The small object argument and Quillen model structure on **Top**

Exercice 1 (**Small Object Argument**)**.** The goal of this exercise is to make precise the small object argument and how to use it in the case of/to construct *cofibrantly generated* model categories.

Let $\mathcal{C}$ be a category that admits all small colimits and I a small set of maps in $\mathcal{C}$. Suppose that the domains of the maps in $\mathcal{C}$ are compact objects [1]. We denote by LLP(I) (resp. RLP(I)) the collection of maps with the left (resp. right) lifting property with respect to I. We let Cell(I) denote the class of maps in $\mathcal{C}$ obtained as transfinite compositions of pushouts of elements of I. More explicitly, a map in Cell(I) is of the form $X_0 \to \operatorname*{colim}_{\alpha<\lambda} X_\alpha$, where λ is an ordinal and $X\colon \lambda \to \mathcal{C}$ is a functor with the following two properties:

- for all $\alpha < \alpha+1 < \lambda$, the map $X_\alpha \to X_{\alpha+1}$ is obtained as a pushout
$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X_\alpha \arrow[r] & X_{\alpha+1} \end{tikzcd}$$
of a map $U \to V$ in I,
- for all limit ordinal $\beta < \lambda$, one has $X_\beta \cong \operatorname*{colim}_{\alpha<\beta} X_\alpha$.

1. Consider a family $\{u_k : A_k \to B_k\}_{k\in K}$ of maps in I, indexed by a set K. Show that any map f in $\mathcal{C}$ obtained as a pushout
$$\begin{tikzcd} \coprod_{k\in K} A_k \arrow[r] \arrow[d, "\coprod_{k\in K} u_k"'] & X \arrow[d, "f"] \\ \coprod_{k\in K} B_k \arrow[r] & Y \end{tikzcd}$$
is in Cell(I).
2. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. Show that we can fonctorially factor f as a composition $\delta \circ \gamma : X \to Z \to Y$, with $\gamma : X \to Z$ in Cell(I) and $\delta : Z \to Y$ in RLP(I).
3. Use this factorization process to prove that every map $f : X \to Y$ in LLP(RLP(I)) is a retract of a map $g : X \to C$ in Cell(I), which fixes X.

Solution 1. 1. The goal is to express a pushout of coproducts of maps in I as a transfinite composition of pushouts of maps in I. Using the axiom of choice, the set K can be given a well-ordering, that is there exists a bijection between K and its cardinal $\kappa = \mathrm{Card}(K)$. Fix such a bijection $\kappa \to K, \alpha \mapsto k_\alpha$. We now define a functor $Y\colon \kappa \to \mathcal{C}$ inductively, so that the resulting $Y_0 \to \operatorname{colim}_{\alpha<\kappa} Y_\alpha$ is f. First, we set $Y_0 := X$. For a sucessor ordinal $\alpha+1 < \kappa$, we define $Y_\alpha \to Y_{\alpha+1}$ as in the following pushout
$$\begin{tikzcd} A_{k_\alpha} \arrow[r] \arrow[d, "u_{k_\alpha}"] & \coprod_k A_k \arrow[r] & X \arrow[r] & Y_\alpha \arrow[d] \\ B_{k_\alpha} \arrow[rrr] & & & Y_{\alpha+1}. \end{tikzcd}$$

[1] Recall that an object X in a category $\mathcal{C}$ is said to be compact if the functor $Hom(X,-)$ commutes with filtered colimits.

Finally, if $\lambda < \kappa$ is a limit ordinal, we define $Y_\lambda := \operatorname{colim}_{\beta<\lambda} Y_\beta$ (the map $Y_\alpha \to Y_\lambda$ is then defined as the canonical map to the colimit). The whole construction yields a map $f' \colon X = Y_0 \to \operatorname{colim}_{\alpha<\kappa} Y_\alpha$. Together with the natural morphism $\coprod_k B_k \to \operatorname{colim}_{\alpha<\kappa} Y_\alpha$, f' satisfies the universal property of the pushout, hence f' is canonically isomorphic to f.

2. This question is only a rewording of the small object argument described in the lecture. The key point is to ensure that the map γ in the factorization of f is indeed in $\mathrm{Cell}(I)$. By definition, γ is defined as a transfinite composition (indexed by ω)[2] of pushouts of *coproducts* of maps in I. For γ to be in $\mathrm{Cell}(I)$, we need it to be a transfinite composition of pushouts of maps in I. By question **1.**, each pushout of coproducts of maps in I can be rewritten as a transfinite composition of pushouts of maps in I. Therefore, we know that γ is a transfinite composition, where each step is itself a transfinite composition of pushouts of maps in I. Now the result follows from the following observation.

