Derived Functors and Homotopy Colimits

Exercice 1 (Composition of Derived Functors)**.** 1. Let $F_1 : \mathcal{C}_1 \to \mathcal{C}_2$ and $F_2 : \mathcal{C}_2 \to \mathcal{C}_3$ be functors and let $\mathcal{W}_i$ be a class of morphisms in $\mathcal{C}_i$. Assuming all the relevant total left derived functors exist, use their universal properties to construct a natural transformation $\mathbb{L}F_2 \circ \mathbb{L}F_1 \to \mathbb{L}(F_2 \circ F_1)$.

2. Suppose now that $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_3$ are model categories and that F_1 and F_2 are left Quillen functors. Show that all derived functors exist and the natural transformation of the previous exercise is a natural isomorphism.

Solution 1. 1. Denote $\pi_i : \mathcal{C}_i \to \mathbf{Ho}(\mathcal{C}_i)$ the canonical functors. Let us recall that the total left derived functor $\mathbb{L}F_1 : \mathbf{Ho}(\mathcal{C}_1) \to \mathbf{Ho}(\mathcal{C}_2)$ come equipped with a natural transformation $\mathbb{L}F_1 \circ \pi_1 \Rightarrow F_1$ which is universal among such (it is a right Kan extension). In particular we have a diagram (which does *not* commute a priori)

$$\begin{tikzcd}
\mathcal{C}_1 \arrow[r, "F_1"] \arrow[dr, "\pi_1"'] & \mathcal{C}_2 \arrow[r, "F_2"] \arrow[dr, "\pi_2"] & \mathcal{C}_3 \arrow[dr, "\pi_3"] & \\
& \mathbf{Ho}(\mathcal{C}_1) \arrow[r, "\mathbb{L}F_1"] & \mathbf{Ho}(\mathcal{C}_2) \arrow[r, "\mathbb{L}F_2"] & \mathbf{Ho}(\mathcal{C}_3)
\end{tikzcd}$$

and natural transformations, given for any $X \in \mathcal{C}_1$ and $Y \in \mathcal{C}_2$ by $\mathbb{L}F_1(\pi_1(X)) \to \pi_2(F_1(X))$ and $\mathbb{L}F_2(\pi_2(Y)) \to \pi_3(F_2(Y))$. Taking $Y = F_1(X)$, the commutativity of the diagram gives a natural transformation

$$\mathbb{L}F_2 \circ \mathbb{L}F_1(\pi_1(X)) \to \pi_3(F_2 \circ F_1(X)).$$

By the universal property of derived functor $\mathbb{L}(F_2 \circ F_1)$, we get a unique natural transformation $\mathbb{L}L_2 \circ \mathbb{L}F_1 \Rightarrow \mathbb{L}(F_2 \circ F_1)$, as depicted in the diagram

$$\begin{tikzcd}
\mathcal{C}_1 \arrow[rr, "F_2 \circ F_1"] \arrow[dr, "\pi_1"'] & & \mathcal{C}_3 \arrow[dr, "\pi_3"] & \\
& \mathbf{Ho}(\mathcal{C}_1) \arrow[ur, phantom, "\Rightarrow"] \arrow[rr, "\mathbb{L}(F_2 \circ F_1)"] \arrow[rrr, bend right, "\mathbb{L}F_2 \circ \mathbb{L}F_1"'] \arrow[rr, phantom, shift right=6, "\Uparrow"] & & & \mathbf{Ho}(\mathcal{C}_3).
\end{tikzcd}$$

2. Let us now address the second question: first we remark that the composition of left Quillen functors is again a left Quillen functor. Indeed, by definition if F_1 preserves both cofibrations and acyclic cofibrations and F_2 also, clearly so does the composition $F_2 \circ F_1$. Therefore, by the theorem given in class, the model structures garantee the existence of $\mathbb{L}F_1$, $\mathbb{L}F_2$ and $\mathbb{L}(F_2 \circ F_1)$. On objects, these derived functors are respectively given by $\mathbb{L}F_1(X) = F_1(Q_1(X))$, $\mathbb{L}F_2(Y) = F_2(Q_2(Y))$ and $\mathbb{L}(F_2 \circ F_1)(X) = F_2(F_1(Q_1(X))$, where Q_1 is a cofibrant replacement functor in $\mathcal{C}_1$ and Q_2 is a cofibrant replacement in $\mathcal{C}_2$. In this case the natural transformation $\mathbb{L}L_2 \circ \mathbb{L}F_1 \Rightarrow \mathbb{L}(F_2 \circ F_1)$ is given on each object $X \in \mathcal{C}$ by a morphism

$$F_2(Q_2(F_1(Q_1(X)))) \longrightarrow F_2(F_1(Q_1(X))).$$

We only have to notice that by construction (in fact, we have to unfold the proof given in class that the formula $\mathbb{L}F = F \circ Q$ has the universal property of total left derived functor) this morphism is the image under F_2 of the cofibrant-replacement

$$Q_2(F_1(Q_1(X))) \longrightarrow F_1(Q_1(X)).$$

By definition, this is a weak equivalence whose source is cofibrant. The target is also cofibrant because F_1 is a left Quillen functor so sends cofibrant objects to cofibrant objects. Therefore by Brown's lemma[1] its image under F_2 is a weak equivalence and therefore an isomorphism in the homotopy category.

Exercice 2 (Homotopy colimits)**.** In this exercise, we first deal with generalities on homotopy pushouts and then specialize to chain complexes with the projective model structure. Let $\mathcal{C}$ be a model category and let I be the category given by the diagram-shape

$$\begin{tikzcd} b \arrow[r] \arrow[d] & c \\ a & \end{tikzcd}$$

1. Let $f : X \to Y$ be a natural transformation of diagrams $X, Y \in \mathrm{Fun}(I, \mathcal{C})$. Show that f has the left lifting property with respect to all projective acyclic fibrations if and only if the the natural maps
$$X_a \bigsqcup_{X_b} Y_b \to Y_a, \quad X_b \to Y_b, \quad X_c \bigsqcup_{X_b} Y_b \to Y_c$$
are cofibrations in $\mathcal{C}$. (Here we mean the usual pushouts in $\mathcal{C}$.)
Deduce that a diagram $Y : I \to \mathcal{C}$ is cofibrant if and only if Y_b is cofibrant in $\mathcal{C}$ and the maps $Y_a \to Y_b$ and $Y_a \to Y_c$ are cofibrations. Moreover, show that $X \to Y$ has the left lifting property with respect to projective fibrations if and only the above three maps are acyclic cofibrations.

2. Show that the category of diagrams $\mathrm{Fun}(I, \mathcal{C})$ admits the projective model structure (without using the result seen in class that such a structure exists since I is very small).

3. Show that the colimit functor $\mathrm{colim}\colon \mathrm{Fun}(I, \mathcal{C}) \to \mathcal{C}$ is a left Quillen functor.

