 0 & A \arrow[l] \arrow[r, tail, "g"] \arrow[d, equal] & B \arrow[d, equal] \\ C(Id_A) \arrow[u] \arrow[d, equal] & A \arrow[l, "t\circ i_0"'] \arrow[r, tail, "g"] \arrow[d, "i_0"'] & B \arrow[d] \\ C(Id_A) & I^1 \otimes A \arrow[l, "t"'] \arrow[r, tail] & Cyl(g) \end{tikzcd}$$
are weak equivalences. For the lower right one, it follows by left properness once we prove that i_0 is. Note that the linear maps $s : I^1 \otimes A \to A$ given in degree n by $s(x, y, w) = x + y$ are a chain complex morphism. Further $s \circ i_0 = Id_A$. To prove that i_0 is a quasi-isomorphism it is thus enough to prove that $i_0 \circ s$ induces the identity in homology and thus it is enough to prove it is homotopic to the identity of $I^1 \otimes A$. Let $h : I^1 \otimes A \to I^1 \otimes A[-1]$ be given by $h(x, y, w) = (0, 0, x)$. Then $dh + hd(x, y, w) = (-x, x, -w) = -(x, y, w) + i_0 \circ s(x, y, w)$ which proves that h is indeed a chain homotopy in between Id and $i_0 \circ s$. Finally, since $C(Id_A)$ is acyclic[2] the upper left vertical map is a quasi-isomorphism.

f. Given Y we can apply (d) to find $Y' \overset{\sim}{\to} Y$ where Y' is of the form $Y' = ($ $\begin{tikzcd} 0 & A' \arrow[l] \arrow[r, tail, "g'"] & B' \end{tikzcd}$ with A' and B' cofibrant. The natural transformation $Y' \overset{\sim}{\to} Y$ being a weak equivalence, it induces a quasi-isomorphism of the homotopy colimits of Y' to the one of Y. Further, by (c) the

[2] as follow from the long exact sequence since id is a quasi-isomorphism

mapping cone of Y' and Y are weak-equivalent. Thus it is enough to prove that the mapping cone of $g' : A' \to B'$ is quasi-isomorphic to the homotopy pushout of the diagram Y'. In other words, we are left to prove (f) in the case where $g; A \to B$ is a cofibration and A, B are cofibrant, which are exactly the assumptions of (e). Now because of left properness we know that the the pushout of such a Y is a homotopy pushout. For the same reason, this is also the case for each horizontal diagram in the string of weak equivalences $Y \overset{\sim}{\leftarrow} Y' \overset{\sim}{\rightarrow} Y$ given by (e). The vertical maps being all weak equivalences we thus have that the homotopy pushout of the top horizontal diagram is equivalent to the one of the lower horizontal diagram. The later is thus the same (again by left properness) as the pushout $C(Id_A) \xleftarrow{t} I^1 \otimes A \rightarrowtail Cyl(g)$, that is by definition the mapping cone $C(g)$ of the original map g while the first was by above quasi-isomorphic to the homotopy pushout.

Exercice 3 (**Bad behavior of Gabriel-Zisman Localization**)**.** Let A be a ring and let $D(A) := \mathbf{Ho}(Ch(A))$ denote the derived category of A; it is the Gabriel-Zisman localization of the category $Ch(A)$ of chain complexes in A localized along quasi-isomorphisms of complexes. We have seen in class that $D(A)$ is the homotopy category of a model structure in $Ch(A)$ with weak-equivalences given by quasi-isomorphisms and fibrations given by levelwise surjections.

1. Show that if E and H are two A-modules seen as complexes concentrated in degree zero, then

$$Hom_{D(A)}(E, H[n]) \simeq Ext^n_A(E, H)$$

2. Show that if A is a field then $D(A)$ is an abelian category[3], equivalent to the category $A^{\mathbb{Z}}$ of $\mathbb{Z}$-graded A-vector spaces.

3. Show that $D(A[X])$ does not admit colimits in general (Hint: Take a non-trivial element $f : A \to A[1]$ and show that if it has a kernel then we get to a condradiction with the fact f is non-trivial);

4. Let A be a field and let I be the category with one object and $\mathbb{N}$ as endomorphisms. Show that $Fun(I, D(A))$ is not equivalent to $D(Fun(I, Ch(A))$. The conclusion is that the theory of diagrams does not interact well with derived categories.

Solution 3. 1. By the fundamental theorem for computing morphisms int he homotopy category, and as every object is fibrant in the projective model structure, $Hom_{D(A)}(E, H[n])$ is in bijection with the set of homotopy classes of maps $Q(E) \to H[n]$ with $Q(E)$ a cofibrant resolution of E. As ,we have seen in class, a projective resolution of A (which is bounded below) is in particular a cofibrant resolution, hence we can take $Q(E) = P$ any projective resolution of E. Then this hom-set is by definition $Hom_{Ch(A)}(P, H[n])/\simeq$ and thus, since $H[n]$ is concentrated in positive degree n, it is the quotient of the set $Z^n(E, H)$ of linear maps $f : E \to H$ such that $E_{n+1} \to E_n \overset{f}{\to} H$ is zero. In other words, this is a degree n-cocyle in the cochain complex $Hom_A(P, H)$. Now, since P is cofibrant and $H[n]$ is fibrant, two maps $f, g : E \to H[n]$ are homotopy equivalent if and only if they are right homotopy equivalent.

