Simplicial Sets

Exercice 1 (*Modules over cdgas*)**.** Let A be a cdga (commutative differential graded algebra) over $\mathbb{Q}$. Let $\mathsf{Mod}(A)$ denote the category of dg modules over A. An object in $\mathsf{Mod}(A)$ is thus a cochain complex M together with a morphism of complexes $A \otimes_{\mathbb{Q}} M \to M$ satisfying the module axioms (i.e. $(a \cdot b) \cdot m = a \cdot (b \cdot m)$, $1 \cdot m = m$).

1. Show that the forgetful functor $U : \mathsf{Mod}(A) \to \mathsf{Ch}(\mathbb{Q})$ is a right adjoint and describe its left adjoint F.
2. Show that there is a model structure on $\mathsf{Mod}(A)$ where
   - weak equivalences are the morphisms f such that $U(f)$ is a quasi-isomorphism,
   - fibrations are the morphisms f such that $U(f)$ is surjective.
3. Show that the functor $- \underset{A}{\otimes} - \colon \mathsf{Mod}(A) \times \mathsf{Mod}(A) \longrightarrow \mathsf{Mod}(A)$ admits a total left derived functor
$$- \overset{\mathbb{L}}{\underset{A}{\otimes}} - : \mathrm{Ho}(\mathsf{Mod}(A) \times \mathsf{Mod}(A)) \cong \mathrm{Ho}(\mathsf{Mod}(A)) \times \mathrm{Ho}(\mathsf{Mod}(A)) \longrightarrow \mathrm{Ho}(\mathsf{Mod}(A)).$$
4. Let $f : A \to B$ be a morphism of cdgas. Show that the functor $f_* : \mathsf{Mod}(B) \to \mathsf{Mod}(A)$, given by $A \otimes_{\mathbb{Q}} M \overset{f \otimes id}{\to} B \otimes_{\mathbb{Q}} M \to M$, is a right Quillen functor.
5. Assume $f : A \to B$ is a quasi-isomorphism of cdgas. Show that f_* is a Quillen equivalence.

Solution 1. One possible reference is the book *Modules over operads and functors*, by Benoît Fresse (sections 11.2.5 – 11.2.10).

Exercice 2 (Playing with simplicial sets)**.** We recall that Δ is the category whose objects are finite ordered sets $[n] = \{0 < 1 < \cdots < n\}$ and morphisms are order-preserving maps. Denote by $\Delta[n] = \Delta^n_\bullet \in \mathsf{sSet}$ the Yoneda embedding: $\Delta[n] := \mathrm{Hom}_\Delta(-, [n])$. We recall that if X is a simplicial set, the data of a n-simplex of X, corresponds to the data of a simplicial set morphism $\Delta[n] \to X$.

1. Write d_i and ϵ_j the face and degeneracies. Check that any map $f : [m] \to [n]$ in Δ can be factored in a unique way as $f =$
$$[m] \xrightarrow{\epsilon_{j_1}} [m-1] \xrightarrow{\epsilon_{j_2}} \dots \xrightarrow{\epsilon_{j_t}} [m-t] \xrightarrow{\partial_{i_1}} [m-t+1] \xrightarrow{\partial_{i_2}} \dots \xrightarrow{\partial_{i_k}} [m-t+k] = [n]$$
where $j_t < j_{t-1} < \cdots < j_1$ are the elements of $[m]$ with $f(j) = f(j+1)$ and $i_1 < i_2 < \cdots < i_k$ are the values in $[n]$ that are not in the image of f. Conclude that Δ is the free category generated by the objects $[n]$ and morphisms ∂_i and ϵ_j submitted to the simplicial relations.
2. Check that a morphism $f : [m] \to [n]$ is an epimorphism if and only if it is a non-decreasing surjection and that the simplicial relations imply that every epimorphism is split.
3. (Eilenberg-Zilber Lemma) Let X be a simplicial set. Show that for each m-simplex $\sigma : \Delta[m] \to X$ there is an epimorphism $s : \Delta[m] \to \Delta[n]$ and a non-degenerate n-simplex $x : \Delta[n] \to X$ such that $y \circ s = \sigma$. Show that the pair (y, s) is unique.

4. (Skeletons) We denote by $\mathrm{sk}_n(X)$ the subsimplicial set of $X \in \mathsf{sSet}$ given by the non-degenerate simplices of X of dimension less than n. Thus its p-simplices are the p-simplices σ of X such that there exists an epimorphism $s : \Delta[p] \to \Delta[q]$ with $q \leq n$ and a q-simplex $x : \Delta[q] \to X$ such that $x \circ s = \sigma$. In other words, for $q \leq n$ the q-cells of $\mathrm{sk}_n(X)$ coincide precisely with the q-cells of X. For $m > n$, the m-cells of $\mathrm{sk}_n(X)$ are given by the m-cells of X which are degenerate.

   The construction $X \mapsto \mathrm{sk}_n(X)$ can be seen as a right adjoint: let $\Delta_{\leq n}$ denote the full subcategory of Δ spanned by those objects $[k]$ with $k \leq n$. Write $i_n : \Delta_{\leq n} \hookrightarrow \Delta$ for the inclusion functor.

   (a) Let $T \in \mathrm{Fun}(\Delta^{op}_{\leq n}, \mathsf{Set})$. Prove that the formula[1] $(i_n)_!(T)_* := \operatorname*{colim}_{* \to k \leq n} T(k)$ defines a functor $(i_n)_! : \mathrm{Fun}(\Delta^{op}_{\leq n}, \mathsf{Set}) \to \mathsf{sSet}$ and that the functor $(i_n)_!$ admits a right adjoint $(i_n)^*$.

   (b) Show that for any $X \in \mathrm{Fun}(\Delta^{op}_{\leq n}, \mathsf{Set})$, the unit of the adjunction $X \to (i_n)^*(i_n)_! X$ is an isomorphism. Conclude that $(i_n)^*$ is fully faithful.

   (c) Show that for any simplicial set X, the co-unit of the adjunction $(i_n)_!(i_n)^*(X) \to X$ is injective and show that its image in X coincides with the sub-simplicial set $\mathrm{sk}_n(X)$;

   (d) Show that the canonical map $\operatorname*{colim}_{n \geq 0} \mathrm{sk}_n(X) \to X$ is an isomorphism.

5. (Boundaries) We give an alternative presentation of $\partial\Delta[n]$ as the result of gluings all the $n-1$-simplices of $\Delta[n]$ along the $n-2$-simplices. Consider the diagram

$$\coprod_{0 \leq i < j \leq n} \Delta[n-2] \overset{u}{\underset{v}{\rightrightarrows}} \coprod_{0 \leq i \leq n} \Delta[n-1] \xrightarrow{p} \Delta[n]$$

   where the map p is induced by the inclusions of the faces of $\Delta[n]$. Each copy of $\Delta[n-2]$ on the l.h.s corresponds to a copy of $[n]$ where both i and j are missing. Similarly, each copy of $\Delta[n-1]$ on the r.h.s corresponds to a copy of $[n]$ where a single element i is missing. The map u is induced by the boundary maps $\partial^{j-1}_{n-1} : \Delta[n-2] \to \Delta[n-1]$ and the maps v are induced by the boundary maps $\partial^{i}_{n-1} : \Delta[n-2] \to \Delta[n-1]$. Check that the image of p is the set of simplices in $\Delta[n]$ belonging to $\partial\Delta[n]$ and conclude that $\partial\Delta[n]$ is isomorphic to the co-equalizer of (u, v).