4. Assume that $\mathcal{C}$ is left proper (i.e. weak equivalences are stable under pushouts along cofibrations). Show that any pushout diagram
$$\begin{tikzcd} B \arrow[r, tail, "f"] \arrow[d] & C \arrow[d] \\ A \arrow[r] & A \bigsqcup_B C \end{tikzcd}$$
where $f\colon B \longrightarrow C$ a cofibration, is also a homotopy pushout diagram.

5. **Case of Topological spaces.** Assume now that $\mathcal{C} = \mathbf{Top}$.
   (a) Using that **Top** is proper (as seen in exercise 3. from the sheet on Quillen model structure), show that there is a canonical isomorphism
   $$\mathbb{L}\,\mathrm{colim}(X \leftarrow A \to Y) \cong X \overset{\mathrm{h}}{\bigsqcup_A} Y = X \bigsqcup_{A \times \{0\}} \mathrm{Cyl}(A \to Y)$$
   in $\mathbf{Ho}(\mathbf{Top})$ between the homotopy pushout computed by the projective model structure and the formula given by the mapping cylinder.
   (b) Give a formula for computing the homotopy colimit of a tower $(X_0 \to X_1 \to X_2 \to \dots)$ as well as the homotopy limit of a tower $(\cdots \to Y_2 \to Y_1 \to Y_0)$.

[1] it is always worth recalling that this lemma does imply that all left Quillen functors send all weak equivalences between cofibrant to weak equivalences and right Quillen functors send weak equivalences between fibrant to weak equivalences

6. **Case of chain complexes.** Assume now that $\mathcal{C}$ is the model category of chain complexes over a ring R.

   (a) Show that $\mathcal{C}$ is left proper.

   (b) Let $g : A \to B$ be a map of chain complexes. Recall that the *mapping cone* of g, denoted $C(g)$, is the chain complex given in level n by $B_n \oplus A_{n-1}$ and whose differential $B_{n+1} \oplus A_n \to B_n \oplus A_{n-1}$ is given $(b, a) \mapsto (\partial_B(b) + g(a), -\partial_A(a))$. Let I denote the chain complex given by $R \oplus R$ in degree 0 and R in degree 1 with differential given by $\partial_R : R \to R \oplus R$ given by $r \mapsto (-r, r)$. We define the *mapping cylinder* of g, denoted $\mathrm{Cyl}(g)$, as the pushout in chain complexes of

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "i_0"] & B \arrow[d] \\ I \otimes A \arrow[r] & \mathrm{Cyl}(g) \end{tikzcd}$$

   where the vertical arrow $A \to I \otimes A$ is induced by the inclusion $i_0 : R \to I$ corresponding to the inclusion of the second factor $R \hookrightarrow R \oplus R$ in degree 0. The differential on $I \otimes A$ is given by $r \otimes a \mapsto \partial_R(r) \otimes a + (-1)^{\deg(r)} r \otimes \partial_A(a)$. Show that the mapping cone of g is the pushout of

$$\begin{tikzcd} I \otimes A \arrow[r] \arrow[d] & \mathrm{Cyl}(g) \arrow[d] \\ C(\mathrm{Id}_A) \arrow[r] & C(g). \end{tikzcd}$$

   (c) Let Δ^1 be the category with two objects and one non trivial morphism in between them. Show that the construction of the mapping cone defines a functor $C : \mathrm{Fun}(\Delta^1, \mathrm{Ch}(R)) \to \mathrm{Ch}(R)$ sending natural transformations objectwise given by quasi-isomorphisms to quasi-isomorphisms.

   (d) Let $Y := (0 \longleftarrow A \xrightarrow{g} B)$ be a diagram in $\mathcal{C}$. Show that there exists a diagram of the form $Y' := (0 \longleftarrow A' \xrightarrow{g'} B')$ with g' a cofibration and A' and B' cofibrant, together with a natural transformation $u : Y' \to Y$ which is objectwise a weak equivalence. Notice that by the previous question the induced map $C(g') \to C(g)$ is a weak equivalence.

   (e) Let $Y := (0 \longleftarrow A \xrightarrow{g} B)$ be a diagram in $\mathcal{C}$ with A and B cofibrant and g a cofibration. Show that $A \to I \otimes A$ is a weak equivalence and show that we can construct a zigzag of diagrams $Y \leftarrow Y' \to Y''$ of the form

$$\begin{tikzcd} 0 & A \arrow[l] \arrow[r, "g"] & B \\ C(A) \arrow[u] \arrow[d] & A \arrow[l] \arrow[r, "g"] \arrow[u] \arrow[d] & B \arrow[u] \arrow[d] \\ C(A) & I \otimes A \arrow[l] \arrow[r, "g"] & \mathrm{Cyl}(g) \end{tikzcd}$$

   where each vertical arrow is a weak equivalence and the map $I \otimes A \to \mathrm{Cyl}(g)$ is a cofibration.

   (f) Let $Y := (0 \longleftarrow A \xrightarrow{g} B)$ be any diagram. Conclude that the mapping cone $C(g)$ is a model for the homotopy colimit of the diagram Y.

Solution 2. First we advise the reader to write down a commutative square of functors in $\mathrm{Fun}(I, \mathcal{C})$, which are given by gluing two commutative cubes on their common face, and in which each face is commutative, as well as to write down what a lifting mean (which is a family of three maps dividing parallel faces into two commutative triangles). A key feature of the diagram we are considering is that the object b has only outgoing non-identity arrows and the other two objects have only incoming non-identity arrows. The object b and its image by a functor will play a specific role.

$$
\begin{tikzcd}
& X_b \arrow[r] \arrow[d] \arrow[ddl] \arrow[rrr, bend left] & X_c \arrow[ddl] \arrow[rrr, bend left] & & U_b \arrow[r] \arrow[d] \arrow[ddl] & U_c \arrow[ddl] \\
& X_a \arrow[ddl] \arrow[rrr, bend left] & & & U_a \arrow[ddl] & \\
Y_b \arrow[r] \arrow[d] \arrow[rrr, bend left] & Y_c \arrow[rrr, bend left] & & V_b \arrow[r] \arrow[d] & V_c & \\
Y_a \arrow[rrr, bend left] & & & V_a & &
\end{tikzcd}
\tag{0.1}
$$

1. Suppose a morphim $X \to Y$ in $\mathrm{Fun}(I, \mathcal{C})$ has the left lifting property with respect to projective acyclic fibrations. We first show that the the map $X_b \to Y_b$ has the left lifting property. Thus, we need to see that for any acyclic fibration $U \to V$ in $\mathcal{C}$, the dotted lifting arrow exists in the diagram

$$
\begin{tikzcd}
X_b \arrow[r] \arrow[d] & U \arrow[d, two heads, "\wr"] \\
Y_b \arrow[r] \arrow[ur, dashed] & V.
\end{tikzcd}
$$