As in Exercise 2, we have a special path object for H; namely $H^I := Hom_A(I^1, H)$ which the chain complex given by $H \oplus H$ in degree 0 and H in degree -1 with differential given by $H \oplus H \to H$ given by $(x, y) \mapsto x - y$. Then we have a chain map $H^I \to H \times H$ given by the dual of i_0 and i_1, which is just the identity map in degree 0 and (necessary) 0 elsewhere. It is surjective levelwise hence a fibration. Finally we also have a canonical map $H \to H^I$ given, in degree 0 by $r \mapsto (r, r)$ and 0 elsewhere. This map is indeed a chain map since $d(r, r) = 0$. Thus H^I with the above maps is a path object for H and so is $H^I[n]$ for $H[n]$.

[3]see links to Homological algebra exercises on the web page if you are not familiar with this

We now prove that if $f \overset{r}{\sim} g$ then there is a right homotopy from f to g with $H^I[n]$ as a cylinder. Indeed, let $R_H \twoheadrightarrow H[n] \times H[n]$ be a path object and $\alpha : P \to R_H$ be a right homotopy. We can factor the structure map $H \overset{\sim}{\to} R_H$ as $H \overset{\sim}{\rightarrowtail} \tilde{R}_H \overset{\sim}{\twoheadrightarrow} R_H$ by the factorisation axiom and the 2 out 3 property. Then the map $\tilde{R}_H \overset{\sim}{\twoheadrightarrow} R_H \twoheadrightarrow H \times H$ makes $\tilde{R}_H$ a path object for H. Since P is cofibrant and $\tilde{R}_H \overset{\sim}{\twoheadrightarrow} R_H$ is an acyclic fibration, the lifting property ensures that there is a lifting $\tilde{\alpha}$ of α:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \tilde{R}_H \arrow[d, "\wr"] \\ P \arrow[r, "\alpha"'] \arrow[ur, dotted, "\tilde{\alpha}"] & R_H \end{tikzcd}$$

and thus we have an homotopy between f, g out of the path object $\tilde{R}_H$. Now the commutative square

$$\begin{tikzcd} H \arrow[r, "\sim"] \arrow[d, tail, "\wr"'] & H^I[n] \arrow[d] \\ \tilde{R}_H \arrow[r] \arrow[ur, dotted] & H[n] \times H[n] \end{tikzcd}$$

provides a map $\tilde{R}_H \to H^I$ so that the composition $P \overset{\tilde{\alpha}}{\to} \tilde{R}_H \to H^I[n]$ is a right homotopy from f to g.

Now we just have to identify what it means to be a right homotopy $P \to H^I$. For degree reason, it has only two possible non-zero components given by a linear map $P_n \to H \oplus H$ which has to be (f, g) (since it is an homotopy) and a map $h : P_{n-1} \to H$. Since the map has to be a chain map, we get that $f - g = h \circ d_{P_n}$ where $d_{P_n} : P_n \to P_{n-1}$ is the differential. In other words, two maps $f, g : P_n \to H$ are right homotopic if they differ by a coboundary in the chain complex $Hom_A(P, H)$. Thus we have indeed a canonical isomorphism $Hom_{D(A)}(E, H[n]) \cong H^n(Hom_A(P, H)) = Ext^n_A(E, H)$

2. One checks that the functor sending a complex $(M_k, \partial_k) \in Ch(A)$ to the $\mathbb{Z}$-graded module $l(M) := \bigoplus_{i \in \mathbb{Z}} H_i(M)$ sends quasi-isomorphisms to isomorphisms and therefore induces a functor $l : D(A) \to A^{\mathbb{Z}}$. One can produce a candidate for the inverse: given a $\mathbb{Z}$-graded module K we consider the associated chain complex with zero differentials $(K, 0)$. This gives a natural functor $A^{\mathbb{Z}} \to Ch(A)$ and we set $t : A^{\mathbb{Z}} \to D(A)$ as the composition with the localization functor. Let us show that l and t form an equivalence of categories. Clearly, the composition $l \circ t$ is isomorphic to the identity. We are left to construct a natural isomorphism between $Id_{D(A)}$ and $t \circ l$, meaning, we should exhibit functorial isomorphisms in $D(A)$ between (M_k, ∂_k) and $(H^k(M), 0)$. For that purpose we construct two morphisms in $Ch(A)$, $f_M : (M_k, \partial_k) \to (H^k(M), 0)$ and $g_M : (H^k(M), 0) \to (M_k, \partial_k)$ which we prove to be isomorphisms in $D(A)$ and behave functorialy with respect to M. To define them let us notice that we always have (by definition) short exact sequences

$$0 \to Ker\partial_n \to M_n \to^{\partial_n} Im\partial_n \to 0$$

and

$$0 \to Im\partial_{n+1} \to Ker\partial_n \to H_n(M) \to 0$$

and as we are working over a field, both exact sequences split (In fact, this proof also shows more generally that the derived category of any semi-simple abelian category is abelian). In this case we have isomorphisms $M_n \simeq Im\partial_{n+1} \oplus H_n(M) \oplus Im\partial_n$ and under this identification the map differential map $M_{n+1} \to M_n$ is identified with the map $(a, b, c) \mapsto (c, 0, 0)$. Now we define the map $f^n_M : M_n \to H_n(M)$ as the projection and the map $g^n_M : H_n(M) \to M_n$ as the inclusion. One checks that these maps are quasi-isomorphisms and therefore become isomorphisms in $D(A)$. Moreover, the choice of the splittings can be made in such a way that these maps provide a natural isomorphism in the homotopy category: has to check that after passing to the homotopy relation on morphisms the choice of the splitting does not matter.