Lemma 1. *Let J be a class of morphisms in $\mathcal{C}$ (e.g. the class of pushouts of maps in I). Suppose $F \colon \kappa \to \mathcal{C}$ is a sequence of transfinite compositions of morphisms in J. Then F can be written as a sequence of morphisms in J.*

We sketch a proof of the lemma. By assumption on F, for each $\alpha < \kappa$, the morphism $F_\alpha \to F_{\alpha+1}$ is of the form $F_{\alpha,0} \to \operatorname{colim}_{\beta<\kappa_\alpha} F_{\alpha,\beta}$ for some sequence $F_\alpha \colon \kappa_\alpha \to \mathcal{C}$. Now, consider the set of pairs (α,β) of ordinals with $\alpha < \kappa$ and $\beta < \kappa_\alpha$. When endowed with the lexicographic order, this set is well-ordered, hence is isomorphic to a unique ordinal λ. This allows to rewrite F as a unique sequence $F' \colon \lambda \to \mathcal{C}$ of elements in J, by which we mean that the map $F_0 \to \operatorname{colim}_{\alpha<\kappa} F_\alpha$ is isomorphic to $F'_0 \to \operatorname{colim}_{\nu<\lambda} F_\nu$.

3. Given $f : X \to Y$ in $\mathrm{LLP}(\mathrm{RLP}(I))$, applying the factorization machine we get $X \to Z \to Y$ such that $Z \to Y$ has the right lifting property with respect to I. Then we can form the commutative square

$$\begin{tikzcd} X \arrow[r, "\gamma"] \arrow[d, "f"'] & Z \arrow[d, "\delta"] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$

and because of the assumption we get a lifting $h \colon Y \to Z$ that makes the diagram commute. But then we have

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[r, equal] \arrow[d, "\gamma"] & X \arrow[d, "f"] \\ Y \arrow[r, "h"] \arrow[rr, equal, bend right] & Z \arrow[r, "\delta"] & Y, \end{tikzcd}$$

which exhibits f as a retract of $\gamma \colon X \to Z$, fixing X.

Exercice 2 (**Model structure on topological spaces).** The goal of this exercice is to show that the category of topological spaces, together with weak homotopy equivalences, Serre fibrations (maps with the right lifiting property with respect to the inclusion $D^n \to D^n \times I$, $n \geq 0$) and cofibrations given by maps with a left lifting property with respect to acyclic Serre fibrations, forms a cofibrantly generated model category.

1. Show that the class of weak equivalences satisfies the 2-out-of-3 property.

2. Show that weak equivalences, fibrations and cofibrations are stable under retracts.

3. Let $\mathcal{C}$ be a category and S be a class of maps. Show that $\mathrm{LLP}(\mathrm{RLP}(\mathrm{LLP}(S))) = \mathrm{LLP}(S)$ and that $\mathrm{RLP}(\mathrm{LLP}(\mathrm{RLP}(S))) = \mathrm{RLP}(S)$.

[2]Here, ω denotes the first infinite ordinal, which can be viewed as the set $\mathbb{N}$ of natural numbers with its usual order.

Let I' denote the collection of all boundary inclusions $\{\partial_n : S^{n-1} \hookrightarrow D^n\}_{n\geq 0}$ and J denote the collection of all maps $\{j_n : D^n \hookrightarrow D^n \times [0,1], x \mapsto (x,0)\}_{n\geq 0}$. Note that by definition, the class of Serre fibrations is $\mathrm{RLP}(J)$.

4. Show that $J \subseteq \mathrm{Cell}(I')$ and deduce that $\mathrm{LLP}(\mathrm{RLP}(J))) \subseteq \mathrm{LLP}(\mathrm{RLP}(I')))$.

5. Show that $\mathrm{LLP}(\mathrm{RLP}(J)) \subseteq \mathcal{W}$. Deduce that $\mathrm{Cell}(J) \subseteq \mathcal{W} \cap \mathrm{LLP}(\mathrm{RLP}(I'))$.

6. Show that every map in $\mathrm{RLP}(I')$ is a trivial Serre fibration.

7. Show that for every set A, the functor $\mathrm{Hom}(A, -) : \mathbf{Set} \to \mathbf{Set}$ commutes with α-sequential colimits for some cardinal α. Use this to prove that every topological space is small with respect to inclusions.
Deduce that the small object argument can be applied both to the class I' and the class J.
NB: In this case, the transfinite induction won't be indexed by ω but by a larger ordinal.

8. Show that if $f : X \to Y$ is a trivial Serre fibration, then f is in $\mathrm{RLP}(I')$. In particular, we get $\mathrm{Cof} = \mathrm{LLP}(\mathrm{RLP}(I'))$.

9. Conclude.

Solution (partielle) 2. For more details on this proof, we refer to the book *Model categories* by Hovey, paragraphs 2.1 and 2.4.

1. The point is that the 2-out-of-3 property is trivial for isomorphisms. Thus we can we deduce the property for π_0 since it is a matter of bijections and then choosing any base point we use again the 2-out-of-3 property for isomorphisms of groups to deduce the property for all π_k.