For this, we notice that the data of such a diagram is equivalent to the data of a morphism of diagrams

$$
\begin{tikzcd}
X \arrow[r] \arrow[d, tail] & (*, U, *) \arrow[d] \\
Y \arrow[r] \arrow[ur, dashed] & (*, V, *)
\end{tikzcd}
$$

where $(*, U, *)$ is a notation for the diagram $* \leftarrow U \to *$ (and $*$ is the terminal object). The lifting exists by the assumption that $X \rightarrowtail Y$. This shows that $X_b \to Y_b$ is a cofibration. We now show that the map $X_a \bigsqcup_{X_b} Y_b \to Y_a$ has the left lifting property

$$
\begin{tikzcd}
X_a \bigsqcup\limits_{X_b} Y_b \arrow[r] \arrow[d] & U \arrow[d, two heads, "\wr"] \\
Y_a \arrow[r] \arrow[ur, dashed] & V
\end{tikzcd}
$$

with respect to any acyclic fibration $U \to V$ in $\mathcal{C}$. We do this using the remark that the data of such a commutative square is equivalent to the data of a commutative square of diagrams

$$
\begin{tikzcd}
X \arrow[r] \arrow[d, tail] & (U, U, *) \arrow[d] \\
Y \arrow[r] \arrow[ur, dashed] & (V, U, *),
\end{tikzcd}
$$

which does have a lift. The case of the remaining map is completely analogous.

We now verify the converse, meaning that if $X \to Y$ is of the form given in the exercise, then it has the left lifting property with respect to projective acyclic fibrations. The idea is again to use first the fact that $X_b \to Y_b$ is a cofibration in $\mathcal{C}$ to construct the lifting in the middle. This is possible since each arrow $U_a \to V_a$, $U_b \to V_b$ and $U_c \to V_c$ are acyclic fibrations, whenever $U \to V$ is an acyclic fibration in $\mathrm{Fun}(I, \mathcal{C})$.

This being done, we see that the lifing $Y_b \to U_b$ gives a commutative diagram

$$\begin{tikzcd}
& X_b \arrow[r] \arrow[d] \arrow[dl, bend right] & U_b \arrow[d] \\
Y_b \arrow[urr] & X_a \arrow[r] & U_a
\end{tikzcd}$$

from which we get a map $X_a \bigsqcup_{X_b} Y_b \to U_a$. By the commutativity of the diagram of squares, the latter map fits into the commuative square

$$\begin{tikzcd}
X_a \bigsqcup_{X_b} Y_b \arrow[r] \arrow[d, tail] & U_a \arrow[d, two heads, "\wr"] \\
Y_a \arrow[r] \arrow[ur, dotted] & V_a
\end{tikzcd}$$

which moreover admits a lift, since $X_a \bigsqcup_{X_b} Y_b \to Y_a$ is a cofibration by assumption. The overall construction can be seen as follows

$$\begin{tikzcd}
& X_b \arrow[rr] \arrow[dl, tail] \arrow[d] & & U_b \arrow[d] \\
Y_b \arrow[urrr] \arrow[d] & X_a \arrow[rr] \arrow[dl] \arrow[ddl] & & U_a \arrow[ddl, two heads, "\sim"] \\
X_a \bigsqcup_{X_b} Y_b \arrow[urrr, dashed] \arrow[d, tail] & & & \\
Y_a \arrow[rr] \arrow[uurrr, dotted] & & V_a. &
\end{tikzcd}$$

The existence of the remaining lift $Y_c \to U_c$ is proved in the same way. This shows the first equivalence.

From this, the characterization of cofibrant diagrams follows, using that $\emptyset \bigsqcup_{\emptyset} C \cong C$ for any object C in $\mathcal{C}$. Finally, the case of acyclic cofibrations is similar, using fibration on the right hand side instead of acyclic ones.

2. One has to check that all the axioms are satisfied. First one checks that $\mathrm{Fun}(I, \mathcal{C})$ admits all limits and colimits: this is true (provided that they exist) in $\mathcal{C}$ because colimits and limits in $\mathrm{Fun}(I, \mathcal{C})$ are computed objectwise in $\mathcal{C}$. Then one has to check the two-out-of-three property of weak equivalences. But again this follows by definition of the weak equivalences as objectwise weak equivalences in $\mathcal{C}$ which verifies this property. Then we have to check that fibrations, cofibrations and weak equivalences are stable under retracts. For fibrations and weak equivalences this follows again from the definitions, so we only have to say something about cofibrations: but since cofibrations are maps defined by a left lifting property, and the latter are stable under retracts, this is also verified (see the proof of the closedness of a model category in class).

The lifting properties have already been dealt with in the previous question so all we have to check is the factorization property. We explain the case where $X \to Y$ is factorized as an acyclic cofibration followed by a fibration. Here is the idea: again, first we factor the middle term $X_b \to Y_b$ as a acyclic cofibration followed by a fibration $X_b \overset{\sim}{\rightarrowtail} Z_b \twoheadrightarrow Y_b$ in $\mathcal{C}$. Then we complete this into a diagram by taking pushouts $X_c \to X_c \bigsqcup_{X_b} Z_b \to Y_b$ and $X_a \to X_a \bigsqcup_{X_b} Z_b \to Y_a$. Now we factor the last two maps $X_c \bigsqcup_{X_b} Z_b \to Z_c \to Y_b$ and $X_a \bigsqcup_{X_b} Z_b \to Z_a \to Y_a$ again in $\mathcal{C}$. The resulting factorization $X \to Z \to Y$ has the required properties.

3. This follows because by definition its right adjoint is the constant diagram functor which is right Quillen as by definition it preserves fibrations and acyclic fibrations.

4. Remark on computations of homotopy pushouts. As we have seen in class, the last point implies in particular that homotopy pushouts exists for any model category $\mathcal{C}$ and are computed as the left total derived functor of the colimit functor $\mathrm{Fun}(I, \mathcal{C}) \to \mathbf{Ho}(\mathcal{C})$, where the diagram category I is given the projective model structure. This means that it is computed by taking the pushout of a cofibrant replacement of $X_a \leftarrow X_b \to X_c$ in $\mathrm{Fun}(I, \mathcal{C})$, that is

$$\mathbb{L}\,\mathrm{colim}\left(X_a \leftarrow X_b \to X_c\right) = \mathrm{colim}\left(L_{X_a} \leftarrow L_{X_b} \to L_{X_c}\right) = L_{X_a} \bigsqcup_{L_{X_b}} L_{X_c}$$

where $L_X \overset{\sim}{\twoheadrightarrow} X$ is the cofibrant replacement. Using the characterization of cofibrant diagrams obtained in question 1., we deduce the following result.

A cofibrant replacement of a diagram X is a diagram

$$L_{X_a} \longleftarrowtail L_{X_b} \rightarrowtail L_{X_c}$$

with L_{X_b} cofibrant, together with a commutative diagram:

$$\begin{tikzcd}
L_{X_a} \arrow[d, two heads, "\wr"] & L_{X_b} \arrow[l, tail] \arrow[r, tail] \arrow[d, two heads, "\wr"] & L_{X_c} \arrow[d, two heads, "\wr"] \\
X_a & X_b \arrow[l] \arrow[r] & X_c.
\end{tikzcd}$$

This question and the proposition below shows that in model categories where weak equivalences are preserved by pushouts, there is an easier formula to compute cofibrant replacement of pushout diagrams.