6. Let X be a simplicial set. Show that for each $n \geq 0$ the squares

$$\begin{tikzcd} \coprod_{\sigma \in X_n,\ \sigma \text{ non-deg}} \partial\Delta[n]_\sigma \arrow[r] \arrow[d, "\text{inclusion}"'] & Sk_{n-1}(X) \arrow[d] \\ \coprod_{\sigma \in X_n,\ \sigma \text{ non-deg}} \Delta[n]_\sigma \arrow[r] & Sk_n(X) \end{tikzcd}$$

   are cocartesian. This allows us to construct X by induction on n.

7. (Horns) Recall the notion of j-horn Λ^j_n the sub simplicial set of $\Delta[n]$ in which the jth face and the interior have been removed.

   (a) Prove that the m-simplices of Λ^j_n are the order preserving maps $p : [m] \to [n]$ whose image does not contain the set $[n] - \{j\}$.

   (b) Describe the horn using boundaries and skeletons.

   (c) Deduce that $\mathrm{Hom}_{\mathsf{sSet}}(\Lambda^r_n, X)$ is in bijection with the set of n-tuples of $(n-1)$-simplices $(x_0, \ldots, \widehat{x_r}, \ldots x_n)$ of X such that for all $i, j \neq r$ and $i < j$, one has $d_i x_j = d_{j-1} x_i$.

[1] this is nothing more than the left Kan extension along the inclusion i_n

(d) Prove that a simplicial set is fibrant if and only if, for any $k \leq n$ and n-tuple of $(n-1)$-simplices $(x_0, \dots, \widehat{x_r}, \dots x_n)$ of X satisfying that, for all $i, j \neq r$ and $i < j$, $d_i x_j = d_{j-1} x_i$, then there exists a n-simplex $x \in X$ such that $d_i(x) = x_i$ for all $i \neq k$.

8. Deduce that the simplicial set $\Delta[n]$ is not fibrant for $n \geq 1$.

Solution 2. The first three questions are really meant to make the reader play with simplicial sets in its categorical incarnation.

1. First note that since the maps in Δ are non-decreasing, then the preimage $f^{(-1)}(\{i\})$ of any integer $i \in [n]$ is either empty or is an interval $\{j, j+1, \dots, j+k\}$. Now let $\{j_t \cdots < j_1\}$ be the (possibly empty) ordered subset of $[m]$ of all integers j such that $f(j) = f(j+1)$. Also, let $\{i_1 < \cdots < i_k\}$ be the (possibly empty) ordered subset of $[n]$ consisting of all values in $[n]$ which are *not* in the image of f. Then, since f is nondecreasing, we have that $f(1)$ is the least integer in $[n] \setminus \{i_1 < \cdots < i_k\}$. If $1 \notin \{j_1 < \cdots < j_t\}$, then $f(2)$ is the least integer in $([n] \setminus \{i_1 < \cdots < i_k\}) \setminus \{f(1)\}$. If $1 = j_1$, then $f(2) = f(1)$ by definition of j_1. We can continue this reasoning inductively to see that the data of the two ordered sets $\{j_t < \cdots < j_1\}$ and $\{i_1 < \cdots < i_k\}$ uniquely determined f. Now, note that the definition of ϵ_k is precisely a map which identifies the value of k and $k+1$ and that ∂_j is precisely the map jumping ahead the value j. Hence, the previous description of f in terms of the two ordered sets gives that

$$f = \partial_{i_k} \circ \cdots \circ \partial_{i_1} \circ \epsilon_{j_t} \circ \cdots \circ \epsilon_{j_1}$$

and further that any composition $g = \partial_{i'_k} \circ \cdots \circ \partial_{i'_1} \circ \epsilon_{j'_t} \circ \cdots \circ \epsilon_{j'_1}$ (where the integers j's are in decreasing order and the i's are in increasing order) satisfies that the ordered set of values not taken by g is $\{i'_1 < \cdots < i'_k\}$ and that the ordered subsets of integers j who take the same value as $j+1$ is $\{j'_t < \cdots < j'_1\}$. This proves the uniqueness of the decomposition while above we have seen the existence.

The uniqueness and existence of the decomposition implies that Δ is freely generated by $[n]$ and the faces and degeneracies together with the cosimplicial identities seen in class. Indeed, let Δ' be the free category generated by those. Then we have a canonical functor $\Delta' \to \Delta$ which is the identity on objects and on the ∂_i, ϵ_j. We are left to prove that this functor is a bijection on the Hom-sets. It is clearly surjective since every map of Δ decomposes as a composition of faces and degeneracies. Let $q \in \mathrm{Hom}_{\Delta'}([m], [n])$. Then q is a finite composition of faces and degeneracies and the cosimplicial identities implies that we can switch around the s_j and ∂_i so that we can first apply only composition of s_j's and then a composition of ∂_i's. Further they allow to reorder the composition of s_j' as well as the one of the faces ∂_i's so that the j are in decreasing order and the i are in increasing order. We do not know *a priori* if such a decomposition is unique. Anyway, assume $q, p \in \mathrm{Hom}_{\Delta'}([m], [n])$ have the same image in Δ. Take a decomposition of q and p as above. Then their image in Δ have the same decomposition and since there is uniqueness of the decomposition in Δ, the two decomposition are the same. Hence the maps are the same. Which concludes the proof that Δ' is isomorphic to Δ.

2. If $f : [n] \to [m]$ is an epimorphism then it is necessarily surjective. Indeed, if it misses the value j, then $\tau_j \circ f = id \circ f$ where $\tau_j : [n] \to [n]$ is the map sending j to $j-1$ and is the identity on other values. This contradicts the epimorphim property. The reciprocical assertion is obvious since a non-surjective map is already an epimorphism f of sets and that maps of Δ are non-decreasing ones. By the first question, an epimorphism is written (uniquely as) a composition $f = \epsilon_{j_t} \circ \cdots \circ \epsilon_{j_1}$ with $j_1 > \cdots > j_t$. The cosimplicial identities implies that $\partial_{j_1} \circ \cdots \circ \partial_{j_t} \circ f = id$ which shows that there is a section of f.

3. Note that a n-simplex $\sigma : \Delta[n] \to X$ is degenerated if σ can be factored as

$$\begin{tikzcd} \Delta[n] \arrow[r, "\sigma"] \arrow[d, "s"'] & X \\ \Delta[m] \arrow[ur, "x"'] & \end{tikzcd}$$

with s an epimorphism for some $m < n$. The existence of the decomposition follows from the factorization properties of Δ. The unicity of the pair is more subtle: let us redo the proof seen in class: Suppose (s, x) and (s', x') are two pairs verifying the hypothesis. Then as every epimorphism splits, both s' and s admit sections, say, t and t' resp. Using the fact s and s' are epimorphisms we deduce the relations $x = \sigma \circ t$ and $x' = \sigma \circ t'$ so that $x = x' \circ s' \circ t$. Since by hypothesis x is non-degenerated, we must have $s' \circ t$ a monomorphism and thus $n' \geq n$. By the symmetric argument we deduce $n \geq n'$ so that $n = n'$. Moreover, as $s' \circ t$ is order preserving, it must be the identity map of $[n]$ so that $x = x'$. This implies $s' = s$.

4. The inclusion functor $i_n : \Delta_{\leq n} \hookrightarrow \Delta$ induces by precomposition a functor $\mathsf{sSet} = Fun(\Delta^{op}, \mathsf{Set}) \to Fun(\Delta^{op}_{\leq n}, \mathsf{Set})$ given by $(\Delta^{op} \xrightarrow{F} \mathsf{Set}) \mapsto (\Delta^{op}_{\leq n} \xrightarrow{i_n} \Delta^{op} \xrightarrow{F} \mathsf{Set})$. We denote $(i_n)^*$ this functor. Thus for a simplicial set $Y_\bullet$, we have, for $k \leq n$ that $(i_n)^*(Y_\bullet)_k = Y_k$ and the structure maps are simply the restriction of the one of Y to the indexes lower than k.