2. For weak equivalences this is essentially the same proofs as for quasi-isomorphisms in the projective model structure seen in class. Again, passing to homotopy set/groups reduces the question to a question of retracts of isomorphisms of sets/groups which are isomorphisms of sets as we have seen in the projective model structure proof. For fibrations this follows because fibrations are by defined as a class of maps with a RLP with respect to the inclusions $D^n \to D^n \times [0,1]$ and as we have seen in class, maps defined by lifting properties either right or left are stable under retracts. For cofibrations this follows because they are defined by a left lifting property in the very same way.

3. Obviously for any class of map K, $\mathrm{LLP}(\mathrm{RLP}(K)) \supseteq K$ since all maps in K have by definition the left lifting properties with respect to the maps in $\mathrm{RLP}(K)$. Thus taking $K = \mathrm{LLP}(S)$ we get $\mathrm{LLP}(\mathrm{RLP}(\mathrm{LLP}(S))) \supseteq \mathrm{LLP}(S)$. Now observe that LLP and RLP are decreasing with respect to inclusion. That is, if $K \subseteq J$, then any map which has lifting properties with respect to all maps in J has the lifting properties with respect to those of K, so that $\mathrm{LLP}(K) \supseteq \mathrm{LLP}(J)$ and $\mathrm{RLP}(K) \supseteq \mathrm{RLP}(J)$. Taking $K = S$ and $J = \mathrm{RLP}(\mathrm{LLP}(S))$, we get $\mathrm{LLP}(S) \supseteq \mathrm{LLP}(\mathrm{RLP}(\mathrm{LLP}(S))$ hence the equality. The same argument works for the other equality (one can also apply the case we just showed to the opposite category $\mathcal{C}^{\mathrm{op}}$).

4. Let us first show that $J \subseteq \mathrm{Cell}(I')$. This follows because the maps in J are relative CW-complexes, which by definition are particular cases of I'-cell objects. Let us show the second claim. First, notice that $\mathrm{Cell}(I') \subseteq \mathrm{LLP}(\mathrm{RLP}(I'))$. This follows from the facts that $I' \subseteq \mathrm{LLP}(\mathrm{RLP}(I'))$ and $\mathrm{LLP}(\mathrm{RLP}(I'))$ is stable under transfinite compositions and retracts. Therefore we obtain the inclusion $J \subseteq \mathrm{LLP}(\mathrm{RLP}(I'))$. This implies that $\mathrm{LLP}(\mathrm{RLP}(J)) \subseteq \mathrm{LLP}(\mathrm{RLP}(\mathrm{LLP}(\mathrm{RLP}(I')))) = \mathrm{LLP}(\mathrm{RLP}(I'))$.

5. By exercise **1.3.**, any map in $\mathrm{LLP}(\mathrm{RLP}(J))$ is a retract of a morphism in $\mathrm{Cell}(J)$. Since weak equivalences are stable under retracts, we are reduced to showing that $\mathrm{Cell}(J) \subseteq \mathcal{W}$. Let f be a J-relative cell complex, so that f is a transfinite composition of pushouts of maps in J. We will show that J is a weak equivalence in two steps: first, each map in the sequence defining f is a weak equivalence, secondly the map to the colimit of the sequence is also a weak equivalence.

Step 1. Note that the maps in J are strong deformation retracts; in particular, they are weak equivalences. The main claim is that homotopy deformation retracts are stable under pushouts. To see this, suppose

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & C \arrow[d, "j"] \\ B \arrow[r] & D \end{tikzcd}$$

is a pushout with i a strong deformation retract. Since I is a locally compact Hausdorff space, the functor $- \times I$ is left adjoint to the mapping space functor $(-)^I$. Therefore $- \times I$ preserves colimits, so in particular pushouts. Hence

$$\begin{tikzcd} A \times I \arrow[r] \arrow[d, "i"] & C \times I \arrow[d] \\ B \times I \arrow[r] & D \times I \end{tikzcd}$$

is also a pushout diagram. Let $r\colon B \to A$ be a retract of i and $H : B \times I \to B$ a homotopy between Id_B and ir. Using the compositions $B \times I \to B \to D$ and $j \circ \mathrm{proj}_C\colon C \times I \to C \to D$ and the universal property of the pushout, we obtain a map from $D \times I \to D$. By construction, this homotopy witnesses that $j\colon C \to D$ is a strong deformation retract.

Step 2. Our goal is to show that f is a weak equivalence, knowing that each step of the transfinite composition is. Write this sequence as $X_*\colon \kappa \to \mathbf{Top}$ so that f is the map $X_0 \to \operatorname{colim}_{\alpha<\kappa} X_\alpha$. The aim is to compare $\operatorname{colim}_{\alpha<\kappa} \pi_n(X_\alpha, x)$ and $\pi_n(\operatorname{colim}_{\alpha<\kappa} X_\alpha, x)$. [3] The key is that the maps $X_\alpha \to X_{\alpha+1}$ are sufficiently nice inclusions. This is formalized by the following definition, taken from Hovey's book. A topological spaces $\iota\colon A \to B$ is called a *closed T_1 inclusion* if it is a homeomorphism onto its image $\iota(A)$, which is a closed subset of B and such that any point in $B \setminus \iota(A)$ is closed. Clearly the maps $j_n\colon D^n \to D^n \times I$ have this property. Moreover, using general point-set topology arguments, one sees that this class of inclusions is stale under pushouts and transfinite compositions. Therefore each $X_\alpha \to X_{\alpha+1}$ is a closed T_1 inclusion.

