Gabriel-Zisman localization and Model structures

Exercice 1 (Model structures on the category of sets)**.** Let Sets denote the category of sets. Show that $(\mathrm{Sets}, \mathcal{W} = \mathrm{bijections}, \mathrm{Fib} = All, \mathrm{Cof} = All)$ determines a model structure.

In fact, there are precisely nine model structures in the category of sets. See link.

Exercice 2 (Whitehead Theorem for model categories)**.** The goal is to prove that in a model category C, *if X, Y are both fibrant and cofibrant objects, then a map $f : X \to Y$ is a weak equivalence if and only if it is an homotopy equivalence.*

1. Let$f \overset{l}{\sim} g$ be left homotopic. Show that f is a weak equivalence if and only if g is a weak equivalence.

2. Let $i : X \overset{\sim}{\hookrightarrow} C$ be an acyclic cofibration where X is both fibrant and cofibrant and C is fibrant. Prove that there is a retraction r of i and then show that r is an homotopy inverse of i.

3. Deduce from the previous question that a weak equivalence between fibrant and cofibrant objects is an homotopy equivalence.

4. Let $f : X \to Y$ be an homotopy equivalence between fibrant and cofibrant objects, and let $f : X \underset{\sim}{\overset{i}{\hookrightarrow}} C \overset{p}{\twoheadrightarrow} Y$ be a factorization where the first map is an acyclic cofibration.

   (a) Prove that C is both fibrant and cofibrant and that if g is an homotopy inverse of f, with left homotopy $H : C' \to Y$ between id_Y and $f \circ g$, there is a lift $H' : C' \to C$ such that $p \circ H' = H$ and $H' \circ i_0 = i \circ g$.

   (b) Deduce that $H' \circ i_1 \circ p$ is homotopical to id_C (one can note that i has an homotopy inverse) and then that it is a weak equivalence.

   (c) Prove that p is a retract of a weak equivalence and then conclude.

Exercice 3 (Gabriel-Zisman localization)**.** Let $\mathcal{C}$ be a small category and $\mathcal{W}$ a subset of the set of morphisms in $\mathrm{Fun}(I, \mathcal{C})$ where I is the category with two objects 0 and 1 and a unique non-trivial morphism $0 \to 1$. A localization of $\mathcal{C}$ with respect to $\mathcal{W}$ is a functor

$$l : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$$

satisfying the following universal property: For any category $\mathcal{D}$, composition with l:

$$\mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, D)$$

is a fully faithful functor and its essential image consists of those functors $\mathcal{C} \to \mathcal{D}$ sending $\mathcal{W}$ to isomorphisms. In other words, l, if it exists if the universal functor sending $\mathcal{W}$ to isomorphisms.

1. Check that $\mathcal{C}[\mathcal{W}^{-1}]$, if it exists, is unique up to canonical equivalences of categories.

2. Show that when $\mathcal{C}$ is the category with a single object $*$ and a monoid M of endomorphisms, and $\mathcal{W} = M$ then $\mathcal{C}[\mathcal{W}^{-1}]$ is equivalent to the category with one object $*$ and M^+ as endomorphisms, with M^+ the group completion of M.

Exercice 4. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor having a right adjoint $G : \mathcal{D} \to \mathcal{C}$[1]. Let $\mathcal{W}$ denote the collection of morphisms f in $\mathcal{C}$ such that $F(f)$ is an isomorphism in $\mathcal{D}$. Show that the following are equivalent:

[1] $\mathcal{D}$ is said to be a reflexive subcategory of $\mathcal{C}$.

1. G is fully faithful;

2. The natural transformation $F \circ G \to Id_{\mathcal{D}}$ is an isomorphism;

3. The natural functor $\mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{D}$ is an equivalence of categories.

Exercice 5. Let $L : \mathcal{C} \to \mathcal{C}$ be a functor and denote by $L\mathcal{C} \subseteq \mathcal{C}$ its essential image. Show that the following are equivalent:

1. There exists a functor $F : \mathcal{C} \to \mathcal{D}$ with a fully faithful right adjoint $G : \mathcal{D} \to \mathcal{C}$ and a natural isomorphism between $G \circ F$ and L;

2. When regarded as a functor $\mathcal{C} \to L\mathcal{C}$, L is a left adjoint to the inclusion $L\mathcal{C} \subseteq \mathcal{C}$;

3. There exists a natural transformation $\alpha : Id_{\mathcal{C}} \to L$ such that for each object $X \in \mathcal{C}$, the natural morphisms $L(\alpha_X)$ and $\alpha_{L(X)}$ are isomorphisms.