Indeed, let

$$\begin{tikzcd}
A' \arrow[d, two heads, "\wr"] & B' \arrow[l, tail] \arrow[r, tail] \arrow[d, two heads, "\wr"] & C' \arrow[d, two heads, "\wr"] \\
A & B \arrow[l] \arrow[r, "f"] & C.
\end{tikzcd}$$

be a cofibrant resolution of the diagram $A \leftarrow B \to C$ (as explained in the remark above). We have to show that the natural map

$$A' \sqcup_{B'} C' \longrightarrow A \sqcup_B C$$

is a weak equivalence. But this map can be obtained as a composition of two maps :

$$A' \sqcup_{B'} C' \longrightarrow A' \sqcup_{B'} C \longrightarrow A \sqcup_B C,$$

where first map is the pushout

$$\begin{tikzcd}
C' \arrow[r] \arrow[d] & A' \bigsqcup_{B'} C' \arrow[d] \\
C \arrow[r] & A' \bigsqcup_{B'} C
\end{tikzcd}$$

The top horizontal arrow is a cofibration (because cofibrations are stable under pushout and $B' \rightarrowtail A'$ is a cofibration) and as $C' \to C$ is a weak equivalence, left properness implies that the right vertical arrow is a weak equivalence. The second map can be obtained as a composition of pushout diagrams:

$$
\begin{tikzcd}
B' \arrow[r, tail] \arrow[d, "\wr"] & A' \arrow[d, "\wr"] \arrow[rd, bend left, "\sim"] & \\
B \arrow[r] \arrow[d, tail, "f"] \arrow[ru, phantom, "\ulcorner"] & B \bigsqcup_A A' \arrow[r, "\text{2-out-of-3}", "\sim"'] \arrow[d, tail] & A \arrow[d] \\
C \arrow[r] \arrow[ru, phantom, "\ulcorner"] & C \bigsqcup_{B'} A' \arrow[r, "\sim"] \arrow[ru, phantom, "\ulcorner"] & A \bigsqcup_B C,
\end{tikzcd}
$$

where we use the left properness of $\mathcal{C}$ to show that $A' \longrightarrow B \sqcup_A A'$ and $C \sqcup_{B'} A' \longrightarrow A \sqcup_B C$ are weak equivalences.

5.

(a) Noticing that the factorisation $A \rightarrowtail A \times [0,1] \bigsqcup_{A \times \{1\}} Y \xrightarrow{\sim} Y$ given by the mapping cylinder is a relative cell complex followed by a weak equivalence, we see that the result will follow from the following general fact:

Proposition (Homotopy pushout in left proper model categories). *If $\mathcal{C}$ is a left proper model category, and $A \rightarrowtail Y' \xrightarrow{\sim} Y$ is a replacement of a morphism $i : A \to Y$ by a cofibration, then there is a natural isomorphism*

$$\mathbb{L}\operatorname{colim}(X \leftarrow A \rightarrow Y) \cong X \bigsqcup_A Y'.$$

Strictly speaking, the proposition asserts that there is an isomorphism in $\mathbf{Ho}(\mathcal{C})$ between the homotopy pushout and the pushout induced by the cofibrant replacement of $A \to Y$ and that this isomorphism is induced by a *natural* zigzag of weak equivalences

$$L_X \bigsqcup_{L_A} L_Y \xleftarrow{\sim} ? \xrightarrow{\sim} X \bigsqcup_A Y'$$

where the $L_X \leftarrow L_A \rightarrow L_Y$ is a cofibrant replacement of $X \leftarrow A \rightarrow Y$ (and thus the source of the weak equivalence is precisely the homotopy pushout) and the question mark ? depends functorially on the diagram.

We now prove the proposition. By question (4.), the target $X \bigsqcup_A Y'$ is the homotopy pushout $\mathbb{L}\operatorname{colim}(X \leftarrow A \rightarrowtail Y')$. The map $Y' \to Y$ induces a map of diagrams

$$
\begin{tikzcd}
X \arrow[d, equal] & A \arrow[l] \arrow[r, tail] \arrow[d, equal] & Y' \arrow[d, "\wr"] \\
X & A \arrow[l] \arrow[r] & Y
\end{tikzcd}
$$

for which all vertical maps are weak equivalences. Hence this is a weak equivalence of diagrams, so that the induced map on homotopy colimits is an isomorphism in $\mathbf{Ho}(\mathcal{C})$.

We thus have a natural isomorphism

$$\mathbb{L}\operatorname{colim}(X \leftarrow A \rightarrow Y) \xleftarrow{\simeq} \mathbb{L}\operatorname{colim}(X \leftarrow A \rightarrowtail Y') \xrightarrow{\simeq} X \bigsqcup_A Y'$$

in $\mathbf{Ho}(\mathcal{C})$ as claimed and the question mark ? can be taken to be the pushout $L_X \bigsqcup_{L_A} L_{Y'}$ where $L_X \leftarrow L_A \rightarrow L_{Y'}$ is the cofibrant replacement of $X \leftarrow A \rightarrow Y'$.

(b) The category $(0 \to 1 \to 2 \to 3 \to \cdots)$ depicting the colimit of towers is the category ω, with exactly one arrow $n \to m$ whenever $n < m$ are two non-negative integers (said differently, it is the category associated to the ordinal ω, or equivalently to the ordered set $(\mathbb{N}, <)$). It is not a very small category, so that the theorem seen in class does not guarantee the existence of homotopy colimits.

However, we can apply the same ideas as in the study of the homotopy pushout. Proceeding exactly as in question 1., we see that for any model category $\mathcal{C}$, a morphism $X \to Y$ in $\mathrm{Fun}(\omega, \mathcal{C})$ is a projective cofibration (resp. acyclic cofibration) if and only if $X_0 \to Y_0$ is a cofibration (resp. acyclic cofibration) and for every $i > 0$, the natural map $X_i \coprod_{Y_{i-1}} X_{i-1} \to Y_i$ is a cofibration (resp. acyclic cofibration). Then one can prove as in 2. that the projective structure on $\mathrm{Fun}(\omega, \mathcal{C})$ makes the category of towers a model category. In particular, the homotopy colimit of towers always exist. Furthermore, a cofibrant replacement of a diagram $X : \omega \to \mathcal{C}$ is given by a cofibrant object L_{X_0} and cofibrations $L_{X_i} \rightarrowtail L_{X_{i+1}}$ (for any $i \in \omega$) together with acyclic fibrations $L_{X_i} \overset{\sim}{\twoheadrightarrow} X_i$ making the obvious squares commutative. In the specific case where X_0 is cofibrant and all the maps $X_i \to X_{i+1}$ are cofibrations, we thus have that X is cofibrant and therefore as seen in class, the canonical map from the homotopy pushout of the tower X to its pushout $\operatorname{colim} X$ is a weak equivalence. It follows that if we have a commutative diagram

$$\begin{tikzcd}
Y_0 \arrow[r, tail] \arrow[d, two heads, "\wr"] & Y_1 \arrow[r, tail] \arrow[d, two heads, "\wr"] & Y_2 \arrow[r, tail] \arrow[d, two heads, "\wr"] & \dots \\
X_0 \arrow[r] & X_1 \arrow[r] & X_2 \arrow[r] & \dots
\end{tikzcd}$$

with Y_0 cofibrant, then it is a weak equivalence of diagrams and by above we thus have a zigzag of weak equivalences

$$\operatorname*{colim}_{i\in\omega} Y_i \overset{\sim}{\longleftarrow} \operatorname*{colim}_{i\in\omega} L_{Y_i} \overset{\sim}{\longrightarrow} \operatorname*{colim}_{i\in\omega} L_{X_i}.$$

This proves that to compute the homotopy colimit of a tower, it is enough to replace it by a weakly equivalent tower consisting of cofibrations and whose first object is cofibrant.