Lemma 2. *If $X_*\colon \lambda \to \mathbf{Top}$ is a sequence of closed T_1 inclusions and λ is a limit ordinal, then any map $u\colon C \to \operatorname{colim} X_*$ from a compact space C factors throught X_α for some $\alpha < \lambda$. If in addition each map $X_\alpha \to X_{\alpha+1}$ is a weak equivalence, then so is $X_0 \to \operatorname{colim} X_*$.*

We sketch a proof of the lemma. For the first claim, suppose towards a contradiction that $u(A)$ is not contained in any X_α with $\alpha < \lambda$. Then construct a sequence $S = \{x_n\}_{0<n<\omega}$ of points in $u(C)$ and a sequence $\{\alpha_n\}_{0<n<\omega}$ of ordinals such that $x_n \in X_{\alpha_n} \setminus X_{\alpha_{n-1}}$ (with $\alpha_0 = 0$). Let $\mu = \sup_n \alpha_n$. Now the intersection of any finite subset of S with any X_{α_n} is finite and avoids X_0, hence is closed (since all maps are closed T_1 inclusions). Therefore $S \hookrightarrow \operatorname{colim}_{n<\omega} X_{\alpha_n}$ has the discrete topology. Since $\operatorname{colim}_{n<\omega} X_{\alpha_n} \hookrightarrow \operatorname{colim} X_*$ is a closed inclusion, S is also discrete in the compact $u(C)$. This contradicts the fact that S is infinite.
For the second, reason by induction on λ. The case $\lambda = 0$ is obvious. If $\lambda = \beta + 1$ is a successor ordinal, then $\operatorname{colim}_{\alpha<\lambda} X_\alpha \cong X_\beta$, so the result holds in that case. Now suppose λ limit. We want the natural map $\operatorname{colim}_{\alpha<\lambda} \pi_n(X_\alpha, x) \to \pi_n(\operatorname{colim} X_*, x)$ to be an isomorphism. We prove the surjectivity. By the first part of the lemma, any element $[\phi] \in \pi_n(\operatorname{colim} X_*, x)$ is represented by some map $S^n \to X_\alpha$ with $\alpha < \lambda$. Hence $[\phi]$ is in the image of $\pi_n(X_\alpha, x)$. Since $X_0 \xrightarrow{\sim} X_\alpha$ is a weak equivalence by induction hypothesis, we get that $\pi_n(X_0) \to \pi_n(\operatorname{colim} X_*)$ is surjective.

[3] Note that homotopy groups do not commute with sequential colimits in general, even for seemingly nice spaces and maps. An interesting counterexample is given by the sequence

$$S^1 \xrightarrow{2} S^1 \xrightarrow{3} S^1 \xrightarrow{4} S^1 \longrightarrow \cdots$$

whose colimit is the indiscrete space $\mathbb{R}/\mathbb{Q}$, which is contractible. However the colimit of the fundamental groups is $\mathbb{Q}$.

Injectivity is proved in a similar manner, using that any homotopy $S^n \times I \to \operatorname{colim} X_*$ factors through some X_α.

We have shown that $\mathrm{Cell}(J) \subseteq \mathcal{W}$. Using the fact that $\mathrm{Cell}(J) \subseteq \mathrm{LLP}(\mathrm{RLP}(J))$ and question **4.**, we obtain that $\mathrm{Cell}(J) \subseteq \mathrm{LLP}(\mathrm{RLP}(I')) \cap \mathcal{W}$.

6. Since $\mathrm{LLP}(\mathrm{RLP}(J))) \subseteq \mathrm{LLP}(\mathrm{RLP}(I')))$ we deduce that

$$\mathrm{RLP}(J) = \mathrm{RLP}(\mathrm{LLP}(\mathrm{RLP}(J)))) \supseteq \mathrm{RLP}(\mathrm{LLP}(\mathrm{RLP}(I')))) = \mathrm{RLP}(I')$$

so that every map in $\mathrm{RLP}(I')$ is a Serre fibration. It remains to show it is also a weak equivalence. Let $f : X \to Y$ in $\mathrm{RLP}(I')$. One must show that for any $x \in X$ the induced maps $\pi_k(f) : \pi_k(X, x) \to \pi_k(Y, f(x))$ are bijections. To show it is surjective, notice that the map $* \to S^n$ can be obtained as the pushout of the map $\partial_n : S^{n-1} \to D^n$ along the map $* \to S^{n-1}$. Therefore $* \to S^n \in \mathrm{LLP}(\mathrm{RLP}(I'))$. Since f is in $\mathrm{RLP}(I')$, any square