Exercice 6. Let $\mathcal{C} = \mathrm{Mod}_{\mathbb{Z}}$ be the category of abelian groups.

1. (Localization at a single prime) Let p be a prime. Show that the base change functor $- \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}] : \mathrm{Mod}_{\mathbb{Z}} \to \mathrm{Mod}_{\mathbb{Z}[\frac{1}{p}]}$ is a localization functor along the class $\mathcal{W}$ of all maps of abelian groups $f : X \to Y$ such that both $\mathrm{Ker} f$ and $\mathrm{coker} f$ are p-torsion groups. (Hint: Use the flatness of $\mathbb{Z}[\frac{1}{p}]$ over $\mathbb{Z}$.)

2. Show that the map $\mathbb{Q} \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ sending $q \mapsto q \otimes 1$ is an isomorphism. Use this and the Exercice 3 to show that the category of $\mathbb{Q}$ vector spaces is a localization of the category of abelian groups.

These examples are really important as we will see later when studying rational homotopy theory.

Exercice 7. Check that $\mathcal{C}[\mathcal{W}^{-1}]$ exists, given by the following pushout in Cat (the category of small categories):

$$\begin{tikzcd}
\coprod_{f \in \mathcal{W}} I \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "l"] \\
\coprod_{f \in \mathcal{W}} J \arrow[r] & \mathcal{C}[\mathcal{W}^{-1}]
\end{tikzcd}$$

where J is the category with two objects 0 and 1 and unique morphism $0 \to 1$ which is an isomorphism.[2]

Exercice 8 (Explicit descripition)**.** In this exercice we review an explicit model for the Gabriel-Zisman localization. Given the pair $(\mathcal{C}, \mathcal{W})$ we construct a new category $\mathcal{D}$ as follows: the objects are the objects of $\mathcal{C}$, morphims from X to Y are given by strings of the form

$$X \to X_1 \leftarrow X_2 \to X_3 \leftarrow ... \to X_n \to Y$$

where all arrows going to the left are in $\mathcal{W}$, submitted to the following equivalence relation: two strings are equivalent if there exists a commutative diagram

$$\begin{tikzcd}
 & X_1 \arrow[dd] & X_2 \arrow[l] \arrow[r] & X_3 \arrow[dd] & \cdots \arrow[l] \arrow[r] \arrow[dd] & X_n \arrow[dd] \arrow[dr] & \\
X \arrow[ur] \arrow[dr] & & & & & & Y \\
 & X'_1 & X'_2 \arrow[l] \arrow[r] & X'_3 & \cdots \arrow[l] \arrow[r] & X'_n \arrow[ur] &
\end{tikzcd}$$

where the vertical arrows are in $\mathcal{W}$. Composition is given by concatenation of strings. Show that this equivalence relation is well-defined and that $\mathcal{D}$, together with the canonical functor $\mathcal{C} \to \mathcal{D}$ sending $X \mapsto X$ and $(f : X \to Y) \mapsto X \to Y = Y$ is a localization of $\mathcal{C}$ along $\mathcal{W}$.

[2] Why do pushouts in Cat exist?

Exercice 9. In this exercice we check that the construction of the the previous exercice can be simplified whenever $\mathcal{W}$ satisfies some additional properties. Suppose that:

1. $\mathcal{W}$ is stable under compositions;

2. For any diagram

$$\begin{tikzcd} X' & \\ X \arrow[u, "s"] \arrow[r, "f"] & Y \end{tikzcd}$$

with $s \in \mathcal{W}$, there exists a way to complete this diagram in a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "g"] & Y' \\ X \arrow[u, "s"] \arrow[r, "f"] & Y \arrow[u, "t"'] \end{tikzcd}$$

with $t \in \mathcal{W}$.

3. Given $f, g : X \to Y$, if there exists $s \in \mathcal{W}$ such that $f \circ s = g \circ s$ then there exists $t : Y \to Z$ such that $t \in \mathcal{W}$ and $t \circ f = t \circ g$.