(a) To prove the formula $(i_n)_!(T)_* := \underset{*\to k\leq n}{\operatorname{colim}} T(k)$ defines a functor we have first to check we have a natural simplicial structure on $(i_n)_!(T)_*$. if $f : [m] \to [\ell]$ is a map in Δ, then by precomposition we get, for any map $[\ell] \to [k]$ another map in Δ defined by $[m] \xrightarrow{f} [\ell] \to [k]$, hence we get a canonical map $f^* : \underset{[\ell]\to[k], k\leq n}{\operatorname{colim}} T([k]) \to \underset{[m]\to[k], k\leq n}{\operatorname{colim}} T([k])$ induced by the identity map $T(k) \to T(k)$ sending the $T(k)$ indexed by $j : [\ell] \to [k]$ to the one indexed by $j \circ f : [m] \to [k]$. We have $(f \circ g)^* = g^* \circ f^*$ hence $(i_n)_!(T)_*$ is indeed a simplicial set. Similarly a natural transformation $(T(k) \to U(k))_{k\leq n}$ induces a map of colimits $(\underset{m\to k\leq n}{\operatorname{colim}} T(k) \longrightarrow \underset{m\to k\leq n}{\operatorname{colim}} U(k))_{m\in\mathbb{N}}$ by post-composition which shows that $(i_n)_!$ is indeed a functor.

We need to check the adjunction formula. Let $g = (g_k : X_k \to (i_n)^*(Y_\bullet)_k = Y_k)_{k\leq n}$ be a map in $Fun(\Delta^{op}_{\leq n}, \mathsf{Set})$ where $X \in Fun(\Delta^{op}_{\leq n}, \mathsf{Set})$ and $Y_\bullet \in \mathsf{sSet}$. We wish to define a map $\theta(g) : (i_n)_!(X)_\bullet \to Y_\bullet$, that is, for every $m \in \mathbb{N}$, a map $\underset{m\to k\leq n}{\operatorname{colim}} X(k) \to Y_m$ commuting with the faces and degeneracies which by universal property of colimits is equivalent to the data of a map $\theta_j(g) : X(k) \to Y_m$ for each $j : [m] \to [k]$ (such that for all composable arrows $[m] \xrightarrow{i} [\ell \leq n] \xrightarrow{j} [k \leq n]$, one has $\theta_{j\circ i}(g) = \theta_j(g) \circ i^*$ where $i^* : X(k) \to X(\ell)$ is the structure map given by the functor X). Since $Y_\bullet$ is a simplicial set, the map $j : [m] \to [k]$ gives the map $j^* : Y_k \to Y_m$, so that we can define

$$\theta_j(g) := X(k) \xrightarrow{g_k} Y_k \xrightarrow{j^*} Y_m.$$

This map passes to the colimit since for all non-decreasing map $i : [\ell \leq n] \to [k \leq n]$ one has $i^* \circ g = g \circ i^*$. That the map $\theta(g)$ induced on the colimit is a map of simplicial sets follows from the formula $(j \circ f)^* \circ g_k = f^* \circ j^* \circ g_k$. We also define, for any simplicial set map $f : (i_n)_!(X)_\bullet \to Y_\bullet$ a natural trasnformation (that is a map of functors) $\psi(f) : X \to (i_n)^*(Y_\bullet)$ as follows. As seen above the map f is equivalent to the data of set maps $X(k) \xrightarrow{f(j)} Y_m$ for any $j : [m] \to [k]$ (compatible with factorisations $[m] \xrightarrow{i} [\ell \leq n] \xrightarrow{j} [k \leq n]$ as above). Taking $id : [k] \to [k]$ yields $g(k) : X(k) \xrightarrow{f([k]\xrightarrow{id}[k])} Y_k$. The compatibility precisely gives that $(g(k))_{k\leq n}$ is a natural transformation. It is immediate (from the definition of θ_{id}) that $\psi(\theta(g)) = g$. For the other direction, note that by definition of the simplicial structure of $(i_n)_!$ given above, in simplicial degree m, the contribution of the component $X(k)$ indexed by $j[m] \to [k \leq n]$ in the colimit is precisely induced by j^* of the component corresponding to $[k] \xrightarrow{id} [k]$ in simplicial degree k. Hence if $f : (i_n)_!(X) \to Y_\bullet$ is a simplicial set morphism, we have that the induced map $X(k) \xrightarrow{f(j)} Y_m$ is equal to $X(k) \xrightarrow{f([k]\xrightarrow{id}[k])} Y_k \xrightarrow{j^*} Y_m$ which shows that $\theta \circ \psi(f) = f$ and the adjunction formula is proved, since the naturality follows from the universal property of colimit.

(b) Let $X : \Delta_{\leq n} \to Fun(\Delta^{op}_{\leq n}, \mathsf{Set})$. By Yoneda, X is a colimit of representables $X([m])$ with $m \leq n$. Using the fact that both $i_!$ and i^* commute with colimits we are reduced to show that $X([m]) \to$

$i^*(\Delta[m])$ is an isomorphism for $m \leq n$. This is an immediate check by the above explicit formulae for the adjunction. The fully faithfulness is implied by the fact that the unit is an isomorphism since we have bijections

$$\mathrm{Hom}((i_n)^*(X), (i_n)^*(Y)) \cong \mathrm{Hom}((i_n)_!((i_n)^*(X)), Y) \xrightarrow{\cong} \mathrm{Hom}(X, Y)$$

where the last one is given by precomposition with the unit.

(c) As we have seen in the explicit description of the adjunction in (a), notice that $(i_n)_!(i_n)^*(X) \to X$ is given by the colimit $\mathrm{colim}_{h:[m]\to(i_n)^*(X)}\Delta[m]$ so that all its p-simplices for $p > n$ are degenerated. Moreover, thanks to the previous exercise, the k-simplices of $(i_n)_!(i_n)^*(X) \to X$ are in bijection with the k-simplices of X for $k \leq n$, so it is injective. To conclude that the map is injective for $p > n$ we use Eilenberg-Zilber's lemma to reduce the question to non-degenerated simplices which by this discussion are necessarily of dimension $\leq n$.

(d) This follows because Δ is the colimit of the categories $\Delta_{\leq n}$ and by the universal property of colimits in categories, presheaves over Δ are the limit of presheaves over each $\Delta_{\leq n}$.

5. Note that the boundary of the standard simplex $\Delta[n]$, denoted as $\partial\Delta[n]$, can be defined as $Sk_{n-1}\Delta[n]$ since it is obtained by removing from $\Delta[n]$ the unique n-simplex non-degenerated (and there are no higher dimensional ones). Thus it is generated by the non-degenerate simplices in dimension $< n$. But each of these simplices contains at most n vertex of $\Delta[n]$ (these vertex each corresponding to an integer $i \in \{0, \dots, n\}$) hence is included in a face $[n] \setminus \{j\}$ and thus in the image of p. On the other hand since the source of p has no simplices in dimension $\geq n$ its image lies necessarily in $Sk_{n-1}\Delta[n] = \partial\Delta[n]$. The fact $p \circ u = p \circ v$ is nothing but the simplicial identity $\partial_{n-1}^{j-1} \circ \partial_n^i = \partial_{n-1}^i \circ \partial_n^j$ for $i < j$. One checks that $\partial\Delta[n]$ is indeed the coequalizer by showing it has the universal property. Any simplicial set map $f : \coprod_{0\leq i\leq n} \Delta[n-1] \to Y$ such that $f \circ u = f \circ v$ is defined uniquely by its restriction in degrees 0 to $n-1$. It defines a map from $\partial\Delta[n] \to Y$ by setting its value on all degree $n-1$-non degenerate simplices to be given by the restriction of f to the component $\Delta[n-1]$ corresponding to i (in other words, one uses the injective increasing maps ∂_n^i to identify $\coprod_{0\leq i\leq n} \Delta[n-1]$ with $\coprod_{0\leq i\leq n} \mathrm{Hom}_{\text{non decreasing}}([\bullet], [n] \setminus \{i\})$). To check that this map is indeed a simplicial set map, we only need to check the faces compatibility since there are no higher non-degenerate simplices. But these faces identities are exactly provided by the simplicial identity on faces which is the coequalizer condition on u and v.