A completely dual analysis shows that the injective model structure is also a model category for $\mathrm{Fun}(\omega, \mathcal{C})$ and hence that homotopy limit of towers exist. Such homotopy limits can then be computed by replacing a tower by a weakly equivalent tower such that all maps are fibrations and the last object Y_0 is fibrant.

Now, recall from class that in **Top**, every object X_0 is fibrant and there exists a CW-complex $\widetilde{X}_0$ weakly equivalent to it: $\widetilde{X}_0 \overset{\sim}{\to} X_0$ (and by composition we have an induced map $\widetilde{X}_0 \to X_1$).

It follows that the homotopy colimit of a tower in **Top** is given by the "telescope"

$$\mathbb{L}\operatorname{colim} X_i \cong \widetilde{X}_0 \times [0,1] \coprod_{\widetilde{X}_0 \times \{1\}} X_1 \times [1,2] \coprod_{X_1 \times \{2\}} X_2 \times [2,3] \coprod_{X_2 \times \{3\}} X_3 \times [3,4] \coprod \dots$$

which is a tower of glued cylinders. Now consider the colimit of almost the same telescope but for which we start at X_0. Then we have a pushout diagram

$$\begin{tikzcd}
\widetilde{X}_0 \times [0,1[\arrow[r, tail] \arrow[d, "\wr"] & \widetilde{X}_0 \times [0,1] \coprod_{\widetilde{X}_0} \left(\coprod_{X_{i-1}} X_i \times [i, i+1] \right) \arrow[d] \\
X_0 \times [0,1[\arrow[r, tail] & X_0 \times [0,1] \coprod_{X_0} \left(\coprod_{X_{i-1}} X_i \times [i, i+1] \right)
\end{tikzcd}$$

in which the right vertical arrow is a weak equivalence by left properness. Hence the homotopy colimit of a tower $X_0 \to X_1 \to \cdots$ is given by the telescope

$$\mathbb{L}\operatorname{colim} X_i \cong X_0 \times [0,1] \coprod_{X_0\times\{1\}} X_1 \times [1,2] \coprod_{X_1\times\{2\}} X_2 \times [2,3] \coprod \ldots.$$

By a similar argument and induction, one can prove that if all the maps in the sequence $X_0 \to X_1 \to \cdots$ are cofibrations, then the colimit of the sequence $\operatorname{colim} X$ is weakly equivalent to its homotopy colimit $\mathbb{L}\operatorname{colim} X$.

Similarly, a homotopy limit of $(\cdots \to Y_2 \to Y_1 \to Y_0)$ is obtained by replacing each map by a fibration and taking the limit (one then gets a limit of path spaces).

6.

(a) Consider the following pushout diagram in $\mathrm{Ch}(R)$

$$\begin{tikzcd} M \arrow[r, tail, "g"] \arrow[d, "f"', "\wr"] & M' \arrow[d, "f'"] \\ N \arrow[r, "g'"] & N' \end{tikzcd}$$

where g is assumed to be a cofibration and f is weak equivalence. We must show that f' is a weak equivalence. But notice that as g is a cofibration and therefore injective, we have a short exact sequence of chain complexes

$$0 \longrightarrow M \xrightarrow{g} M' \longrightarrow M'/M \longrightarrow 0$$

and therefore a long exact sequence of homology groups, and finally we have maps of exact sequences

$$\begin{tikzcd} H_{n+1}(M'/M) \arrow[r] \arrow[d, "\cong"] & H_n(M) \arrow[r] \arrow[d, "\cong"] & H_n(M') \arrow[r] \arrow[d] & H_n(M'/M) \arrow[r] \arrow[d, "\cong"] & H_{n-1}(M) \arrow[d, "\cong"] \\ H_{n+1}(N'/N) \arrow[r] & H_n(N) \arrow[r] & H_n(N') \arrow[r] & H_n(N'/N) \arrow[r] & H_{n-1}(N) \end{tikzcd}$$

where the first and fourth vertical maps are isomorphisms because the diagram is a pushout and the second and last vertical maps are isomorphisms because f is a weak equivalence. So f' is also a weak equivalence.

(b) Note that in degree n, one has $(I \otimes A)_n = A_n \oplus A_n \oplus A_{n-1}$. The formula given for the differential gives $d(x, y, w) = \big(\partial_A(x) - w, \partial_A(y) + w, -\partial_A(w)\big)$. Hence the pushout $\mathrm{Cyl}(g) := B \coprod_A I \otimes A$ is given in degree n by $B_n \oplus A_n \oplus A_{n-1}$ and the map $I \otimes A \to B \coprod_A I \otimes A$ is given in degree n by $(x, y, w) \mapsto (g(y), x, w)$. Thus the differential on the pushout $\mathrm{Cyl}(g)$ is given by

$$(b, x, w) \mapsto \big(\partial_B(b) + g(w), \partial_A(x) - w, -\partial_A(w)\big).$$

The formula for the differential of $I \otimes A$ above shows that the linear maps $(I \otimes A)_n = A_n \oplus A_n \oplus A_{n-1} \to A_n \oplus A_{n-1}$ given by $(x, y, z) \mapsto (y, z)$ defines a chain map $t : I \otimes A \to C(\mathrm{Id}_A)$.

Now we compute the pushout $\mathrm{Cyl}(g) \coprod_{I\otimes A} C(\mathrm{Id}_A)$. In degree n, we have

$$\Big(\mathrm{Cyl}(g) \coprod_{I\otimes A} C(\mathrm{Id}_A)\Big)_n = (B_n \oplus A_n \oplus A_{n-1}) \oplus (A_n \oplus A_{n-1})/\big((g(y), x, w, 0, 0) \sim (0, 0, 0, y, w)\big)$$

and hence it is isomorphic to $B_n \oplus A_{n-1}$ (the terms corresponding to x being killed off in the quotient). The differential then reads $(b, w) \mapsto (\partial_B(b) + g(w), -\partial_A(w))$ which proves that the pushout is indeed the cone $C(g)$.