In this case $\mathcal{W}$ is said to be a calculus of (right) fractions. Under these hypothesis we consider for each $X \in \mathcal{C}$ the category $\mathcal{W}_{X/.}$ whose objects are morphisms $s : X \to X'$ with $s \in \mathcal{W}$ and morphisms are commutative triangles over X. Assume that $\mathcal{W}$ forms a calculus of fractions. Show that:

1. For each $X \in \mathcal{C}$, $\mathcal{W}_{X/.}$ is a filtered category.

2. The category $\mathcal{C}_{\mathcal{W}}$ whose objects are given by the objects of $\mathcal{C}$, hom-sets $Hom_{\mathcal{C}_{\mathcal{W}}}(X, Y)$ are given by $\mathrm{colim}_{u:Y\to Y' \in \mathcal{W}_{Y/.}} Hom_{\mathcal{C}}(X, Y')$ and compositions are induced from compositions in $\mathcal{C}$, is well-defined.

   In other words, morphisms in $\mathcal{C}_{\mathcal{W}}$ between X and Y are given by equivalence classes of strings of lenght one

$$X \to Y' \leftarrow Y$$

   where the left arrow belongs to $\mathcal{W}$. This simplifies the general explicit description given in (6).

3. Show that the canonical functor $Q : \mathcal{C} \to \mathcal{C}_{\mathcal{W}}$ induced by the identity on objects and by the canonical map

$$Hom_{\mathcal{C}}(X, Y) \to \mathrm{colim}_{u:Y\to Y' \in \mathcal{W}_{Y/.}} Hom_{\mathcal{C}}(X, Y')$$

   on morphisms, is well-defined;

4. Show that if $s : X \to X'$ is a map in $\mathcal{W}$ and Y is an object in $\mathcal{C}$ then the composition map $- \circ s$

$$Hom_{\mathcal{C}_{\mathcal{W}}}(X', Y) \to Hom_{\mathcal{C}_{\mathcal{W}}}(X, Y)$$

   is a bijection. Conclude that Q sends $\mathcal{W}$ to isomorphisms.

5. Show that Q is a localization of $\mathcal{C}$ along $\mathcal{W}$.

6. Show that if $\mathcal{C}$ is an additive category and $\mathcal{W}$ is a calculus of fractions then the localization functor Q preserves finite colimits and $\mathcal{C}[\mathcal{W}^{-1}]$ is also additive.

Exercice 10. The following generalizes the exercice (4): Let $\mathcal{C}$ be an abelian category and let $\mathcal{D} \subseteq \mathcal{C}$ be a thick subcategory, ie, a full subcategory such that for each exact sequence

$$0 \to X_1 \to X_2 \to X_3 \to 0$$

in $\mathcal{C}$, X_2 is in $\mathcal{D}$ if and only if X_1 and X_3 are in $\mathcal{D}$. Let $\mathcal{W}$ denote the colllection of morphisms f in $\mathcal{C}$ such that $\mathrm{Ker} f$ and $\mathrm{coker} f$ are in $\mathcal{D}$. Show that $\mathcal{W}$ admits a calculus of fractions and that $\mathcal{C}[\mathcal{W}^{-1}]$ is equivalent to the pushout $\mathcal{C}/\mathcal{D}$ in Cat given by

$$\begin{tikzcd} \mathcal{D} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ 0 \arrow[r] & \mathcal{C}/\mathcal{D} \end{tikzcd}$$

Exercice 11 (The canonical model structure in Cat)**.** Let Cat denote the 1-category of small categories and morphisms given by functors between them. Let $\mathcal{W}$ be the collection of functors which are equivalences of categories.

1. Show that the Gabriel-Zisman localization $\mathrm{Cat}[\mathcal{W}^{-1}]$ is equivalent to the category whose objects are small categories and morphisms are isomorphism classes of functors.

2. A functor $F : \mathcal{C} \to \mathcal{D}$ between small categories is said to be an isofibration if for every object $c \in \mathcal{C}$ and every isomorphism $f : F(c) \to d$ in $\mathcal{D}$, there exists an object $c' \in \mathcal{C}$ and an isomorphism $u : c \to c'$ such that $d = F(c')$ and $f = F(u)$. Show that an isofibration that is an equivalence of categories is surjective on objects. Conversely, show that if a functor F is fully faithful and surjective on objects then it is an isofibration.

3. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be a cofibration if it is injective on objects. Let Fib denote the collection of all isofibrations and Cof the class of cofibrations. Show that $(\mathrm{Cat}, \mathcal{W}, \mathrm{Fib}, \mathrm{Cof})$ is a model structure and identify its fibrant-cofibrant objects.