In concrete terms the main point is that the diagram

$$\begin{tikzcd}
[n-2] \arrow[r, "\partial^i"] \arrow[d, "\partial^{j-1}"'] & [n-1] \arrow[d, "\partial^j"] \\
[n-1] \arrow[r, "\partial^i"] & [n]
\end{tikzcd}$$

is a pullback in Δ (which boils down to intersection). Hence $\Delta[n-2] \cong \Delta[n-1] \times_{\Delta[n]} \Delta[n-1]$ and the latter is isomorphic to $\Delta[n-1] \times_{\partial\Delta[n]} \Delta[n-1]$ because the map to $\delta[n]$ factors through $\partial\Delta[n]$.

6. To check that this is true it is enough to check that each restriction $(i_k)^*$ is cocartesian for $k \geq 0$. More easily, as all the simplicies are n-truncated, it is enough to check that the diagram is cocartesian after applying $(i_k)^*$ for $k \leq n$. This is clear when $k < n$ as in this case the vertical left arrow is an isomorphism. In other words, the diagram is a pushout for what concerns simplexes of dimension $< n$. It is enough to check that this holds for $k = n$. But this is true as in k=n the top line of the diagram does not have non-degenerated cells so that the only cells added at the non-degenerated cells of X encoded by the $(\coprod_{\sigma\in X_n, \sigma\, non-deg} \Delta[n]_\sigma)_n$. This makes the skeleton n of X.

7.

(a) Recall that $\Delta[n]_k = \mathrm{Hom}_\Delta([k], [n])$, hence a k-simplex of $\Delta[n]$ is a non-decreasing application with value in $\{0, \dots, n\}$. The n-simplex is the identity map $id : [n] \to [n]$. Note that if $f : [k] \to [n]$

is a k-simplex, then all its iterated degeneracies $\epsilon_{j_k} \circ \cdots \epsilon_{j_1} \circ f$ have the samie image in $[n]$ as f (as $\epsilon_k \circ f$ does not change the image because it is obtained as $f \circ \epsilon^k$ where the last map is surjective). Hence $\partial\Delta[n]_k$ is included in the subset of non-surjective non decreasing map from $[k]$ to $[n]$. Similarly the i-face of $id : [n] \to [n]$ is the map $\partial^i : [n-1] \cong [n] \setminus \{i\} \hookrightarrow [n]$ and, reasoning in the same way, this allow to see the sub-simplicial set generated by the i-face has the subset of all non decreasing maps with values in $[n] \setminus \{i\}$; its only non-degenerate $n-1$-simplex is thus identified with ∂_n^i and its iterated degeneracies are precisely the maps $f : [k] \to [n]$ whose image is exactly $[n] \setminus \{i\}$. It follows that $\partial\Delta[n]$ is the subset of all non-surjective maps and further, since Λ_n^i is obtained by removing the i-face in $\partial\Delta[n]$, we get that the k-simplices of Λ_n^i are the the non-decreasing maps $p : [k] \to [n]$ whose image does not contain the set $[n] - \{j\}$.

(b) We claim that Λ_n^r is the coequalizer of the diagram

$$\coprod_{0\le i<j\le n, i,j\neq r} \Delta[n-2] \underset{v}{\overset{u}{\rightrightarrows}} \coprod_{0\le i\neq r\le n} \Delta[n-1] \,.$$

the difference with question 5 being that we remove one component $\Delta[n-1]$ in the right (precisely the one associated to the k-th face (as well as those that maps onto it). The maps u and v are defined as in question 5. Then we get the claim by noticing as in question 5 that $\Delta[n-2] \cong \Delta[n-1] \times_{\Delta[n]} \Delta[n-1] \cong \Delta[n-1] \times_{\Lambda_n^r} \Delta[n-1]$ where the maps to $\Delta[n-1] \to \Delta[n]$ are induced by ∂^i and ∂^j with $i, j \neq r$. Hence the above coequalizer is isomorphic to $\coprod_{0\le i<j\le n, i,j\neq r} \Delta[n-1] \times_{\Lambda_n^r} \Delta[n-1] \underset{v}{\overset{u}{\rightrightarrows}} \coprod_{0\le i\neq r\le n} \Delta[n-1]$ which is Λ_n^r by definition.

(c) Using that $\mathrm{Hom}_{\mathsf{sSet}}(\Delta[m], X) \cong X_m$ and moving colimits out of Hom, the previous question shows that

$$\mathrm{Hom}_{\mathsf{sSet}}(\Lambda_n^r, X) \cong \mathrm{Equalizer}\Big(\prod_{i\neq r} X_{n-1} \underset{v^*}{\overset{u^*}{\rightrightarrows}} \prod_{i,j\neq r, i<j} X_{n-2}\Big)$$

where u^* and v^* are the boundary maps d_{j-1} and d_i. This is exactly the result.

(d) Let $p : \coprod_{0\le i\neq r\le n} \Delta[n-1] \to \Delta[n]$ be thr map induced on the component i by ∂^i. Then as in question 5. its image is precisley the r-horn Λ_n^r. It follows that the restriction $\mathrm{Hom}_{\mathsf{sSet}}(\Delta[n], X) \to \mathrm{Hom}_{\mathsf{sSet}}(\Lambda_n^r, X))$ is precisely the map onto the equalizer of (c) induced by the maps $X_n \xrightarrow{\prod_{i\neq r} d_i} \prod_{i\neq r} X_{m-1}$. The result now follows from the previous question.

8. It suffices to prove that $\Delta[1]$ is not. Consider the map $\Lambda_0^2 \to \Delta[1]$ given by the pair $\big(\epsilon_0(\{0\}), [1] \overset{id}{\to} [1]\big)$. Then $d_1([1] \overset{id}{\to} [1]) = 0 \mapsto 0 = \{0\} = d_1(\epsilon_0(\{0\})$ which proves by (c) that it does define a simmplicial set maps.

Assume a lift exists $\Delta[2] \to \Delta[1]$ exists, then by (d), we shall have $x \in \Delta[1]_2$ such that $d_2(x) = [1] \overset{id}{\to} [1]$ and $d_1(x) = \epsilon_0(\{0\})$. The simplicial identity further tells that $d_1(d_0(x)) = d_0(d_2(x)) = [0] \overset{\partial id \circ d_1^0}{\longrightarrow} [1] = 0 \mapsto 1$ and furtehr that $d_0(d_0(x)) = d_0(d_1(x)) = \{0\}$. Hence $d_0(x) : [1] \to [1]$ is a non-decreasing map f such that $d_1(f) = f(0) = 1$ but $d_0(f) = f(1) = 0$ which is absurd.

Exercice 3 (Detailled construction of the Nerve of a category)**.** In this exercise and the following, we establish a link between the theory of categories and the theory of simplicial sets. More precisely, we check that we can translate the information provided by a category $\mathcal{C}$ into a simplicial set, called the nerve of $\mathcal{C}$ and denoted by $N(\mathcal{C})$. We will see that this translation does not lose any information and that in fact the theory of categories can be seen as a sub-theory of that of simplicial sets.