3. As we have seen in 1., we have $Hom_{D(A[X])}(A, A[1]) = Ext^1_{A[X]}(A, A) \simeq A$ (we leave this computation as an exercise: take the obvious projective resolution $A[X] \overset{\times X}{\to} A[X]$). Let $1 \in A$ and take $f : A \to A[1]$ be a $D(A[X])$ non zero map (corresponding to $1 \in A$ in the above identification of Ext. Assume there is a kernel $Kerf$. Then by definition of a kernel, we have a long exact sequence of abelian groups

$$0 \to [A[X], kerf[i]] = H^i(Kerf) \to [A[X], A[i]] = H^i(A) \to [A[X], A[i+1]] = H^{i+1}(A) = 0$$

This implies $Kerf$ is quasi-isomorphic to A hence $f = 0$. This is absurd.

4. Assume A is a field. Let I be the category with one object and $\mathbb{N}$ as its monoid of endomorphism. Then the category $Fun(I, C(A))$ is isomorphic to $C(A[X])$ hence $D(Fun(I, C(A))) \simeq D(A[X])$ which by the previous question has no (co)limits in general. However, since A is a field (in particular is semi-simple) the category $Fun(I, D(A))$ is abelian since $D(A)$ is itslef abelian.

Exercice 4 (Small Object Argument). The goal of this exercise is to make precise the small object argument and how to use it in the case of/to construct *cofibrantly generated* model categories.

Let $\mathcal{C}$ be a category having all small colimits and I a small set of maps in $\mathcal{C}$. Suppose that the domains of the maps in $\mathcal{C}$ are compact objects [4]. We let $Cell(I)$ denote the class of maps in $\mathcal{C}$ obtained as transfinite compositions of pushouts of elements of I, ie, a map $f : A \to B$ is in $Cell(I)$ if there exists a diagram $X : \mathbb{N} \to \mathcal{C}$ such that f is the composition of X and for all $n < n+1$ the map $X_n \to X_{n+1}$ is obtained via a pushout

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X_n \arrow[r] & X_{n+1} \end{tikzcd}$$

where the map $U \to V$ is in I. Also, we denote by $LLP(I)$ (resp. $RLP(I)$) the collection of maps with the left (resp. right) lifting property with respect to I.

1. Show that $Cell(I)$ is contained in $LLP(RLP(I))$;

2. Show that $Cell(I)$ is closed under transfinite compositions.

3. Show that if K is a set and $\{u_k : A_k \to B_k\}_{k\in K}$ is a K-family of maps in I then any map f in $\mathcal{C}$ obtained as a pushout
$$\begin{tikzcd} \coprod_{k\in K} A_k \arrow[r] \arrow[d, "\coprod_{k\in K} u_k"'] & X \arrow[d, "f"] \\ \coprod_{k\in K} B_k \arrow[r] & Y \end{tikzcd}$$
is in $Cell(I)$;

4. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. Show that we can always factor f as a composition $\delta \circ \gamma : X \to Z \to Y$ where $\gamma : X \to Z$ is in $Cell(I)$ and $\delta : Z \to Y$ is in $RLP(I)$

5. Explain why this factorization process can be exhibited as a functor $Arr(\mathcal{C}) \to Arr(\mathcal{C}) \times_{\mathcal{C}} Arr(\mathcal{C})$ sending $f \mapsto (\gamma(f), \delta(f))$.

6. Use this factorization process to show that every map $f : X \to Y$ in $LLP(RLP(I))$ is a retract of a map $g : X \to C$ in $Cell(I)$, which fixes X.

Solution (partielle) 4. 1. This is done by induction using the universal property of pushouts and the definition of left lifting property and is analogous to the fact that maps with a left lifting properties are preserved along pushouts.

[4] Recall that an object X in a category $\mathcal{C}$ is said to be compact if the functor $Hom(X, -)$ commutes with filtered colimits.

2. This part is essentially set theoretic. The point is that if $F : \kappa \to \mathcal{C}$ is a sequence (indexed by an ordinal κ) of relative I-cell complexes. That is for each $k \in \kappa$, the map $F(k) \to f(k+1)$ is a relative cell complex, that is a diagram $F(k) = F(k)_0 \to F(k)_1 \to \cdots$ where each $F(k)_i \to F(k)_{i+1}$ is obtained via a pushout along a map in I and the sequence is indexed by τ_κ in general (and $\mathbb{N}$ in this exercise). Taking the lexicographic on the set of pairs (α, β) of ordinals such that $\alpha < \kappa$ and $\beta < \tau_\beta$ we obatin a well ordered set hence it is isomorphic to a unique ordinal. Using this ordinal we can redinex all the sequence into a transfinite compositions of pushouts (provided we get rid of useless isomorphisms). If we only consider things indexed by $\mathbb{N}$, then we still get a trasnfinite compositions indexed by integers.