$$\begin{tikzcd} * \arrow[r, "x"] \arrow[d] & X \arrow[d, "f"] \\ S^n \arrow[r] & Y \end{tikzcd}$$

has a lifting, so that the maps on homotopy groups are surjective. We show that the maps are injective. Suppose we have two maps $\phi, \phi' : (S^n, *) \to (X, x)$ representing the same homotopy class in $\pi_n(Y, f(x))$. Then we have an homotopy $H : S^n \times I \to Y$ between the two maps and we can build up a commutative diagram

$$\begin{tikzcd} S^n \vee S^n \arrow[r, "\phi\vee\phi'"] \arrow[d] & X \arrow[d, "f"] \\ S^n \wedge I \arrow[r] & Y. \end{tikzcd}$$

Here $S^n \wedge I = S^n \times I/(* \times I)$ can be obtained as the pushout

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & S^n \vee S^n \arrow[d] \\ D^{n+1} \arrow[r] & S^n \wedge I. \end{tikzcd}$$

In particular, the map $S^n \vee S^n \to S^n \wedge I$ is in $\mathrm{LLP}(\mathrm{RLP}(I'))$, which ensures the existence of a lift in the previous square. Therefore ϕ and ϕ' are homotopic in X, as desired.

7. First, we show that every set A is α-small, in the sense that $\mathrm{Hom}(A, -)$ commutes with any colimit indexed by α, for some ordinal α depending on A. For this, we can actually take α to be any ordinal that is $\mathrm{Card}(A)$-*filtered*, meaning any limit ordinal with the property that whenever $B \subseteq \alpha$ is a subset of cardinality at most $\mathrm{Card}(A)$, we have $\sup B < \alpha$. For example, one can take α to be the first cardinal that is strictly bigger that $\mathrm{Card}(A)$. Let $X : \alpha \to \mathbf{Set}$ be a sequence and $f : A \to \operatorname{colim} X$ a map. Since α is limit, for each $a \in A$ there exists $\beta_a \in \alpha$ such that $f(a) \in X_{\beta_a}$. Let $\beta := \sup_{a\in A} \beta_a$. Because α is $\mathrm{Card}(A)$-filtered, we have $\beta < \alpha$. Therefore f factors through $A \to X_\beta$, showing that A is small with respect to inclusions.

Now let A be a topological space, and choose α as above (with respect to the underlying set of A). We will show that A is α-small with respect to the inclusions of topological spaces. Let $X : \alpha \to \mathbf{Top}$ be a sequence of inclusions and $A \to \operatorname{colim} X$ be a morphism. By transfinite induction, every map $X_\beta \to \operatorname{colim} X$ is also an inclusion for all $\beta < \alpha$. By the first part of the question, we know that $A \to \operatorname{colim} X$ factors through some $A \to X_\beta$ *as a map of underlying sets*. But since all maps involved are inclusions, this factorization actually gives a *continuous* map $A \to X_\beta$, as desired.

Finally, to show that the small object argument applies to I' and J, it is enough to know that morphisms in $\mathrm{Cell}(I')$ and in $\mathrm{Cell}(J)$ are inclusions. This is easily seen to hold, as inclusions of topological spaces are stable under pushouts and transfinite compositions.

8. Let $f\colon X \to Y$ be an acyclic Serre fibration. We want to show it has the right lifting property against the maps $\partial_n\colon S^{n-1} \to D^n$. For $n = 0$ and $n = 1$, this is equivalent to the assertions that $\pi_0(f)$ is surjective and injective (respectively), which hold true. Now consider a commutative square of the form

$$\begin{tikzcd} S^{n-1} \arrow[r, "g"] \arrow[d, "\partial_n"'] & X \arrow[d, "f"] \\ D^n \arrow[r, "k"] & Y, \end{tikzcd} \tag{0.1}$$

where $n \geqslant 2$. In particular, we have $f_*[g] = 0$ in $\pi_{n-1}(Y)$. As f is a weak equivalence, $[g] = 0 \in \pi_{n-1}(X)$ so that there is a morphism k' making the triangle

$$\begin{tikzcd} S^{n-1} \arrow[r, "g"] \arrow[d, "\partial_n"'] & X \\ D^n, \arrow[ur, "k'"'] & \end{tikzcd}$$

commute. The idea is to deform k' to get a lift in the previous square. Note that at least the following diagram is commutative:

$$\begin{tikzcd} S^{n-1} \arrow[r, "\partial_n"] \arrow[d, "\partial_n"'] & D^n \arrow[d, "f\circ k'"] \\ D^n \arrow[r, "k"] & Y. \end{tikzcd}$$

Since the pushout of $D^n \leftarrow S^{n-1} \to D^n$ is S^n, the universal property of the pushout gives us a map $(k, f \circ k')\colon S^n \to Y$, hence an element $[(k, f \circ k')] \in \pi_n(Y)$. Since f_* is an isomorphism on π_n, there exists $[p] \in \pi_n(X)$ such that $f_*[p] = [(k, f \circ k')]$. Equivalently, p is a lift up to homotopy

$$\begin{tikzcd} & X \arrow[d, "f"] \\ S^n \arrow[ur, "p"] \arrow[r, "{(k,f\circ k')}"'] & Y. \arrow[ul, phantom, "\Swarrow" near end] \end{tikzcd}$$