1. The category of simplexes Δ can be canonically identified with a full subcategory of Cat, spanned by the categories of the form $[n] := [0 \to 1 \to \cdots \to n]$. Use this inclusion and the previous exercise to produce an adjunction $\mathsf{sSet} \underset{N}{\overset{\tau}{\rightleftarrows}} \mathsf{Cat}$ sending $\tau(\Delta[n]) = [n]$.

2. Let $\mathcal{C}$ be a small category. Check that the functor N is characterized as follows: $N(\mathcal{C})_n$ consists of composable strings of morphims in $\mathcal{C}$ of lenght n:
$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \ldots \xrightarrow{f_{n-1}} X_n.$$
In particular, the 0-simplexes of $N(\mathcal{C})$ are the objects of $\mathcal{C}$ and the 1-cells are morphisms in $\mathcal{C}$. Describe the face and degeneracy maps in terms of compositions and identity morphims.

3. Show that the canonical morphism induced by the inclusion $\tau(\partial\Delta[n]) \to \tau(\Delta[n]) = [n]$ is an isomorphism of categories for $n \geq 3$. Describe both $\tau(\partial\Delta[1])$ and $\tau(\partial\Delta[2])$.
(*Hint:* use the construction of $\partial\Delta[n]$ as a cokernel).

4. Deduce that the canonical map $\tau(\mathrm{sk}_2(X)) \to \tau(X)$ is an isomorphism of categories for every simplicial set X. In other words, the category $\tau(X)$ only depends on the 2-skeleton of X.

5. Let X be a simplicial set. Check that the category $\tau(\mathrm{sk}_2(X))$ is isomorphic to the quotient of the free category with X_0 as objects and X_1 as morphisms under the following relation on morphisms:
   - for every 2-simplex $\sigma : \Delta[2] \to X$, we identify $\partial_1(\sigma)$ with the composition $\partial_0(\sigma) \circ \partial_2(\sigma)$.
   - for every $x \in X_0$, identify $\epsilon_0(x)$ with Id_x

6. Let $\mathcal{C}$ be a category and describe the category $\tau(\mathrm{sk}_2(N(\mathcal{C}))$. Conclude that the adjunction map $\tau(N(\mathcal{C})) \to \mathcal{C}$ is an isomorphism of categories and that N is fully faithful.

7. Let I_n denote the sub-simplicial set (subfunctor) of $\Delta[n]$ given by $\bigcup_i^n \mathrm{im}\ \alpha_i \subseteq \Delta[n]$ where $\alpha_i : \Delta[1] \to \Delta[n]$ is the map sending $0 \to i$ and $1 \mapsto i+1$. Show that I_n is the colimit of the diagram
$$\begin{tikzcd}
\Delta[1] & & \Delta[1] & & \cdots & & \Delta[1] \\
& \Delta[0] \arrow[ul, "\partial_1"] \arrow[ur, "\partial_0"'] & & \Delta[0] \arrow[ul, "\partial_1"] \arrow[ur, "\partial_0"'] & \cdots & \Delta[0] \arrow[ul, "\partial_1"] \arrow[ur, "\partial_0"'] &
\end{tikzcd}$$
where $\Delta[1]$ appears n times.

8. Let $\mathcal{C}$ be a category and let $N(\mathcal{C})$ denote its nerve. Show that the composition with the inclusion $I_n \subseteq \Delta[n]$ produces a bijection
$$\mathrm{Hom}_{\mathsf{sSet}}(\Delta[n], N(\mathcal{C})) \cong \mathrm{Hom}_{\mathsf{sSet}}(I_n, N(\mathcal{C}))$$
for all $n \geq 2$. Conclude that the canonical map $\tau(I_n) \to \tau(\Delta[n]) = [n]$ is an isomorphism of categories for $n \geq 2$.

Solution 3. Note that the category $[n] := [0 \to 1 \to ... \to n]$ is just the category associated to the poset $0 < 1 \cdots < n$. In particular an order preserving map $[n] \to [m]$ is a functor from the category $[n]$ to the category $[m]$ (as one can simply check by hand).

1. Let us define $N : \mathrm{Cat} \to \mathrm{sSet}$ as the functor defined as follows. To a small category $\mathcal{C}$ we associate the family of sets $N(\mathcal{C})_n := \mathrm{Hom}_{\mathrm{Cat}}([n], \mathcal{C})$, in other words the set of functors from the category $[n]$ to $\mathcal{C}$. Since the morphisms of Δ are precisely the non-decreasing application which as we have seen are functors between categories of the form $[n]$, it is immediate that any non-decreasing map $f[n] \to [m]$ induces a map $f^* : N(\mathcal{C})_m = \mathrm{Hom}_{\mathrm{Cat}}([m], \mathcal{C}) \xrightarrow{-\circ f} \mathrm{Hom}_{\mathrm{Cat}}([n], \mathcal{C}) = N(\mathcal{C})_m$ by composition of functors. By functoriality of composition of functors, we obtain a well defined functor $N(\mathcal{C}) = \mathrm{Hom}_{\mathrm{Cat}}(-, \mathcal{C}) : \Delta^{op} \to \mathrm{sSet}$ given by the collection of the $(N(\mathcal{C})_n)_{n\in\mathbb{N}}$. The construction shall really look like the construction of $\mathrm{Sing}_\bullet(X)$ for a space. We use the collection of the categories $[n]$ as an natural cosimplicial category where in the latter we were using the natural cosimplicial space $(\Delta^n)_{n\geq 0}$. That being seen, it is natural to find the left adjoint of the functor N by mimicking the definition of the geometric realization. More concretely, we set $\tau : \mathrm{sSet} \to \mathrm{Cat}$ by setting $\tau(X_\bullet) := (\coprod_{n\in\mathbb{N}} X_n \times [n]) \coprod_{(\coprod_{f:[n]\to[m]\in\Delta} X_m \times [n])} (\coprod_{m\in\mathbb{N}} X_m \times [m])$ to be the pushout[2] in Cat given by the diagram

$$\tau(X_\bullet) := \coprod_{n\in\mathbb{N}} X_n \times [n] \xleftarrow{f^*} \coprod_{f:[n]\to[m]\in\Delta} X_m \times [n] \xrightarrow{f_*} \coprod_{m\in\mathbb{N}} X_m \times [m]$$

where f^* is just induced by the map $f^* : X_m \to X_n$ given by the simplicial structure of $X_\bullet$ and f_* is just induced by $f : [n] \to [m]$. Note that since $\Delta[n]$ has a unique non-degenerate n-simplex ($id : [n] \to [n]$) and all others non-degenerate simplices are faces of it, then, it is immediate that the pushout defining $\tau(\Delta[n])$ is nothing more than the category $[n]$ itself. The proof that the two constructions are indeed an adjunction can be done in a similar way to the proof of the geometric realisation case seen in class; one only needs to replace continuous maps $\Delta^n \to Y$ by functors, and elements $\bar{t} \in \Delta^n$ by objects of $[n]$.

A companion proof is to use that every simplicial set is the colimit $X_\bullet \cong \mathrm{colim}_{\Delta[n]\to X_\bullet} \Delta[n]$. Since τ is defined by a colimit, it shows that to prove the adjunction it is enough to check it on all $\Delta[n]$. But then we have

$$\mathrm{Hom}_{\mathrm{sSet}}(\Delta[n], N(\mathcal{C})) \cong N(\mathcal{C})_n = \mathrm{Hom}_{\mathrm{Cat}}([n], \mathcal{C}) \cong \mathrm{Hom}_{\mathrm{Cat}}(\tau(\Delta[n]), \mathcal{C})$$

where the first identity is given by the Yoneda Lemma for $\Delta[n]$ as seen in class.
Remark: the pushout formula defining τ shows that for every degenerate simplex $\sigma \in X_n$, the category $\{\sigma\} \times [n]$ is collapsed into the category $\{y\} \times [j]$ corresponding to the unique non-degenerate simplex $y \in X_j$ that σ is an iterate degeneracy of. Hence, as for the geometric realisation of spaces, the category $\tau(X_\bullet)$ is uniquely defined by the non-degenerate simplices. Further, the set of objects of $\tau(X_\bullet)$ is exactly the set X_0 of vertices and every non-degenerate 1-simplex σ yields a morphism $d_1(\sigma) \to d_0(\sigma)$ in $\tau(X_\bullet)$.