3. This is again a game of reindexing the pushout so that it becomes a transfinite composition: since K is a set, it isomorphic to an ordinal (corresponding to its cardinal), written κ. Then we define a κ-sequence by setting $X_0 = X$ and then, for any $i \in \kappa$, we set $X_i \to X_{i+1}$ to be the pushout $B_{k_i} \coprod_{A_{k_i}} X_i$ along g_{k_i} (where $k_i \in K$ is the element corresponding to $i \in \kappa$).

4. Starting with f, the goal is in a way to find the best factorization of f having the lifting property with respect to all maps $g : A \to B$ in I, ie

$$\begin{tikzcd} A \arrow[r] \arrow[dd] & X \arrow[d] \\ & Z \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Let $g \in I$ and let $K(g)$ denote the set of all commutative diagrams

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\ B \arrow[r] & Y \end{tikzcd}$$

We define a new object X_1 as the pushout

$$\begin{tikzcd} \coprod_{g \in I} \coprod_{u \in K(g)} Source(g) \arrow[r] \arrow[d, "\coprod_{g \in I} g"'] & X \arrow[d, "f"] \\ \coprod_{g \in I} \coprod_{u \in K(g)} Target(g) \arrow[r] & X_1 \end{tikzcd}$$

By definition, it comes canonically equipped with a map $X_1 \to Y$ which provides a factorization of f. Moreover, by (3) the map $X \to X_1$ is in $Cell(I)$. We now iterate this construction setting X_2 as the result of applying this to the map $X_1 \to Y$. By induction what we obtain is a commutative diagram $X_\bullet : \mathbb{N} \to \mathcal{C}$ together with a natural transformation to the constant diagram $X_\bullet \to Y$ and such that each map $X_n \to X_{n+1}$ is in Cell(I). By (2), the map $X_0 \to Z := colim_{n \in \mathbb{N}} X_n$ is in $Cell(I)$. We claim that the canonical map $Z \to Y$ is in $RLP(I)$. Consider $g : A \to B$ a map in I. We want to show the lifting property

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "g"'] & Z := colim_{n \in \mathbb{N}} X_n \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

But as by assumption all objects in the source of maps inI are compact, given $h : A \to Z$ there exists an $N \in \mathbb{N}$ such that h factors through $u : A \to X_N$ (Notice that more generally we don't need the source objects to be compact, we only need them to verify this property). The the previous diagram can be written as

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[dd, "g"'] & X_N \arrow[d] \\ & Z \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

which we can also write as

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[dd, "g"'] & X_N \arrow[r] \arrow[dr] & X_{N+1} \arrow[d] \\ & & Z \arrow[d] \\ B \arrow[rr] & & Y \end{tikzcd}$$

where $X_N \to X_{N+1}$ is the canonical map. But now by construction of X_{N+1} out of X_N we know that we can find a lifting $B \to X_{N+1}$ that makes the diagram commute - take the canonical map in the pushout diagram defining X_{N+1}.

5. Since we have been using the small object argument and only natural construction, the whole process is natural.

6: given $f : X \to Y$ in $LLP(RLP(I))$, applying the factorization machine we get $X \to Z \to Y$ such that $Z \to Y$ has the right lifting property with respect to I. Then we can form the commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Z \arrow[d] \\ Y \arrow[r] & Y \end{tikzcd}$$

and because of the assumption we get a lifting $Y \to Z$ that makes the diagram commute. But then we have

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \arrow[r] & Y \end{tikzcd}$$

where the top maps are identities.

Exercice 5 (**Model structure on topological spaces).** The goal of this exercice is to show that the category of topological spaces, together with homotopy weak-equivalences, Serre fibrations (maps with the right lifiting property with respect to the inclusion $i_0 : D^n \mapsto D^n \times I$, $n \geq 0$) and cofibrations given by maps with a left lifting property with respect to acyclic Serre fibrations, forms a (cofibrantly generated) model category:

1. Show that the class of weak-equivalences satisfies the 2 out of 3 property;

2. Show that weak-equivalences, fibrations and cofibrations are stable under retracts;

3. Let $\mathcal{C}$ be a category and S be a class of maps. Show that $LLP(RLP(LLP(S))) = LLP(S)$ and that $RLP(LLP(RLP(S))) = RLP(S)$.

4. Let I' denote the collection of all boundary inclusions $\{\partial : S^{n-1} \hookrightarrow D^n\}_{n\geq 0}$ and J denote the collection of all maps $\{i_0 : D^n \hookrightarrow D^n \times [0,1], x \mapsto (x,0)\}_{n\geq 0}$. Notice that Serre fibrations are then defined as $RLP(J)$. Show that $J \subseteq Cell(I')$ and deduce that $LLP(RLP(J))) \subseteq LLP(RLP(I')))$.

5. Show that $LLP(RLP(J)) \subseteq \mathcal{W}$. Deduce that $Cell(J) \subseteq \mathcal{W} \cap LLP(RLP(I'))$.

6. Show that every map in $RLP(I')$ is a trivial Serre fibration.

7. Show that for every set A, the functor $Hom(A, -) : Sets \to Sets$ commutes with α-filtered colimits for some cardinal α. Use this to show that the small object argument can be applied both to the class I' and the class J because the elements of I' and J are inclusions of topological spaces. In this case, the transfinite induction won't be indexed by ω but by a larger ordinal.