(c) A functor from Δ^1 to any category is simply the data of two objects and one morphism between them, that is the data of an arrow $A \xrightarrow{g} B$. A map between functors is simply a natural transformation thus a commutative diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "\alpha"] & B \arrow[d, "\beta"] \\ A' \arrow[r, "g'"] & B'. \end{tikzcd}$$

Now since we consider chain complexes, the linear maps $\beta_n \oplus \alpha_{n-1} : B_n \oplus A_{n-1} \to B'_n \oplus A'_{n-1}$ yield a map $C(g) \to C(g')$ of chain complexes (because α and β commutes with differential and the diagram is commutative). And it is easy to check that the assignement $g \mapsto C(g)$ gives a functor $\mathrm{Fun}(\Delta^1, \mathrm{Ch}(R)) \to \mathrm{Ch}(R)$. It remains to prove that it sends objectwise weak equivalences to weak equivalences. To see this, we note that given $f : A \to B$ a morphism of chain complexes, one has an exact sequence of complexes $0 \to B \to C(f) \to A[1] \to 0$. Hence a map of morphisms produces a map of exact sequences and if the maps are quasi-isomorphisms, by the five-lemma, the middle terms will also be.

(d) Take $u_a : A' \overset{\sim}{\twoheadrightarrow} A$ a cofibrant replacement of A. Then choose a factorization of $A' \to A \to B$ as a cofibration $g' : A' \rightarrowtail B'$ followed by an acyclic fibration $u_b : B' \overset{\sim}{\twoheadrightarrow} B$ and set $u_c : 0 \to 0$ as the identity. The commutative diagram

$$\begin{tikzcd} 0 \arrow[d, equal] & A' \arrow[l] \arrow[r, tail, "g'"] \arrow[d, "u_a", "\wr"'] & B' \arrow[d, "u_b", "\wr"'] \\ 0 & A \arrow[l] \arrow[r, "g"] & B \end{tikzcd}$$

gives us the required natural transformation of diagrams.

(e) First note that the composition $A \xrightarrow{i_0} I \otimes A \to C(\mathrm{Id}_A)$ is given $y \mapsto (0, y, 0) \mapsto (y, 0)$. Since g is assumed to be a cofibration and cofibrations are stable under pushouts, by definition of the mapping cylinder, the map $I \otimes A \to \mathrm{Cyl}(g)$ is a cofibration as well. Now we are only left to prove the vertical arrows in the diagram

$$\begin{tikzcd} 0 & A \arrow[l] \arrow[r, tail, "g"] & B & & Y \\ C(\mathrm{Id}_A) \arrow[u] \arrow[d, equal] & A \arrow[l, "t \circ i_0"'] \arrow[r, tail, "g"] \arrow[u, equal] \arrow[d, "i_0"] & B \arrow[u, equal] \arrow[d] & & Y' \arrow[u] \arrow[d] \\ C(\mathrm{Id}_A) & I \otimes A \arrow[l, "t"'] \arrow[r, tail] & \mathrm{Cyl}(g) & & Y'' \end{tikzcd} \tag{0.2}$$

are weak equivalences. For the lower right one, it follows by left properness once we prove that i_0 is. Note that the linear map $s : I \otimes A \to A$ given in degree n by $s(x, y, w) = x + y$ is a chain complex morphism. Further $s \circ i_0 = \mathrm{Id}_A$. To prove that i_0 is a quasi-isomorphism, it is then enough to show that $i_0 \circ s$ is homotopic to the identity of $I \otimes A$. Let $h : I \otimes A \to I \otimes A[-1]$ be given by $h(x, y, w) = (0, 0, x)$. Then

$$(d \circ h + h \circ d)(x, y, w) = (-x, x, -w) = -(x, y, w) + i_0 \circ s(x, y, w)$$

which proves that h is indeed a chain homotopy in between Id and $i_0 \circ s$. Finally, since $C(\mathrm{Id}_A)$ is acyclic[2], the upper left vertical map is a quasi-isomorphism.

[2] as follow from the long exact sequence since Id is a quasi-isomorphism

(f) Given a diagram $Y = (0 \longleftarrow A \xrightarrow{g} B)$, we can apply question (d) to find $\widetilde{Y} \overset{\sim}{\to} Y$, where $\widetilde{Y}$ is of the form $\widetilde{Y} = (0 \longleftarrow \widetilde{A} \xrightarrow{\widetilde{g}} \widetilde{B})$ with $\widetilde{A}$ and $\widetilde{B}$ cofibrant. The natural transformation $\widetilde{Y} \overset{\sim}{\to} Y$ being a weak equivalence, it induces a quasi-isomorphism on the respective homotopy colimits. Further, by (c) the mapping cone of $\widetilde{g}$ and g are weakly equivalent. Thus it is enough to prove that the mapping cone of $\widetilde{g} \colon \widetilde{A} \to \widetilde{B}$ is quasi-isomorphic to the homotopy pushout of the diagram $\widetilde{Y}$. In other words, we have reduced question (f) to the case where $g \colon A \to B$ is a cofibration and A, B are cofibrant, which are exactly the assumptions of question (e).
Consider the zigzag of weak equivalences of diagrams $\widetilde{Y} \overset{\sim}{\leftarrow} Y' \overset{\sim}{\to} Y''$ given by question (e) (i.e. the diagram 0.2). By left properness of $\mathcal{C}$ and question 4., we know that the pushout of each horizontal level $Y^{(*)}$ is also a homotopy pushout. The vertical maps being all weak equivalences, we have that the homotopy pushout of the top horizontal diagram $(\widetilde{Y})$ is equivalent to the one of the lower one (Y''). By question (b), the pushout of the latter diagram

$$C(\mathrm{Id}_A) \xleftarrow{\;t\;} I \otimes A \rightarrowtail \mathrm{Cyl}(g),$$

is the mapping cone $C(g)$ of the original map g. This concludes the proof that the mapping cone computes the homotopy pushout of Y.

Exercice 3 (Bad behavior of Gabriel-Zisman Localization). Let A be a ring and let $D(A) := \mathbf{Ho}(\mathrm{Ch}(A))$ denote the derived category of A; it is the Gabriel-Zisman localization of the category $\mathrm{Ch}(A)$ of chain complexes in A along quasi-isomorphisms of complexes. We have seen in class that $D(A)$ is the homotopy category of a model structure in $\mathrm{Ch}(A)$ with weak equivalences given by quasi-isomorphisms and fibrations given by levelwise surjections.

1. Show that if E and H are two A-modules seen as complexes concentrated in degree zero, then

$$\mathrm{Hom}_{D(A)}(E, H[n]) \simeq \mathrm{Ext}^n_A(E, H).$$

2. Show that if A is a field, then $D(A)$ is an abelian category[3], equivalent to the category $A^{\mathbb{Z}}$ of $\mathbb{Z}$-graded A-vector spaces.

3. Show that $D(A[X])$ does not admit limits in general. (*Hint*: take a non-trivial element $f : A \to A[1]$ and show that the existence of a kernel for f gives a contradiction.)

4. Let A be a field and let I be the category with one object and $\mathbb{N}$ as endomorphisms. Show that $\mathrm{Fun}(I, D(A))$ is not equivalent to $D(\mathrm{Fun}(I, \mathrm{Ch}(A)))$.
The conclusion is that the theory of diagrams does not interact well with derived categories.

