# Derived Functors, Homotopy Colimits and Model structures

**Exercice 1** (Composition of Derived Functors)**.**

1. Let $F_1 : \mathcal{C}_1 \to \mathcal{C}_2$ and $F_2 : \mathcal{C}_2 \to \mathcal{C}_3$ be functors and let $\mathcal{W}_i$ be a class of morphisms in $\mathcal{C}_i$. Assuming all the relevant total left derived functors exist, use their universal properties to construct a natural transformation $\mathbb{L}L_2 \circ \mathbb{L}F_1 \to \mathbb{L}(F_2 \circ F_1)$.

2. Suppose now that $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_3$ are model categories and that $F_1$ and $F_2$ are left Quillen functors. Show that all derived functors exist and the natural transformation of the previous exercise is a natural isomorphism.

**Exercice 2** (Homotopy colimits)**.** In this exercise we first deal with generalities on homotopy pushouts and then specialized to chain complexes with the projective model structure. Let $\mathcal{C}$ be a model category and let $I$ be the category given by the diagram-shape

$$\begin{tikzcd} b \arrow[r] \arrow[d] & c \\ a & \end{tikzcd}$$

1. Let $f : X \to Y$ be a natural transformation of diagrams $X, Y \in \mathrm{Fun}(I, \mathcal{C})$. Show that $f$ has the left lifting property with respect to all projective acyclic fibrations if and only if the the natural maps
$$X(a) \coprod_{X(b)} Y(b) \to Y(a), \quad X(b) \to Y(b), \quad X(c) \coprod_{X(b)} Y(b) \to Y(c)$$
are cofibrations in $\mathcal{C}$ (Here we mean the usual pushouts in $\mathcal{C}$). Conclude that a diagram $Y : I \to \mathcal{C}$ is cofibrant if and only if $Y(b)$ is cofibrant in $\mathcal{C}$ and each map $Y(a) \to Y(b)$ and $Y(a) \to Y(c)$ is a cofibration. Moreover, show that $X \to Y$ has the left lifting property with respect to projective fibrations if and only the same three maps are acyclic cofibrations.

2. Show that the category of diagrams $\mathrm{Fun}(I, \mathcal{C})$ admits the projective model structure (without using the result seen in class that such structure exists since $I$ is very small).

3. Show that the colimit functor $\mathrm{colim} : Fun(I, \mathcal{C}) \to \mathcal{C}$ is a left Quillen functor.

4. A model category $\mathcal{C}$ is said to be left proper if weak-equivalences are stable under pushouts along cofibrations. Show that $\mathcal{C}$ be left proper and
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ C \arrow[r] & C \coprod_A B \end{tikzcd}$$
is a pushout diagram with $A \to B$ a cofibration, then the diagram is also a homotopy pushout.

5. **Case of Topological spaces.** Assume now that $\mathcal{C} = \mathbf{Top}$.
   (a) Using that **Top** is proper (see exercise 5), deduce that that there is a canonical isomorphism
   $$\mathbb{L}\,\mathrm{colim}(X \leftarrow A \to Y) \cong X \coprod_A^h Y = X \coprod_{A \times \{0\}} \mathrm{Cyl}(A \to Y)$$
   in $\mathbf{Ho}(\mathbf{Top})$ between the homotopy pushout computed by the projective model structure and the formula given by the mapping cylinder.

(b) Give a formula for computing the homotopy colimit of a tower $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_2 \to \dots.$ as well as the homotopy limit of a tower $\dots Y_2 \to Y_1 \to Y_0$.

6. **Case of chain complexes.** Assume now that $\mathcal{C}$ is the model category of chain complexes over a ring $R$.

(a) Show that $\mathcal{C}$ is left proper.

(b) Let $g : A \to B$ be a map of chain complexes. Recall that the *mapping cone* of $g$, denoted $C(g)$, is the chain complex given in level $n$ by $B_n \oplus A_{n-1}$ and whose differential $B_{n+1} \oplus A_n \to B_n \oplus A_{n-1}$ is given $(b, a) \mapsto (\partial_B(b) + g(a), -\partial_A(a))$. Let $I^1$ denote the chain complex given by $R \oplus R$ in degree 0 and $R$ in degree 1 with differential given by $\partial_R : R \to R \oplus R$ given by $r \mapsto (-r, r)$. We define the mapping cylinder of $g$, $Cyl(g)$ to be the pushout in chain complexes of

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "i_0"'] & B \arrow[d] \\ I^1 \otimes A \arrow[r] & Cyl(g) \end{tikzcd}$$

where the vertical arrow $A \to I^1 \otimes A$ is the induced by the inclusion $i_0 : R \to I^1$ corresponding to the inclusion of the second factor $R \hookrightarrow R \oplus R$ in degree 0 and the differential on $I^1 \otimes A$ is given by $r \otimes a \mapsto \partial_R(x) \otimes a + (-1)^{\deg(r)} r \otimes \partial_A(a)$. Show that the mapping cone of $g$ is the pushout of

$$\begin{tikzcd} I^1 \otimes A \arrow[r] \arrow[d] & Cyl(g) \arrow[d] \\ C(Id_A) \arrow[r] & C(g) \end{tikzcd}$$

(c) Let $\Delta^1$ be the category with two objects and one non trivial morphism in between them. Show that the construction of the mapping cone defines a functor $C : Fun(\Delta^1, Ch(R)) \to Ch(R)$ sending natural transformations objectwise given by quasi-isomorphisms to quasi-isomorphisms.

(d) Let $Y := (\, 0 \longleftarrow A \xrightarrow{g} B \,)$ be a diagram in $\mathcal{C}$. Show that there exists a diagram of the form $Y' := (\, 0 \longleftarrow A' \xrightarrow{g'} B' \,)$ with $g'$ a cofibration and $A'$ and $B'$ cofibrant, together with a natural transformation $u : Y' \to Y$ which is objectwise a weak-equivalence. Notice that by the previous question the induced map $C(g') \to C(g)$ is a weak-equivalence.