8. Show that if $f : X \to Y$ is a trivial Serre fibration then it is in $RLP(I')$. In particular, we get $Cof = LLP(RLP(I'))$.

9. Conclude.

Solution (partielle) 5. A complete solution with the full level of details can be found in Hovey's book. In this sketch we will essentially ignore set-theoretical and size issues and only sketch the main arguments in the proof.

1. The point is that the 2-out of 3 property is trivial for isomorphisms. Thus we can we deduce the property for π_0 since it is a matter of bijections and then choosing any base point we use again the 2 out of 3 property for isomorphisms of groups to deduce the property for all π_k

2. For weak-equivalences this is essentially the same proofs as for quasi-isomorphisms in the projective model structure seen in class. Again, passing to homotopy set/groups reduces the question to a question of retracts of isomorphisms of sets/groups which are isomorphisms of sets as we have seen in the projective model structure proof. For fibrations this follows because fibrations are by defined as a class of maps with a RLP with respect to the inclusions $D^n \to D^n \times [0,1]$ and as we have seen in class, maps defined by lifting properties either right or left are stable under retracts. For cofibrations this follows because they are defined by a left lifting property in the very same way.

3. Obviously for any class of map K, $LLP(RLP(K)) \supset K$ since all maps in K have by definition the left lifting properties with respect to the maps in $RLP(K)$. Similarly $RLP(LLP(K)) \supset K$. Now if $K \subset J$, then any maps which has lifting properties with respect to all maps in J has the lifting properties with respect to those of K. Hence $LLP(K) \supset LLP(J)$ and $RLP(K) \supset RLP(J)$. Thus taking $K = LLP(S)$ we get $LLP(RLP(LLP(S))) \supset LLP(S)$. On the other hand, taking $K = S$ and $J = RLP(LLP(S))$, we get $LLP(S) \supset LLP(RLP(LLP(S)))$ hence the equality. The same works for the other equality.

4. Let us first show that $J \subseteq Cell(I')$. This follows because the maps in J are relative CW-complexes which are particular cases of I'-cell objects by definition. Let us show the second claim: As seen in the exercise on the small object argument, $Cell(I') \subseteq LLP(RLP(I')))$ so that we deduce $J \subseteq LLP(RLP(I')))$. This implies that $LLP(RLP(J))) \subseteq LLP(RLP(LLP(RLP(I'))))) = LLP(RLP(I'))$.

5. The maps in J are homotopy deformation retracts, meaning, the inclusions $i_0 : D^n \to D^n \times [0,1]$ admits a retract r and the the composition $r \circ i_0$ is homotopic to the identity. In particular, each map i_0 is a weak-equivalence. The main claim is that homotopy deformation retracts are stable under pushouts. Indeed, Suppose

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\ B & \longrightarrow & D \end{array}$$

is a pushout with i an homotopy deformation retract. Then the product

$$\begin{tikzcd} A \times I \arrow[r, "f"] \arrow[d, "i"] & C \times I \arrow[d] \\ B \times I \arrow[r] & D \times I \end{tikzcd}$$

is also a pushout. It remains to use the homotopy $H : B \times I \to B$ and the composition $C \times I \to C \to D$ where the first is the projection and the second is j, to deduce a map from $D \times I \to D$ by the universal property of the colimit. Here we just need to remark that by definition of homotopy deformation retract, the homotopy H fixes A. Finally, using this one obtains that all maps in $Cell(J)$ are homotopy equivalences and using the fact that $LLP(RLP(J))$ are (see (6) of previous exercise) retracts of maps in $Cell(J)$, and weak-equivalences are stable under retracts, one concludes the proof. The second statement follows from (1) of the previous exercise and from the combination of the result in this item and the previous item.

6. Since $LLP(RLP(J))) \subseteq LLP(RLP(I')))$ we deduce that

$$RLP(J) = RLP(LLP(RLP(J)))) \supseteq RLP(LLP(RLP(I')))) = RLP(I')$$

so that every map in $RLP(I')$ is a Serre fibration. It remains to show it is also a weak-equivalence. Let $f : X \to Y$ in $RLP(I')$. One must show that for any $x \in X$ the induced maps $\pi_k(f) : \pi_k(X, x) \to \pi_k(Y, f(x))$ is a bijection. To show it is surjective, notice that as the map $\partial : S^{n-1} \to D^n$ is in I', then the pushout $* \to S^n$ is in $LLP(RLP(I'))$ so that if $f : X \to Y$ is in $RLP(I')$ any square

$$\begin{tikzcd} * \arrow[r, "x"] \arrow[d] & X \arrow[d, "f"] \\ S^n \arrow[r] & Y \end{tikzcd}$$

as a lifting so that the maps on homotopy groups are surjective. To show that the maps are injective: suppose we have two maps $(S^n, *) \to (X, x)$ representing the same homotopy class in $\pi_n(Y, f(x))$. Then we have an homotopy $H : S^n \times I \to Y$ between the two maps and we can build up a commutative diagram

$$\begin{tikzcd} S^n \vee S^n \arrow[r, "x"] \arrow[d] & X \arrow[d, "f"] \\ S^n \wedge I \arrow[r] & Y \end{tikzcd}$$

and the left vertical map is in $LLP(RLP(I'))$ because it can be shown to be an I'-cell map obtained by attaching a n+1-disk to $S^n \vee S^n$.$(S^n \wedge I = S^n \times I / * \times I)$

7. In fact, essentially, we only need to use the ordinal associated to the cardinal of the set underlying the source topological space. Then we can apply this to our maps defining I' and J since they are the topoogical spaces D^n and S^{n-1} which have the same cardinal.