Solution 3. 1. By the fundamental theorem for computing morphisms in the homotopy category, and as every object is fibrant in the projective model structure, $\mathrm{Hom}_{D(A)}(E, H[n])$ is in bijection with the set of homotopy classes of maps $Q(E) \to H[n]$, with $Q(E)$ a cofibrant resolution of E. As we have seen in class, a projective resolution of A (which is bounded below) is in particular a cofibrant resolution, hence we can take $Q(E) := P$ any projective resolution of E. Then this hom-set is by definition $\mathrm{Hom}_{\mathrm{Ch}(A)}(P, H[n])/\simeq$ and since $H[n]$ is concentrated in positive degree n, it is the quotient of the set $Z^n(P, H)$ of linear maps $f : P_n \to H$ such that $P_{n+1} \to P_n \xrightarrow{f} H$ is zero. In other words, the hom-set consists of degree n-cocyles in the cochain complex $\mathrm{Hom}_A(P, H)$, modulo the homotopy equivalence relation. Note that, since P is cofibrant and $H[n]$ is fibrant, two maps $f, g : P \to H[n]$ are homotopy equivalent if and only if they are right homotopy equivalent.

As in Exercise 2, we have a special path object for H; namely $H^I := \mathrm{Hom}_A(I, H)$ which is given by $H \oplus H$ in degree 0 and H in degree -1, with differential given by $d_{H^I} : H \oplus H \to H$ given by

[3] see links to homological algebra exercises on the web page, if you are not familiar with this.

$(x, y) \mapsto x - y$. Then we have a chain map $H^I \to H \times H$ given by the dual of i_0 and i_1, which is just the identity map in degree 0 and (necessary) 0 elsewhere. It is surjective levelwise hence a fibration. We also have a canonical map $H \to H^I$ given in degree 0 by $r \mapsto (r, r)$ (and by 0 elsewhere). Thus H^I with the above maps is a path object for H and so is $H^I[n]$ for $H[n]$.

We now prove that if $f \overset{r}{\sim} g \colon P \to H[n]$, then there is a right homotopy from f to g *with* $H^I[n]$ *as a cylinder*. Indeed, let $R_H \twoheadrightarrow H[n] \times H[n]$ be any path object and $\alpha : P \to R_H$ be a right homotopy. We can factor the structure map $H[n] \xrightarrow{\sim} R_H$ as $H[n] \overset{\sim}{\rightarrowtail} \widetilde{R}_H \overset{\sim}{\twoheadrightarrow} R_H$ by the factorisation axiom and the 2 out 3 property. Then the map $\widetilde{R}_H \overset{\sim}{\twoheadrightarrow} R_H \twoheadrightarrow H[n] \times H[n]$ makes $\widetilde{R}_H$ a path object for $H[n]$. Since P is cofibrant and $\widetilde{R}_H \overset{\sim}{\twoheadrightarrow} R_H$ is an acyclic fibration, the lifting property ensures that there is a lifting $\widetilde{\alpha}$ of α:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, tail] & \widetilde{R}_H \arrow[d, two heads, "\wr"] \\ P \arrow[r, "\alpha"] \arrow[ur, dashed, "\widetilde{\alpha}"] & R_H \end{tikzcd}$$

and thus we have an homotopy between f, g out of the path object $\widetilde{R}_H$. Now the commutative square

$$\begin{tikzcd} H[n] \arrow[r, "\sim"] \arrow[d, tail, "\wr"] & H^I[n] \arrow[d, two heads] \\ \widetilde{R}_H \arrow[r] \arrow[ur, dashed] & H[n] \times H[n] \end{tikzcd}$$

provides a map $\widetilde{R}_H \to H^I[n]$ so that the composition $P \xrightarrow{\widetilde{\alpha}} \widetilde{R}_H \to H^I[n]$ is a right homotopy from f to g.

Now we just have to identify what it means to be a right homotopy $P \to H^I$. For degree reasons, it has only two possible non-zero components given by a linear map $P_n \to H \oplus H$ which has to be (f, g) (since it is an homotopy) and a map $h : P_{n-1} \to H$. Since the map has to be a chain map, we get that $f - g = d_{H^I} \circ (f, g) = h \circ d_{P_n}$, where $d_{P_n} : P_n \to P_{n-1}$ is the differential. In other words, two maps $f, g : P_n \to H$ are right homotopic if they differ by a coboundary in the chain complex $\mathrm{Hom}_A(P, H)$. Thus we have a canonical isomorphism

$$\mathrm{Hom}_{D(A)}(E, H[n]) \cong H^n(\mathrm{Hom}_A(P, H)) = \mathrm{Ext}^n_A(E, H).$$

2. One checks that the functor sending a complex $(M_k, \partial_k) \in \mathrm{Ch}(A)$ to the $\mathbb{Z}$-graded module $l(M) := \bigoplus_{i \in \mathbb{Z}} H_i(M)$ sends quasi-isomorphisms to isomorphisms and therefore induces a functor $l : D(A) \to A^{\mathbb{Z}}$. One can produce a candidate for the inverse: given a $\mathbb{Z}$-graded module K, we consider the associated chain complex with zero differential $(K, 0)$. This gives a natural functor $A^{\mathbb{Z}} \to \mathrm{Ch}(A)$ and we set $t : A^{\mathbb{Z}} \to D(A)$ the composition with the localization functor. Let us show that l and t form an equivalence of categories. Clearly, the composition $l \circ t$ is isomorphic to the identity. We are left to construct a natural isomorphism between $\mathrm{Id}_{D(A)}$ and $t \circ l$, meaning, we should exhibit functorial isomorphisms in $D(A)$ between (M_k, ∂_k) and $(H^k(M), 0)$. For that purpose we construct two morphisms in $\mathrm{Ch}(A)$, $f_M : (M_k, \partial_k) \to (H^k(M), 0)$ and $g_M : (H^k(M), 0) \to (M_k, \partial_k)$ which we prove to be isomorphisms in $D(A)$ and behave functorialy with respect to M. To define them, let us notice that we always have (by definition) short exact sequences

$$0 \to \ker \partial_n \longrightarrow M_n \xrightarrow{\partial_n} \mathrm{im}\, \partial_n \to 0$$

and

$$0 \to \mathrm{im}\, \partial_{n+1} \longrightarrow \ker \partial_n \longrightarrow H_n(M) \to 0$$

for every $n \in \mathbb{Z}$. As we are working over a field, both exact sequences split.[4] We thus have isomorphisms $M_n \simeq \operatorname{im}\partial_{n+1} \oplus H_n(M) \oplus \operatorname{im}\partial_n$ and under this identification, the map differential map $M_{n+1} \to M_n$ is given with the map $(a,b,c) \mapsto (c,0,0)$. Now we define the map $f_M^n : M_n \to H_n(M)$ as the projection and the map $g_M^n : H_n(M) \to M_n$ as the inclusion. One checks that these maps are quasi-isomorphisms and therefore become isomorphisms in $D(A)$. Moreover, the choice of the splittings can be made in such a way that these maps provide a natural isomorphism in the homotopy category: the claim is that after passing to the homotopy relation on morphisms, the choice of the splitting does not matter.