2. We have seen that $N(\mathcal{C})_n = \mathrm{Hom}_{\mathrm{Cat}}([n], \mathcal{C})$. Since $[n]$ has only $n+1$ objects (the integers $0, \dots, n$) and exactly one non-identity morphisms $i \to j$ between two objects i, j such that $i < j$ (and no such morphism for $j \geq i$, a functor is given by $n+1$-objects $X_0, \dots X_n \in \mathcal{C}$ and one morphism $f_i : X_i \to X_{i+1}$ for any $i < n$.

3. Since $\partial\Delta[1] = \Delta[0] \coprod \Delta[0]$ has only two non-degenerates simplices both of degrees 0, we find from the explicit description above of τ that $\tau(\partial\Delta[1])$ is a discrete category with two objects. Similarly, $\partial\Delta[2]$ has three non-degenerates vertices and three non-degenerates 1-simplices linking them. $\tau(\partial\Delta[2])$ is a category with three objects and four morphisms - the three morphisms given directly as the images of the 1-simplexes and a new morphism generated by the free composition of the only possible composable arrow: namely the one linking $0 \to 1 \to 2$; this composition has no reason to be the the arrow $0 \to 2$ corresponding to the third edge since there are no higher non-degenerate simplices to identify them.

Now, for $n \geq 3$, we use the result of question 5 of Exercise **??** and that, since τ is a left adjoint, it commutes with colimits. Hence we get that $\tau(\partial\Delta[n])$ is the coequalizer of

$$\coprod_{0\leq i<j\leq n} \tau(\Delta[n-2]) \underset{\tau(v)}{\overset{\tau(u)}{\rightrightarrows}} \coprod_{0\leq i\leq n} \tau(\Delta[n-1]) \cong \coprod_{0\leq i<j\leq n} [n-2] \underset{b}{\overset{a}{\rightrightarrows}} \coprod_{0\leq i\leq n} [n-1] .$$

[2]this pushout is precisely the coequalizer of the maps f_*, f^*

Here the maps a, b are simply induced by the obvious inclusions missing either i or $j-1$ (that is d^i or d^{j-1}). The colimit is exactly $[n]$ because, unlike for $n=2$, any two arrows in the colimit category corresponding to $i \to i+1 \to i+2$ whose composition shall be an arrow in the colimit is actually equal to the arrow $i \to i+2$ in one category $[n] \setminus j$ (just take $j < i$ or $j > i+2$).

4. We know that X is the colimit of its skeletons and that each skeleton is built by induction via the pushouts along the inclusions $\partial\Delta[n] \to \Delta[n]$. As τ commutes with colimits the previous exercise solves the question.

5. Since $Sk_2(X)$ has no non-degenerates simplices of degree ≥ 3, we only have to understand the contributions of non-degenerate simplices of degree 1 and 2. We have seen in question 1 that the objects of $\tau(Sk_2(X)$ are X_0 and that X_1 generates morphisms. Note that if $\sigma \in X_1$ is degenerate, that is $\sigma = \epsilon_0(x)$, then, in the colimit defining $\tau(Sk_2(X)$, we have that $\tau(\sigma) = \epsilon_0(\tau(x)) = Id_x$. Now it remains to understand the two simplices. But in the two simplex $[2]$ we have that the unique morphism $0 \to 2$ is the composition $0 \to 1 \to 2$. But $0 \to 2$ is just the image $d_1([1])$ by the functor associated to d_1 while the subcategory $0 \to 1 \subset [2]$ is the image of d_2 and $1 \to 2$ the one of d_0. Hence the explicit formula of the colimit defining τ shows that every two simplex σ imposes a relation $\partial_1(\sigma) = \partial_0(\sigma) \circ \partial_2(\sigma)$. We have no other relations since we only need to consider non-degenerates simplices of degree less than 2.

6. By question 2., $N(\mathcal{C})_0$ is the set of objects of $\mathcal{C}$ and $N(\mathcal{C})_1$ is the set of morphisms in $\mathcal{C}$ and $N(\mathcal{C})_2$ is the set of all composable two arrows. Its faces are given by $\partial_0(\, X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2\,) = f_1$, $\partial_1(\, X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2\,) = f_1 \circ f_0$ and $\partial_2(\, X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2\,) = f_0$. Hence by question 5., we have that $\tau(Sk_2(N(\mathcal{C})))$ is the free category generated by the objects and arrows of $\mathcal{C}$ quotiented by the relation of composition in $\mathcal{C}$. It is thus isomorphic to $\mathcal{C}$ itself. By direct inspection, the adjunction map $\tau(N(\mathcal{C})) \to \mathcal{C}$ is the map taking the category $\tau(N(\mathcal{C}))$ which is the identity on objects and maps string of arrows to their class in $\mathcal{C}$. By the previous computations it is thus an isomorphism. This being proved we thus have the isomorphisms

$$\mathrm{Hom}_{\mathrm{Cat}}(\mathcal{D},\mathcal{C}) \cong \mathrm{Hom}_{\mathrm{Cat}}(\tau(N(\mathcal{D})),\mathcal{C}) \cong_{\mathrm{sSet}} (N(\mathcal{D}), N(\mathcal{C}))$$

which proves the fully faithfulness of N.

7. This essentially reduces to a computation of colimits of sets.

8. The simplicial set I_n has exactly $n+1$ non-degenerate 1-simplices, denoted $\alpha_i \cong \{i, i+1\}$, and no higher non-degenerates ones. The only relation between these 1-simplices are that $d_1(\alpha_{i+1}) = d_0(\alpha_i)$ hence, a simplicial set map from I_n to $X_\bullet$ is given by a n-tuple $(x_1, \dots, x_n)$ satisfying that $d_0(x_1) = d_1(x_2)$ and so on. In other words, $\mathrm{Hom}_{\mathrm{SSet}}(I_n, X) \cong X_1 \times_{X_0} X_1 \times \cdots \times_{X_0} X_1$. Applying this to $X_\bullet = N(\mathcal{C})$, we obtain that a map from I_n to $N(\mathcal{C})$ is exactly a string of n-composable arrows, hence the claimed isomorphism. We take $\mathcal{C} = [n] = \tau(\Delta[n])$. The canonical map $\tau(I_n) \to \tau(\Delta[n]) = [n]$ is by definition the image of the identity of $[n]$ under the map $\mathrm{Hom}_{\mathrm{Cat}}([n],[n]) \xrightarrow{-\circ\tau(i)} \mathrm{Hom}_{\mathrm{Cat}}(\tau(I_n),[n])$ where $i : I_n \hookrightarrow \Delta[n]$ is the inclusion. But by adjunction, and since $\tau(\Delta[n]) = [n]$, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Cat}}([n],[n]) \arrow[r, "\cong"] \arrow[d, "-\circ\tau(i)"'] & \mathrm{Hom}_{\mathsf{sSet}}(\Delta[n], N([n])) \arrow[d, "-\circ i"] \\
\mathrm{Hom}_{\mathrm{Cat}}(\tau(I_n),[n]) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathsf{sSet}}(I_n, N([n]))
\end{tikzcd}$$

where the right vertical map is a bijection by above. Hence the left vertical one is bijective too.

Exercice 4 (Segal conditions and categories)**.** Let $N : \mathsf{Cat} \to \mathsf{sSet}$ be the nerve functor.