One can prove that p can be chosen of the form (p', k') for some $p'\colon S^n \to X$. Now consider the map $(f \circ p', k)\colon S^n \to Y$. Using the definition of the sum on $\pi_n(Y)$, we can compute the its value on homotopy to find

$$[(f \circ p', k)) = [(f \circ p', f \circ k')] + [(f \circ k', k)] = [(k, f \circ k')] + [(f \circ k', k)] = 0.$$

Therefore there exists a homotopy $H\colon D^n \times I \to Y$ between $f \circ p' \Rightarrow k$ fixing the boundary $\partial D^n = S^{n-1}$. We are now in the situation

$$\begin{tikzcd} S^{n-1} \arrow[r, "g"] \arrow[d, "\partial_n"'] & X \arrow[d, "f"] \\ D^n \arrow[ur, "p'"] \arrow[r, "k"'] & Y. \arrow[ul, phantom, "\Swarrow H" near end] \end{tikzcd}$$

where we have a lift of k up to homotopy, which fixes the boundary. We now need to lift $H\colon f \circ p' \simeq k$ to a homotopy of the form $\widetilde{H}\colon p' \simeq p''$ that fixes the boundary and such that $f \cdot \widetilde{H} = H$. Once we have such a $\widetilde{H}$, then p'' will be a lift in the diagram 0.1.

Since H fixes the boundary of D^n, it extends to a map

$$S^{n-1} \coprod_{S^{n-1}\times I} D^n \times I \xrightarrow{(f\circ g, H)} Y.$$

Therefore finding the desired $\widetilde{H}$ is equivalent to having a lift h in the diagram

$$
\begin{tikzcd}
D^n \arrow[rr, "p'"] \arrow[d] & & X \arrow[d, "f"] \\
S^{n-1} \coprod_{S^{n-1}\times I} D^n \times I \arrow[rr, "(f\circ g, H)"] & & Y,
\end{tikzcd}
\tag{0.2}
$$

for then $\widetilde{H}$ is given by the composite $D^n \times I \to S^{n-1} \coprod_{S^{n-1}\times I} D^n \xrightarrow{h} X$. Observe that the left map in the square (0.2) is homeomorphic to $j_n\colon D^n \to D^n \times I$. Since f is a Serre fibration, there is such a lift h. This concludes the proof that $\mathcal{W} \cap \mathrm{RLP}(J) \subseteq \mathrm{RLP}(I')$. Since the converse inclusion was proved in question 6., we deduce that $\mathcal{W} \cap \mathrm{RLP}(J) = \mathrm{RLP}(I')$. In particular, one has $\mathrm{Cof} = \mathrm{LLP}(\mathrm{RLP}(I'))$.

9. We want to use the main theorem we have seen in class to guarantee that **Top** with weak equivalences and generating set of cofibrations I' and generating acyclic cofibrations J is a cofibrantly generated model category. Then the resulting model structure will be the same as Quillen's one. This last point follows from the previous questions: 4. gives $\mathrm{Fib} = \mathrm{RLP}(J)$ and 8. yields $\mathrm{Fib} \cap \mathcal{W} = \mathrm{RLP}(I')$. In particular, we have that $\mathrm{Cof} := \mathrm{LLP}(\mathrm{Fib} \cap \mathcal{W}) = \mathrm{LLP}(\mathrm{RLP}(I')) = I' - \mathrm{Inj}$.

We have to check the assumptions of the theorem: the (co)completness of **Top** is given by (co)limit topologies on the (co)limits of the underlying sets and the cardinality issues of the source of the generating (acyclic) cofibrations have been checked in question 7.

It remains to see that $\mathrm{Cell}(J) \subseteq \mathcal{W} \cap \mathrm{LLP}(\mathrm{RLP}(I'))$, which is exactly question 5. Similarly, we have to prove that $I' - \mathrm{Inj} = \mathrm{RLP}(I') \subseteq \mathcal{W} \cap J - \mathrm{Inj} = \mathcal{W} \cap \mathrm{RLP}(J)$. But since $\mathrm{RLP}(J)$ are Serre fibrations by definition, the result is precisely question 6.