(e) Let $Y := (\, 0 \longleftarrow A \xrightarrow{g} B \,)$ be a diagram in $\mathcal{C}$ with $A$ and $B$ cofibrant and $g$ a cofibration. Show that $A \to A \otimes \Delta^1$ is a weak-equivalence and show that we can construct a zigzag of diagrams $Y \leftarrow Y' \to Y''$

$$\begin{tikzcd} 0 & A \arrow[l] \arrow[r, "g"] & B \\ C(A) \arrow[u] \arrow[d] & A \arrow[l] \arrow[r, "g"] \arrow[u] \arrow[d] & B \arrow[u] \arrow[d] \\ C(A) & I^1 \otimes A \arrow[l] \arrow[r, "g"] & Cyl(g) \end{tikzcd}$$

where each vertical arrow is a weak-equivalence and the map $I^1 \otimes A \to Cyl(g)$ is a cofibration.

(f) Let $Y := ( 0 \longleftarrow A \xrightarrow{g} B )$. Conclude that the mapping cone $C(g)$ is a model for the homotopy colimit of the diagram $Y$.

**Exercice 3** (**Bad behavior of Gabriel-Zisman Localization**). Let $A$ be a ring and let $D(A) := \mathbf{Ho}(Ch(A))$ denote the derived category of $A$; it is the Gabriel-Zisman localization of the category $Ch(A)$ of chain complexes in $A$ localized along quasi-isomorphisms of complexes. We have seen in class that $D(A)$ is the homotopy category of a model structure in $Ch(A)$ with weak-equivalences given by quasi-isomorphisms and fibrations given by levelwise surjections.

1. Show that if $E$ and $H$ are two $A$-modules seen as complexes concentrated in degree zero, then
$$Hom_{D(A)}(E, H[n]) \simeq Ext^n_A(E, H)$$
2. Show that if $A$ is a field then $D(A)$ is an abelian category[1], equivalent to the category $A^{\mathbb{Z}}$ of $\mathbb{Z}$-graded $A$-vector spaces.
3. Show that $D(A[X])$ does not admit colimits in general (Hint: Take a non-trivial element $f : A \to A[1]$ and show that if it has a kernel then we get to a condradiction with the fact $f$ is non-trivial);
4. Let $A$ be a field and let $I$ be the category with one object and $\mathbb{N}$ as endomorphisms. Show that $Fun(I, D(A))$ is not equivalent to $D(Fun(I, Ch(A))$. The conclusion is that the theory of diagrams does not interact well with derived categories.

**Exercice 4** (**Model structure on topological spaces).** The goal of this exercice is to show that the category of topological spaces, together with homotopy weak-equivalences, Serre fibrations (maps with the right lifiting property with respect to the inclusion $i_0 : D^n \mapsto D^n \times I$, $n \geq 0$) and cofibrations given by maps with a left lifting property with respect to acyclic Serre fibrations, forms a (cofibrantly generated) model category:

1. Show that the class of weak-equivalences satisfies the 2 out of 3 property;
2. Show that weak-equivalences, fibrations and cofibrations are stable under retracts;
3. Let $\mathcal{C}$ be a category and $S$ be a class of maps. Show that $LLP(RLP(LLP(S))) = LLP(S)$ and that $RLP(LLP(RLP(S))) = RLP(S)$.
4. Let $I'$ denote the collection of all boundary inclusions $\{\partial : S^{n-1} \hookrightarrow D^n\}_{n\geq 0}$ and $J$ denote the collection of all maps $\{i_0 : D^n \hookrightarrow D^n \times [0,1], x \mapsto (x, 0)\}_{n\geq 0}$. Notice that Serre fibrations are then defined as $RLP(J)$. Show that $J \subseteq Cell(I')$ and deduce that $LLP(RLP(J))) \subseteq LLP(RLP(I')))$.
5. Show that $LLP(RLP(J)) \subseteq \mathcal{W}$. Deduce that $Cell(J) \subseteq \mathcal{W} \cap LLP(RLP(I'))$.
6. Show that every map in $RLP(I')$ is a trivial Serre fibration.
7. Show that for every set $A$, the functor $Hom(A, -) : Sets \to Sets$ commutes with $\alpha$-filtered colimits for some cardinal $\alpha$. Use this to show that the small object argument can be applied both to the class $I'$ and the class $J$ because the elements of $I'$ and $J$ are inclusions of topological spaces. In this case, the transfinite induction won't be indexed by $\omega$ but by a larger ordinal.

[1] see links to Homological algebra exercises on the web page if you are not familiar with this

8. Show that if $f : X \to Y$ is a trivial Serre fibration then it is in $RLP(I')$. In particular, we get $Cof = LLP(RLP(I'))$.

9. Conclude.

**Exercice 5 (Top is proper).** We endow **Top** with Quillen model category structure.

1. Prove that the category is right proper, that is that the pullback of a weak-equivalence under a fibration is a weak equivalence.

2. Prove that **Top** is left proper that is that the pushout of a weak equivalence by a cofibration is a weak equivalence.

**Exercice 6.** Let $\mathbf{Top}_*$ be the category of *pointed* topological spaces and $U : \mathbf{Top}_* \to \mathbf{Top}$ be the functor forgetting the base point.

1. Prove that $U$ is a right adjoint and compute its left adjoint.

2. We endow **Top** with Quillen model structure. Find a model structure on $\mathbf{Top}_*$ such that $U$ is right Quillen.

3. Generalize the previous construction to any model category $\mathbb{C}$ ?