8. We want to use the main theorem we have seen in class to guarantee that **Top** with weak equivalences and generating set of cofibrations I' and generating acyclic cofibrations J is a cofibrantly generated model category, whose undelrying model structure is Quillen model structure. This last point follows by (4) $Fib = RLP(J)$ and by (8) $Fib \cap W = RLP(I')$. In particular, we already have that $Cof := LLP(Fib \cap W) = LLP(RLP(I')) = I' - Inj$.

Now we have to check the assumptions of the Theorem: the (co)completness of **Top** is given by (co)limit topologies on underlying (co)limits and the cardinality issues of the source of the generating (acyclic) cofibrations have been checked in (7).

It remains to show $Cell(J) \subset W \cap LLP(RLP(I'))$ which is exactly (5). Similarly, we have to prove that $I' - Inj = RLP(I') \subset W \cap J - Inj = W \cap RLP(J)$. But since $RLP(J)$ are Serre fibrations by definition, the result is precisely (6).

Now, it only remains to show $Cof \cap W \subset LLP(RLP(J))$. Let $f : X \to Y$ be in $Cof \cap W$ and use the small object argument applied to J to produce a factorization $u \circ v : X \to X' \to Y$ with v in $Cell(J) \subseteq LLP(RLP(J))$ and $u \in RLP(J)$. As we already know $LLP(RLP(J)) \subseteq W$ we have that v is in W. By the 2 out of 3 property, as f is assumed to be in W, u is in W. So $u \in W \cap RLP(J) = RLP(I')$. As f is supposed to be a cofibration and we already know that $Cof = LLP(RLP(I'))$ we deduce the existence of a map $Y \to X'$ which splits the factorization. This makes f a retract of v and as v is in $LLP(RLP(J))$ and this class is stable under retracts, we deduce that f is also in $LLP(RLP(J))$.

Exercice 6 (**Top is proper**)**.** We endow **Top** with Quillen model category structure.

1. Prove that the category is right proper, that is that the pullback of a weak-equivalence under a fibration is a weak equivalence.

2. Prove that **Top** is left proper that is that the pushout of a weak equivalence by a cofibration is a weak equivalence.

Solution 6. 1. Consider a pullback diagram

$$\begin{tikzcd} C \times_Y X \arrow[r, "f"] \arrow[d, two heads] & X \arrow[d, two heads] \\ C \arrow[r, "\sim"] & Y \end{tikzcd}$$

where the lower map is a weak equivalence. Since it is a pullback of a fibration the left vertical map is also a fibration and the diagram induces a map of fibration. Since the diagram is a pullback, the induced map on the fibers are homeomorphisms. Since the lower map is a weak equivalence, passing to the long exact homotopy sequences of Serre fibrations, we get a commutative diagram

$$\begin{tikzcd} \dots \arrow[r] & \pi_i(F, f_0) \arrow[r] \arrow[d, "\cong"] & \pi_i(C \times_Y X, *) \arrow[r] \arrow[d] & \pi_i(C, c_0) \arrow[r] \arrow[d, "\cong"] & \pi_{i-1}(F, f_0) \arrow[r] \arrow[d, "\cong"] & \dots . \\ \dots \arrow[r] & \pi_i(F, f_0) \arrow[r] & \pi_i(X, x_0) \arrow[r] & \pi_i(Y, y_0) \arrow[r] & \pi_{i-1}(F, f_0) \arrow[r] & \dots \end{tikzcd}$$

Then the 5-lemma ensures that $f : C \times_Y X \to X$ is a weak equivalence.

2. Let $\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f", "\wr"'] & B \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$ be a pushout square with $i : A \to B$ a cofibration and $f : A \to X$ a weak equivalence. We wish to prove that $f \coprod_A B : B \to Y$ is a weak equivalence. Note that $X \to Y$ is a cofibration as well since it is a pushout of cofibration. Since the cofibrations of the Quillen structure are retract of relative cell complexes (as the model category is cofibrantly generated), we have a retract $\begin{tikzcd} A \arrow[r] \arrow[d, tail, "j"] & T \arrow[r] \arrow[d, tail, "p"] & A \arrow[d, tail] \\ B \arrow[r, "k"] & U \arrow[r, "q"] & B \end{tikzcd}$ where the middle vertical map is a relative cell complex. Denoting $X \coprod_T U$ the pushout of the maps $T \overset{p}{\to} A \overset{f}{\to} X$ and $T \to U$, the universal property of pushout yield the dotted arrow and a commutative diagram

$$\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, "j"'] & B \arrow[d, "k"] \arrow[dddrr] & & \\
T \arrow[r, tail] \arrow[d, "p"'] & U \arrow[d, "q"] \arrow[ddr] & & \\
A \arrow[r, tail, "i"] \arrow[d, "f"'] & B \arrow[d] & & \\
X \arrow[r, tail] \arrow[rr, bend right] \arrow[rrr, bend right] & Y & X \coprod_T U \arrow[l, dotted] & Y \arrow[l, dotted]
\end{tikzcd} .$$