From there, it only remains to show that $\mathrm{Cof} \cap \mathcal{W} \subseteq J - \mathrm{Inj} = \mathrm{LLP}(\mathrm{RLP}(J))$. Let $f : X \to Y$ be in $\mathrm{Cof} \cap \mathcal{W}$ and use the small object argument applied to J to produce a factorization $u \circ v : X \to X' \to Y$ with v in $\mathrm{Cell}(J) \subseteq \mathrm{LLP}(\mathrm{RLP}(J))$ and $u \in \mathrm{RLP}(J)$. As we already know $\mathrm{LLP}(\mathrm{RLP}(J)) \subseteq \mathcal{W}$ we have that v is in $\mathcal{W}$. By the 2-out-of-3 property, as f is assumed to be in $\mathcal{W}$, u is also a weak equivalence. So $u \in \mathcal{W} \cap \mathrm{RLP}(J) = \mathrm{RLP}(I')$. Since f is a cofibration and we already know that $\mathrm{Cof} = \mathrm{LLP}(\mathrm{RLP}(I'))$, we deduce the existence of a map $Y \to X'$ which splits the factorization. This makes f a retract of v and as v is in $\mathrm{LLP}(\mathrm{RLP}(J))$ and this class is stable under retracts, we deduce that f is also in $\mathrm{LLP}(\mathrm{RLP}(J))$.

Exercice 3 (**Top is proper**)**.** We endow **Top** with Quillen model category structure.

1. Prove that the category is *right proper*, that is the pullback of a weak equivalence under a fibration remains a weak equivalence.

2. Prove that **Top** is *left proper*, that is the pushout of a weak equivalence by a cofibration remains a weak equivalence.

Solution 3. **1.** Consider a pullback diagram

$$
\begin{tikzcd}
C \times_Y X \arrow[r, "f"] \arrow[d] & X \arrow[d, two heads] \\
C \arrow[r, "\sim"] & Y
\end{tikzcd}
$$

where the lower map is a weak equivalence. Since it is a pullback of a fibration the left vertical map is also a fibration and the diagram induces a map of fibration. Since the diagram is a pullback, the induced map on the fibers are homeomorphisms. Since the lower map is a weak equivalence, passing to the long exact homotopy sequences of Serre fibrations, we get a commutative diagram

$$\begin{tikzcd}
\dots \arrow[r] & \pi_i(F, f_0) \arrow[r] \arrow[d, "\cong"] & \pi_i(C \times_Y X, *) \arrow[r] \arrow[d] & \pi_i(C, c_0) \arrow[r] \arrow[d, "\cong"] & \pi_{i-1}(F, f_0) \arrow[r] \arrow[d, "\cong"] & \dots . \\
\dots \arrow[r] & \pi_i(F, f_0) \arrow[r] & \pi_i(X, x_0) \arrow[r] & \pi_i(Y, y_0) \arrow[r] & \pi_{i-1}(F, f_0) \arrow[r] & \dots
\end{tikzcd}$$

Then the 5-lemma ensures that $f : C \times_Y X \to X$ is a weak equivalence.

2. Consider a pushout square

$$\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, "f"', "\sim"] & B \arrow[d] \\
X \arrow[r] & Y
\end{tikzcd}$$

with $i : A \to B$ a cofibration and $f : A \to X$ a weak equivalence. We wish to prove that $f \sqcup_A B : B \to Y$ is a weak equivalence. Note that $X \to Y$ is a cofibration as well since it is a pushout of cofibration. Since the cofibrations of the Quillen structure are retract of relative cell complexes (as the model category is cofibrantly generated), we have a retract

$$\begin{tikzcd}
A \arrow[r] \arrow[d, tail, "j"] & T \arrow[r] \arrow[d, tail, "p"] & A \arrow[d, tail] \\
B \arrow[r, "k"] & U \arrow[r, "q"] & B
\end{tikzcd}$$

where the middle vertical map is a relative cell complex. Denoting $X \sqcup_T U$ the pushout of the maps $T \overset{p}{\to} A \overset{f}{\to} X$ and $T \to U$, the universal property of pushout yield the dotted arrow and a commutative diagram

$$\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, "j"] & B \arrow[d, "k"] \arrow[dddrr] & & \\
T \arrow[r, tail] \arrow[d, "p"] & U \arrow[d, "q"] \arrow[ddr] & & \\
A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[d] & & \\
X \arrow[r, tail] \arrow[rr, bend right] \arrow[rrr, bend right] & Y & X \sqcup_T U \arrow[l, dotted] & Y \arrow[l, dotted]
\end{tikzcd}$$

The uniqueness of the lift proves that the composition of the dotted arrow is the identity of Y, hence the induced commutative diagram

$$\begin{tikzcd}
B \arrow[r, "k"] \arrow[d] & U \arrow[r, "q"] \arrow[d] & B \arrow[d] \\
Y \arrow[r] & X \sqcup_T U \arrow[r] & Y
\end{tikzcd}$$

is a retract so that it suffices to prove that $U \to X \sqcup_T U$ is a weak equivalence to conclude for $B \to Y$. That is we are reduced to the case where $A \rightarrowtail B$ is a relative cell complex.