1. (Grothendieck-Segal condition) Show that a simplicial set X belongs to the essential image of the functor N (ie, X encodes the information of a category) if and only if the composition map

$$\mathrm{Hom}_{\mathsf{sSet}}(\Delta[n], X) \longrightarrow \mathrm{Hom}_{\mathsf{sSet}}(I_n, X)$$

is a bijection for all $n \geq 2$.

2. (Grothendieck-Segal condition via horns) Let X be a simplicial set. Show that X is in the essential image of N if and only if for all $n \geq 2$, all $0 < i < n$ and for any map of simplicial sets $u : \Lambda_n^i \to X$ there exists a unique factorization of u along the canonical inclusion

$$\begin{tikzcd} \Lambda_n^i \arrow[r, "u"] \arrow[d, hook] & X \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

In other words, the composition map gives a bijection

$$\mathrm{Hom}_{\mathsf{sSet}}(\Delta[n], X) \cong \mathrm{Hom}_{\mathsf{sSet}}(\Lambda_n^i, X).$$

3. Following the previous question, show that a simplicial set X is the nerve of a groupoid if and only if for all $n \geq 1$ and $0 \leq i \leq n$ and for any map of simplicial sets $u : \Lambda_n^i \to X$ there exists a unique factorization of u along the canonical inclusion (as in the above diagram).
In particular, if $\mathcal{C}$ is a groupoid, then $N(\mathcal{C})$ is a Kan complex.

Solution 4. 1. The previous exercise shows that if X is in the essential image of N then the map is a bijection. To prove the reciprocal assertion, it is enough[3] to show that if X satisfies the condition then the canonical map $X \to N(\tau(X))$ is an isomorphism. Since it means that it is a bijection in every degree, by Yoneda it is enough to check that the induced map

$$\mathrm{Hom}_{\mathsf{sSet}}(\Delta[n], X) \to \mathrm{Hom}_{\mathsf{sSet}}(\Delta[n], N(\tau(X)))$$

is a bijection (since the left hand side is nothing more than X_n and the right hand side is $N(\tau(X))_n$). Using the hypothesis, the l.h.s is in bijection with

$$X_1 \times_{X_0} \times X_1 \times_{X_0} \ldots\ldots \times_{X_0} X_1$$

n-times. Also because $N(\tau(X))$ is a nerve, the r.h.s by the previous exercise is equivalent to

$$N(\tau(X))_1 \times_{N(\tau(X))_0} \times N(\tau(X))_1 \times_{N(\tau(X))_0} \ldots\ldots \times_{N(\tau(X))_0} N(\tau(X))_1$$

We conclude that the two are the same using the description of $\tau(X)$ given above.

2. Consider the commutative square

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{sSet}}(\Delta[n], X) \arrow[r, "u_*"] \arrow[d, "r_i^*"'] & \mathrm{Hom}_{\mathsf{sSet}}(\Delta[n], N(\tau(X))) \arrow[r, "\cong"] \arrow[d, "r_i^*"'] & \mathrm{Hom}_{\mathrm{Cat}}(\tau(\Delta[n], \tau(X)) \arrow[d, "\tau(r_i)^*"] \\ \mathrm{Hom}_{\mathsf{sSet}}(\Lambda_n^i, X) \arrow[r, "u_*"] & \mathrm{Hom}_{\mathsf{sSet}}(\Lambda_n^i, N(\tau(X))) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathrm{Cat}}(\tau(\Lambda_n^i), \tau(X)) \end{tikzcd}$$

[3]the fact that N is fully faithful and $\tau(N(\mathcal{C}) \to \mathcal{C}$ is an isomorphism actually implies that the condition is necessary as well because it implies the right vertical arrow is an isomorphism in the commutative square
$$\begin{tikzcd} X \arrow[r, "\cong"] \arrow[d] & N(\mathcal{C}) \arrow[d] \\ N(\tau(X)) \arrow[r, "\cong"] & N(\tau(N(\mathcal{C})) \end{tikzcd}$$
(whose horizontal arrows are induced by an isomorphism between X and an object in the image of N) so that the left is an isomorphism as well; hence if X is in the essential image of N then $X \to N(\tau(X))$ is an isomorphism

induced by the restrictions to $r_i : \Lambda^i_n \hookrightarrow \Delta[n]$ for the vertical arrows and composition with the canonical unit map $u : X \to N(\tau(X))$ for the horizontal ones.

To prove that X is in the essential image of N it is enough to prove that $u : X \to N(\tau(X))$ is an isomorphism.

The diagram tells that if u is an isomorphism then to and to prove that the left vertical arrow is an isomorphism is equivalent to proving that the right one is an isomorphism. Conversely, assume the right one is an isomorphism and further assume that $\mathrm{Hom}_{\mathsf{sSet}}(\Delta[n], X) \to \mathrm{Hom}_{\mathsf{sSet}}(\Lambda^n_i, X)$ is an isomorphism for any $0 < i < n$ and $n \geq 2$., then all vertical arrows in the diagram are isomorphisms. Let us prove that u is an isomorphism. As we have seen in the previous question it is the same as saying as the $u_* : \mathrm{Hom}_{\mathsf{sSet}}(\Delta[n], X) \to \mathrm{Hom}_{\mathsf{sSet}}(\Delta[n], N(\tau(X)))$ is an isomorphism. By commutativity of the diagram and since all vertical maps are equivalences, it is equivalent to proving that $\mathrm{Hom}_{\mathsf{sSet}}(\Lambda^i_n, X) \to \mathrm{Hom}_{\mathsf{sSet}}(\Lambda^i_n, N(\tau(X))$ is an isomorphism.

By our direct description of N and τ in questions 1 and 5, we have that u is an isomorphism in degree 0 and 1. Now, assume the result $u_n : X_n \to N(\tau(X))_n$ has been proven for $n \geq 1$ and let us prove it for $n+1$. Since u is an isomorphism in degree $\leq n$ Λ^i_n has no non-degenerate simplices in degree n and above, it implies that $u_* : \mathrm{Hom}_{\mathsf{sSet}}(\Lambda^i_n, X) \to \mathrm{Hom}_{\mathsf{sSet}}(\Lambda^i_n, N(\tau(X)))$ is an isomorphism. Hence by the above argument it is also the case for $u_* : \mathrm{Hom}_{\mathsf{sSet}}(\Delta[n], X) \to \mathrm{Hom}_{\mathsf{sSet}}(\Delta[n], N(\tau(X)))$ which concludes by induction. since .

So that we are left to show that the canonical map $\tau(\Lambda^i_n) \to \tau(\Delta[n]) = [n]$ is an isomorphism of categories for $n \geq 2$. For $n = 2$ the only possibility is Λ^1_2 which is isomorphic as a simplicial set to I_2 and we have already seen that $\tau(I_2) \to \tau(\Delta[2])$ is an isomorphism. Notice that $Sk_2(\partial\Delta[n]) \simeq Sk_2(Sk_{n-1}(\Delta[n])) \simeq Sk_2(\Delta[n])$. Let us show that $\tau(\Lambda^i_n)) \to \tau(\partial\Delta[n]))$ is an isomorphism of categories for $n > 3$. This follows because $\partial\Delta[n]$ can be obtained from Λ^i_n, for any i and n, as a pushout diagram where we attach the missing n-1-simplex non-degenerated

$$\begin{tikzcd}
\partial\Delta[n-1] \arrow[r] \arrow[d] & \Lambda^i_n \arrow[d] \\
\Delta[n-1] \arrow[r] & \partial\Delta[n]
\end{tikzcd}$$