The uniqueness of the lift proves that the composition of the dotted arrow is the identity of Y, hence the induced commutative diagram

$$\begin{tikzcd}
B \arrow[r, "k"] \arrow[d] & U \arrow[r, "q"] \arrow[d] & B \arrow[d] \\
Y \arrow[r] & X \coprod_T U \arrow[r] & Y
\end{tikzcd}$$

is a retract so that it suffices to prove that $U \to X \coprod_T U$ is a weak equivalence to conclude for $B \to Y$. That is we are reduced to the case where $A \rightarrowtail B$ is a relative cell complex.

Let us first prove now that if B is obtained as a pushout $B \cong A \coprod_{S^{n-1}} D^n$ along an inclusion $S^{n-1} = \partial D^n \hookrightarrow D^n$, then the result holds. That is, since a pushout of pushout is a psuhout, we have to prove that the induced map $\tilde{f} : B \cong A \coprod_{S^{n-1}} D^n \overset{f \coprod_{S^{n-1}} D^n}{\longrightarrow} X \coprod_{S^{n-1}} D^n \cong Y$ is a weak-equivalence when f is. First, taking an open deformation retract of S^{n-1} in D^n and the complement $V = D^{n-1} \backslash S^{n-1}$, we have a decomposition $B = U_A \cup V$ where $U_A \supset A$ is a (strong) deformation retract of X in $B \cong A \coprod_{S^{n-1}} D^n$. There is a similar decomposition $Y = U_X \cup V$. Since V is a deformation retract of a point, it is an easy consequence of Van Kampen theorem to prove that the induced map $\tilde{f}_* : \pi_1(B, x) \to \pi_1(Y, \tilde{f}(x))$ is an isomorphism for any base point. It also induces a bijection on π_0. Now we can use a theorem in the lecture notes of the class asserting that to prove that $\tilde{f}$ is a weak homotopy equivalence it is now enough to prove in addition that $\tilde{f}$ induces an isomorphism in homology with local coefficient; the latter follows by the 5-lemma applied to the map between the Mayer-Vietoris long exact sequence applied to the same open decomposition of the two spaces. For the reader not so used to homology with local coefficient we give an alternate proof. First, working on each path component, we assume the spaces are arcwise connected. Let $T_B \overset{\sim}{\twoheadrightarrow} B$ be a cofibrant replacement so that T_B is a cell complex (we can even, as we have seen in class require that it is a CW-complex) and factor $T_B \to B \to Y$ into $T_B \overset{\bar{f}}{\rightarrowtail} T_Y \overset{\sim}{\twoheadrightarrow} Y$. By above, the map $\bar{f}$ is an isomorphism on π_0 and π_1 and it is enough to prove that $\overline{(f)}$ is a weak equivalence to ensure $\tilde{f}$ by the 2 out of 3 property. Take the universal covers of T_Y (the only reason we introduce them was to ensure we have universal covers) so that we have a pullback diagram

$$\begin{tikzcd}
\tilde{T_B} \arrow[r, "\check{f}"] \arrow[d, two heads] & \tilde{T_Y} \arrow[d, two heads] \\
T_B \arrow[r, "\bar{f}"] & T_Y
\end{tikzcd}$$

in which the vertical arrows are covering maps (by stability of covering maps under pullbacks), hence with discrete fibers. Since we already now that $\bar{f}$ is an isomorphism on π_1, we have that $\tilde{T_B}$ is simply connected and now by the long homotopy exact sequence of a fibration, it is enough to prove that $\check{f} : \tilde{T_B} \to \tilde{T_Y}$ is a weak homotopy equivalence. In other words, we are back to prove the result for a map between simply connected spaces. The pullbacks along $\tilde{T_B} \twoheadrightarrow T_B \overset{\sim}{\twoheadrightarrow} B$ of U_A and V gives an open covering $\tilde{U}_A$, $\tilde{V}$ of T_B and similarly for $\tilde{T_Y}$; further $\check{f}$ preserves the decomposition. Hence we can apply Mayer-Vietoris exact sequence to these covers. But since V is contractible (and that the restriction of an acyclic fibration is an acyclic fibration by pullback invariance of those) $T_{B|V}$ and $T_{Y|V}$ are weakly homotopy equivalent to a point. It follows that $\tilde{V} \cong V \times \pi_1(Y)$ is a trivialisable covering space and the restriction of $\check{f}$ to those open is thus a weak homotopy equivalence. Since U_A and U_X are deformation retracts of A, X respectively, and $f : A \to X$ is a weak equivalence, we obtain in a same way that $\tilde{U}_A \to \tilde{U}_X$ is a weak homotopy equivalence. Hence by Mayer Vietoris we get that $\check{f}$ is an isomorphism in all homology groups with coefficient in $\mathbb{Z}$ and by Whitehead theorem it is thus a weak homotopy equivalence as required.