Let us first prove now that if B is obtained as a pushout $B \cong A \sqcup_{S^{n-1}} D^n$ along an inclusion $S^{n-1} = \partial D^n \hookrightarrow D^n$, then the result holds. That is, since a pushout of pushout is a psuhout, we have to prove that the induced map $\tilde{f} : B \cong A \sqcup_{S^{n-1}} D^n \overset{f \sqcup_{S^{n-1}} D^n}{\longrightarrow} X \sqcup_{S^{n-1}} D^n \cong Y$ is a weak equivalence when f is. First, taking an open deformation retract of S^{n-1} in D^n and the complement $V = D^{n-1} \backslash S^{n-1}$, we have a decomposition $B = U_A \cup V$ where $U_A \supseteq A$ is a (strong) deformation retract of X in $B \cong A \sqcup_{S^{n-1}} D^n$. There is a similar decomposition $Y = U_X \cup V$. Since V is a deformation retract of a point, it is an easy consequence of Van Kampen theorem to prove that the induced map $\tilde{f}_* : \pi_1(B, x) \to \pi_1(Y, \tilde{f}(x))$ is an isomorphism for any base point. It also induces a bijection on π_0. Now we can use a theorem in the lecture notes of the class asserting that to prove that $\tilde{f}$ is a weak homotopy equivalence it is now enough to prove in addition that $\tilde{f}$ induces an isomorphism in homology with local coefficient; the latter follows by the 5-lemma applied to the map between the Mayer-Vietoris long exact sequence applied to the same open decomposition of the two spaces. For the reader not so used to homology with local coefficient we give an alternate proof. First, working on each path component, we assume the spaces are arcwise connected. Let $T_B \overset{\sim}{\twoheadrightarrow} B$ be a cofibrant replacement so that T_B is a cell complex (we can even, as we have seen in class require that it is a CW-complex) and factor $T_B \to B \to Y$ into $T_B \overset{\bar{f}}{\rightarrowtail} T_Y \overset{\sim}{\twoheadrightarrow} Y$. By above, the map $\bar{f}$ is an isomorphism on π_0 and π_1 and it is enough to prove that $\overline{(f)}$ is a weak equivalence to ensure $\tilde{f}$ by the 2-out-of-3 property. Take the universal covers of T_Y (the only reason we introduce them was to ensure we have universal covers) so that we have a pullback diagram

$$\begin{tikzcd}
\tilde{T_B} \arrow[r, "\check{f}"] \arrow[d, two heads] & \tilde{T_Y} \arrow[d, two heads] \\
T_B \arrow[r, "\bar{f}"] & T_Y
\end{tikzcd}$$

in which the vertical arrows are covering maps (by stability of covering maps under pullbacks), hence with discrete fibers. Since we already now that $\bar{f}$ is an isomorphism on π_1, we have that $\tilde{T_B}$ is simply connected and now by the long homotopy exact sequence of a fibration, it is enough to prove that $\check{f} : \tilde{T_B} \to \tilde{T_Y}$ is a weak homotopy equivalence. In other words, we are back to prove the result for a map between simply connected spaces. The pullbacks along $\tilde{T_B} \twoheadrightarrow T_B \overset{\sim}{\twoheadrightarrow} B$ of U_A and V gives an open covering $\tilde{U}_A$, $\tilde{V}$ of T_B and similarly for $\tilde{T_Y}$; further $\check{f}$ preserves the decomposition. Hence we can apply Mayer-Vietoris exact sequence to these covers. But since V is contractible (and that the restriction of an acyclic fibration is an acyclic fibration by pullback invariance of those) $T_{B|V}$ and $T_{Y|V}$ are weakly homotopy equivalent to a point. It follows that $\tilde{V} \cong V \times \pi_1(Y)$ is a trivialisable covering space and the restriction of $\check{f}$ to those open is thus a weak homotopy equivalence. Since U_A and U_X are deformation retracts of A, X respectively, and $f : A \to X$ is a weak equivalence, we obtain in a same way that $\tilde{U}_A \to \tilde{U}_X$ is a weak homotopy equivalence. Hence by Mayer Vietoris we get that $\check{f}$ is an isomorphism in all homology groups with coefficient in $\mathbb{Z}$ and by Whitehead theorem it is thus a weak homotopy equivalence as required.

By induction on the last step, we see that in the case where $A \rightarrowtail B$ is obtained by a finite number of attachment of cells, the induced map $B \to Y$ is a weak homotopy equivalence. Now assume $A \rightarrowtail B$ is any relative cell complex so that $f := B \cong \operatorname{colim}_\kappa B_k \overset{\operatorname{colim}_\kappa f_k}{\longrightarrow} \operatorname{colim}_\kappa Y_k \cong Y$ where $B_k \overset{f_k}{Y}_k$ is the map induced by the cell attachments. We need to prove that $f_* : \pi_n(B, b_0) \to \pi_n(Y, f(b_0))$ is an isomorphism. But let $S^n \to B$ be a continuous map. By compacity of S^n and colimit topology, its image intersects only finitely many cells in B. The same holds too for Y and for any homotopy $S^n \times [0, 1]$ to either B or Y. So that to prove injectivity and surjectivity of f_*, we are left to prove it for a finite attachment of cells for which we have already seen that f_* is an isomorphism.