If $n \geq 4$ then we know by the exercise 2.3 that τ applied to the left vertical map is an isomorphism of categories. As τ preserves pullbacks, we deduce that $\tau(\Lambda^i_n)) \to \tau(\partial\Delta[n]))$ is an isomorphism of categories for $n \geq 4$. Finally, for $n = 3$ one can show $\tau(Sk_2(\Lambda^i_3)) \simeq \tau(Sk_2(\Delta[3])) = [3]$ directly. These arguments work for all $0 \leq i \leq n$

3. In view of the previous question we have that if X satisfies the lifting properties, then it is in the essential image of the nerve of a category $\mathcal{C}$; and we can further take $\mathcal{C}$ to be $\tau(X)$ and in fact $X \to N(\tau(X))$ is an isomorphism so that $N(\tau(X))$ has the same lifting properties as X. We are left to prove that the additional lifting properties implies that all morphisms in the category $\tau(X)$ are invertible. This is check by looking at the degree 1 of the nerve. Consider an arrow $x \xrightarrow{f} y$ in $\tau(X)$ and let $\phi_f : \Lambda^2_2 \to N(\tau(X))$ defined by sending $\{0,1\}$ onto f (seen as a 1-simplex of $N(\tau(X))$) and $\{0,2\}$ onto $s_0(x) = id_x$ (by question 5). The lifting property yields a 2-simplex $\sigma \in N(\tau(X))_2$ such that $d_2(\sigma) = f$, $d_1(\sigma) = id_x$ hence $d_0(\sigma) \circ f = id_x$ and f has a left inverse. Using the horn Λ^0_2 we obtain similarly a right inverse of f. Hence f is invertible as claimed and $\tau(X)$ is a groupoid.

Conversely, assume $\mathcal{C}$ is a groupoid. Then we claim that $N(\mathcal{C})$ has all the lifting properties. By the previous question we have seen that $\tau(\Lambda^i_n) \to \tau(\Delta[n])$ is an isomorphism for $n \geq 3$ which proved that the vertical maps in the commutative diagram where isomorphisms, hence the lifting properties and further have checked the lifting property for Λ^1_2. The only remaining parts are Λ^0_2 and Λ^2_2 but e have just seen how to use the inversibility of maps to provide the liftings.

Exercice 5 (*Classifying space)*. Let G be a group and let $\mathcal{G}$ be the category with one object, whose endomorphisms are given by G.

1. Verify that $\mathcal{G}$ is a category and describe the n-simplices of its nerve $N\mathcal{G}$.
2. Show that $N\mathcal{G}$ is a Kan complex.
3. Let $\mathcal{E}_G$ be the category whose objects are the elements of G and with a unique morphism between every two objects. Show that $\mathcal{E}_G$ is well-defined and describe the n-simplices of its nerve.
4. Show that there exists a functor $\phi : \mathcal{E}_G \to \mathcal{G}$ sending a morphism $g \to g'$ to $g' \cdot g^{-1} \in \mathrm{End}_{\mathcal{G}}(*)$.
5. Prove that the induced morphism of simplicial sets $N\phi : N\mathcal{E}_G \to N\mathcal{G}$ is a Kan fibration.
6. Show that $N\mathcal{E}_G$ is contractible. Deduce the homotopy type of $N\mathcal{G}$.

Exercice 6 (Universal Property of Presheaves of Sets)**.** Let $\mathcal{C}$ be a small category. We denote by $\mathcal{P}(\mathcal{C})$ the category of functors $\mathcal{C}^{op} \to \mathsf{Set}$ with natural transformations as morphisms. Objects in this category are called *presheaves* (of sets) over $\mathcal{C}$. We have a canonical way to go from $\mathcal{C}$ to $\mathcal{P}(\mathcal{C})$, namely, to each object $X \in \mathcal{C}$ we assign the functor $h(X) := \mathrm{Hom}_{\mathcal{C}}(-, X) : \mathcal{C}^{op} \to \mathsf{Set}$. We let h denote this functor.

1. Let $F : \mathcal{C}^{op} \to \mathsf{Set}$ be a presheaf over $\mathcal{C}$ and let $X \in \mathcal{C}$. Given a natural transformation $u : h(X) \to F$ we can produce an element in $F(X)$ as follows: evaluating both $h(X)$ and F at the object X, u induces a map $u_X : h(X)(X) := \mathrm{Hom}_{\mathcal{C}}(X, X) \to F(X)$. We consider the element $u_X(\mathrm{Id}_X) \in F(X)$. Show that the assignment $\mathrm{Hom}_{\mathcal{P}(\mathcal{C})}(h(X), F) \to F(X)$ given by $u \mapsto u_X(\mathrm{Id}_X)$ is an isomorphism of sets. Use this to deduce that h is fully faithful. [4]
2. Show that $\mathcal{P}(\mathcal{C})$ admits all small colimits. (Hint: Construct the colimits objectwise.)
3. Let $F \in \mathcal{P}(\mathcal{C})$ and denote by $\mathcal{C}/F$ the full subcategory of objects over F in $\mathcal{P}(\mathcal{C})$ whose source is of the form $h(X)$ for some $X \in C$. Consider the diagram $\mathcal{C}/F \to \mathcal{P}(\mathcal{C})$ sending $(h(X) \to F)$ to $h(X)$. Show that the canonical arrow

$$\operatorname*{colim}_{u: h(X) \to F} h(X) \longrightarrow F$$

is an isomorphism in $\mathcal{P}(\mathcal{C})$ [5]. In other words, every presheaf F is the colimit of all representable presheaves defined over F.

4. Show that h has the following universal property: for any functor $\Phi : \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ is a category which admits all small colimits, there exists a unique functor (up to canonical equivalence of categories) $\widetilde{\Phi} : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ which commutes with small colimits and makes the following diagram commute up to natural isomorphism

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\Phi"] \arrow[d, "h"] & \mathcal{D} \\ \mathcal{P}(\mathcal{C}) \arrow[ur, dashed, "\widetilde{\Phi}"'] & \end{tikzcd}$$

In other words, $\mathcal{P}(\mathcal{C})$ is the universal way to complete $\mathcal{C}$ with all small colimits.

5. Check that the previous universal property can be reformulated as follows: if $\mathcal{D}$ is a category having all small colimits, then composition with h induces an equivalence of categories

$$\mathrm{Fun}^L(\mathcal{P}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

where $\mathrm{Fun}^L(\mathcal{P}(\mathcal{C}), \mathcal{D})$ is by definition the full subcategory of $\mathrm{Fun}(\mathcal{P}(\mathcal{C}), \mathcal{D})$ spanned by all functors which commute with colimits.

[4] h is also called the Yoneda functor.

[5] This is a small colimit because the indexing category is small as by hypothesis $\mathcal{C}$ is small.

6. Show that in the context of the previous question, the functor $\widetilde{\Phi}$ always admits a right adjoint.

7. Let F be a presheaf over $\mathcal{C}$. Show that the category of presheaves over the comma category of representables over F is equivalent to the category of all presheaves over F, that is,

$$\mathcal{P}(\mathcal{C}/F) \simeq \mathcal{P}(\mathcal{C})/F.$$

8. Use the conclusion of this exercise to show that in order to produce an adjunction

$$F : \mathsf{sSet} \rightleftarrows \mathcal{D} : G$$

where $\mathcal{D}$ is a category having all small colimits, it is enough to give a functor $\Delta \to \mathcal{D}$ (also called a *cosimplicial object* in $\mathcal{D}$). Observe that the following examples arise in this way:

- $(|-| \dashv \mathrm{Sing})$: the geometric realization of simplicial sets and the singular chain functors,
- $(\tau \dashv N)$: the truncation (or categorical realization) and the nerve of categories,
- $(\mathfrak{C} \dashv \mathfrak{N})$: the rigidification functor and the homotopy-coherent nerve of simplicial categories.