By induction on the last step, we see that in the case where $A \rightarrowtail B$ is obtained by a finite number of attachment of cells, the induced map $B \to Y$ is a weak homotopy equivalence. Now assume $A \rightarrowtail B$ is any relative cell complex so that $f := B \cong \mathrm{colim}_\kappa B_k \xrightarrow{\mathrm{colim}_\kappa f_k} \mathrm{colim}_\kappa Y_k \cong Y$ where $B_k \overset{f_k}{Y_k}$ is the map induced by the cell attachments. We need to prove that $f_* : \pi_n(B, b_0) \to \pi_n(Y, f(b_0))$ is an isomorphism. But let $S^n \to B$ be a continuous map. By compacity of S^n and colimit topology, its image intersects only finitely many cells in B. The same holds too for Y and for any homotopy $S^n \times [0,1]$ to either B or Y. So that to prove injectivity and surjectivity of f_*, we are left to prove it for a finite attachment of cells for which we have already seen that f_* is an isomorphism.

Exercice 7. Let $\mathbf{Top}_*$ be the category of *pointed* topological spaces and $U : \mathbf{Top}_* \to \mathbf{Top}$ be the functor forgetting the base point.

1. Prove that U is a right adjoint and compute its left adjoint.
2. We endow $\mathbf{Top}$ with Quillen model structure. Find a model structure on $\mathbf{Top}_*$ such that U is right Quillen.
3. Generalize the previous construction to any model category $\mathcal{C}$?

Solution 7. 1. Let $U : \mathbf{Top}_* \to \mathbf{Top}$ be the forget functor which sends a pointed space/map to the underlying space/map of topological space. Let also $P : \mathbf{Top} \to \mathbf{Top}_*$ be the functor that sends a space X to $P(X) := X \coprod \{*\}$ where we take the additional point $*$ as the base point. Similarly $P(X \overset{f}{\to} Y)$ is the map that send $*$ onto $*$ and whose restriction to X is f. It is clear that it is continuous and defines a functor. Since X and $\{*\}*$ are open subsets in $P(X)$, a continuous pointed map from $X \coprod \{*\} \to (Y, y_0)$ is by definition a map sending $*$ to y_0 together with a continuous map from X to Y. Hence there is an natural bijection

$$Hom_{\mathbf{Top}_*}(P(X), Y_+) \cong Hom_{\mathbf{Top}}(X, U(Y, y_0))$$

which proves that $P : \mathbf{Top} \rightleftarrows \mathbf{Top}_* : U$ is an adjonction.

2. In order for U to be Quillen, we need to have a model structure in $\mathbf{Top}_*$ for which U preserves fibrations and acyclic fibrations. Let us define these classes in teh simplest possible way for this to work. We define a weak equivalence (resp. fibration, resp. cofibration) in $\mathbf{Top}_*$ to be a map f such that $U(f)$ is a weak equivalence (resp. fibration, resp. cofibration) in $\mathbf{Top}$. In general such a simple definition do not work[5], but here it will (essentially because limits and colimits in both categories are very close).

It is immediate to check the axiom MC2, and MC3 since the forget of a retract is a retract. For the lifting properties, note that if we have a commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] & X \arrow[d, two heads, "\wr"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

in $\mathbf{Top}_*$, then the underlying diagram is a square in $\mathbf{Top}$ with an acyclic fibration on the right and a cofibration on the left. Thus the lift, that is the dotted arrow, exists. The only question is wether or not it preserves the base point. But since the left vertical does and the top horizontal one as well, then necessarily it does. It is thus a lft in $\mathbf{Top}_*$. Of course, the other lifting property is the same. In teh same way assume

[5] but often imposing two classes do

$f : (X, x_0 \to (Y, y_0)$ is a pointed map and let $X \rightarrowtail C_f \overset{\sim}{\twoheadrightarrow} Y$ be a factorisation in **Top**. by setting the base point of C_f to be the image of x_0, then the factorosation is pointed and still natural.

Thus the only thing left is the (co)completness of the category. A limit of pointed topological space is naturally pointed (since the base point gives a canonical map from $\{*\}$ to every space in the diagram which does commute with all maps in the diagram since they are all pointed). The only difference is with the colimit. But colimit on $\mathbf{Top}_*$ are obtained as follows: if $\mathcal{D}$ is a diagram, consider the diagram $\mathcal{D}_+$ obtained by adding an object $+$ to $\mathcal{D}$ and exactly one map from $+$ to every object (that is we create an initial object in $\mathcal{D}$). To a diagram $X : \mathcal{D} \to \mathbf{Top}_*$ of pointed spaces we add $X(+) = \{*\}$ and $X(+ \to d) = * \mapsto X(d)_0$ the abse point of $X(d)$. It is a diagram of space and we set $\mathrm{colim}_{\mathbf{Top}_*}(X) := \mathrm{colim}_{\mathcal{D}_+} X$. This is well defined since the image of $\{*\} = X(+)$ defines a basepoint in $\mathrm{colim}_{\mathcal{D}_+} X$ and any natural transformation of diagram is pointed.

3. The idea is to define the category $\mathcal{C}_+$ to be the category whose objects are maps $* \to X$ in $\mathcal{C}$ from the terminal object to any object and maps $(* \xrightarrow{f} X) \to (* \xrightarrow{h} Y$ in $\mathcal{C}_+$ are just maps $g : X \to Y$ in $\mathcal{C}$ such that $h = g \circ f$ (draw the commutative triangle). Then the previous question is extended in analogous way.