3. As we have seen in question 1., we have

$$\operatorname{Hom}_{D(A[X])}(A, A[1]) = \operatorname{Ext}^1_{A[X]}(A,A) \simeq A$$

where the second isomorphism is a simple computation (for example, one can take the obvious projective resolution $A[X] \xrightarrow{\cdot X} A[X]$ of A). Let $f : A \to A[1]$ be the morphism in $D(A[X])$ that corresponds to $1 \in A$ via the above identification. In particular, f is non-zero. Assume f admits a kernel $i\colon K \to A$, in the sense of an equalizer of f and the zero morphism. Then we have a long exact sequence of abelian groups

$$0 \longrightarrow [A[X], K[n]] = H^n(K) \xrightarrow{i_*} [A[X], A[n]] = H^n(A) \longrightarrow [A[X], A[n+1]] = H^{n+1}(A) = 0.$$

This implies that $i\colon K \to A$ is an isomorphism in the derived category $D(A[X])$. Hence $f = f \circ i \circ i^{-1} = 0$, which contradicts the fact that f is nonzero.

4. Assume A is a field. Let I be the category with one object and $\mathbb{N}$ as its monoid of endomorphism. Then the category $\operatorname{Fun}(I, \operatorname{Ch}(A))$ is isomorphic to $\operatorname{Ch}(A[X])$, hence $D(\operatorname{Fun}(I, \operatorname{Ch}(A))) \simeq D(A[X])$ which by the previous question has no (co)limits in general. However, since A is a field (in particular is semi-simple), the category $\operatorname{Fun}(I, D(A))$ is abelian (as $D(A)$ is). This shows that this category cannot be equivalent to $D(\operatorname{Fun}(I, \operatorname{Ch}(A)))$.

Exercice 4. Let $\mathbf{Top}_*$ be the category of *pointed* topological spaces and $U : \mathbf{Top}_* \to \mathbf{Top}$ be the functor forgetting the base point.

1. Prove that U is a right adjoint and compute its left adjoint.
2. We endow $\mathbf{Top}$ with Quillen model structure. Find a model structure on $\mathbf{Top}_*$ such that U is right Quillen.
3. Generalize the previous construction to any model category $\mathcal{C}$.

Solution 4. **1.** Let $U : \mathbf{Top}_* \to \mathbf{Top}$ be the forget functor which sends a pointed space/map to the underlying space/map of topological space. Let also $P : \mathbf{Top} \to \mathbf{Top}_*$ be the functor that sends a space X to $P(X) := X \bigsqcup \{*\}$ where we take the additional point $*$ as the base point. Similarly $P(X \xrightarrow{f} Y)$ is the map that sends $*$ onto $*$ and whose restriction to X is f. It is clear that it is continuous and defines a functor. Since X and $\{*\}$ are open subsets in $P(X)$, a continuous pointed map from $X \bigsqcup \{*\} \to (Y, y_0)$ is by definition a map sending $*$ to y_0 together with a continuous map from X to Y. Hence there is an natural bijection

$$\operatorname{Hom}_{\mathbf{Top}_*}(P(X), (Y, y_0)) \cong Hom_{\mathbf{Top}}(X, U(Y, y_0))$$

which proves that $P : \mathbf{Top} \rightleftarrows \mathbf{Top}_* : U$ is an adjonction.

2. In order for U to be a right Quillen functor, we need to have a model structure in $\mathbf{Top}_*$ for which U preserves fibrations and acyclic fibrations. Let us define these classes in the simplest possible

[4] Since this is the only place we use our assumption that A is a field, the proof shows more generally that the derived category of any *semi-simple* abelian category is abelian.

way for this to work. We define a weak equivalence (resp. fibration, resp. cofibration) in $\mathbf{Top}_*$ to be a map f such that $U(f)$ is a weak equivalence (resp. fibration, resp. cofibration) in $\mathbf{Top}$. In general such a simple definition do not work[5], but here it will (essentially because limits and colimits in both categories are very close).

It is immediate to check the axiom MC2, and MC3. For the lifting properties, note that if we have a commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] & X \arrow[d, two heads, "\wr"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

in $\mathbf{Top}_*$, then the diagram of underlying unpointed spaces is a square in $\mathbf{Top}$ with an acyclic fibration on the right and a cofibration on the left. Thus the lift, that is the dotted arrow, exists in $\mathbf{Top}$. The only question is whether it preserves the base point. But since the left vertical does and the top horizontal one as well, then it necessarily does. It is thus a lift in $\mathbf{Top}_*$. The other lifting property is completely similar. In the same way assume $f : (X, x_0 \to (Y, y_0)$ is a pointed map and let $X \rightarrowtail C_f \overset{\sim}{\twoheadrightarrow} Y$ be a factorisation in $\mathbf{Top}$. By declaring the base point of C_f to be the image of x_0, then the factorisation is pointed and still natural.

Thus the only thing left is the (co)completness of the category. A limit of pointed topological space is naturally pointed, because the base point gives a canonical map from $\{*\}$ to every space in the diagram (which does commute with all maps in the diagram since they are all pointed). The only difference is with the colimit. The construction of colimits in $\mathbf{Top}_*$ is as follows: given a category $\mathcal{D}$, consider the new category $\mathcal{D}_+$ obtained by adding an object $+$ to $\mathcal{D}$ and exactly one map from $+$ to every object (that is we create an initial object in $\mathcal{D}$). To a diagram $X : \mathcal{D} \to \mathbf{Top}_*$ of pointed spaces, we add $X(+) = \{*\}$ and $X(+ \to d) = (* \mapsto X(d)_0)$, the base point of $X(d)$. This yields a diagram of spaces $X^+ \colon \mathcal{D}_+ \to \mathbf{Top}$, which is well defined since the image of $\{*\} = X(+)$ is a basepoint in $\operatorname{colim}_{\mathcal{D}_+} X^+$ and any natural transformation of diagrams is pointed. One then verifies that $\operatorname{colim}_{\mathcal{D}}(X) \cong \operatorname{colim}_{\mathcal{D}_+} X^+$ in $\mathbf{Top}_*$.

3. Any model category is complete, hence has a terminal object $*$. The idea is then to define a category $\mathcal{C}_{*/}$ whose objects are maps $* \to X$ in $\mathcal{C}$ from the terminal object to any object and whose morphisms $(* \xrightarrow{f} X) \to (* \xrightarrow{h} Y)$ in $\mathcal{C}_{*/}$ are just maps $g : X \to Y$ in $\mathcal{C}$ such that $h = g \circ f$ (draw the commutative triangle). Then the arguments of the previous question apply and show that $\mathcal{C}_{*/}$ admits a model structure for which the forgetful functor $\mathcal{C}_{*/} \longrightarrow \mathcal{C}$ is a right Quillen functor.

[5]but often imposing two classes do