The small object argument and Quillen model structure on **Top**

Exercice 1 (Small Object Argument). The goal of this exercise is to make precise the small object argument and how to use it in the case of/to construct *cofibrantly generated* model categories.

Let $\mathcal{C}$ be a category that admits all small colimits and I a small set of maps in $\mathcal{C}$. Suppose that the domains of the maps in $\mathcal{C}$ are compact objects [1]. We denote by LLP(I) (resp. RLP(I)) the collection of maps with the left (resp. right) lifting property with respect to I. We let Cell(I) denote the class of maps in $\mathcal{C}$ obtained as transfinite compositions of pushouts of elements of I. More explicitly, a map in Cell(I) is of the form $X_0 \to \operatorname{colim}_{\alpha<\lambda} X_\alpha$, where λ is an ordinal and $X\colon \lambda \to \mathcal{C}$ is a functor with the following two properties:

- for all $\alpha < \alpha + 1 < \lambda$, the map $X_\alpha \to X_{\alpha+1}$ is obtained as a pushout

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X_\alpha \arrow[r] & X_{\alpha+1} \end{tikzcd}$$

  of a map $U \to V$ in I,

- for all limit ordinal $\beta < \lambda$, one has $X_\beta \cong \operatorname{colim}_{\alpha<\beta} X_\alpha$.

1. Consider a family $\{u_k : A_k \to B_k\}_{k\in K}$ of maps in I, indexed by a set K. Show that any map f in $\mathcal{C}$ obtained as a pushout

$$\begin{tikzcd} \coprod_{k\in K} A_k \arrow[r] \arrow[d, "\coprod_{k\in K} u_k"'] & X \arrow[d, "f"] \\ \coprod_{k\in K} B_k \arrow[r] & Y \end{tikzcd}$$

   is in Cell(I).

2. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. Show that we can fonctorially factor f as a composition $\delta \circ \gamma : X \to Z \to Y$, with $\gamma : X \to Z$ in Cell(I) and $\delta : Z \to Y$ in RLP(I).

3. Use this factorization process to prove that every map $f : X \to Y$ in LLP(RLP(I)) is a retract of a map $g : X \to C$ in Cell(I), which fixes X.

Exercice 2 (Model structure on topological spaces). The goal of this exercice is to show that the category of topological spaces, together with weak homotopy equivalences, Serre fibrations (maps with the right lifiting property with respect to the inclusion $D^n \to D^n \times I$, $n \geq 0$) and cofibrations given by maps with a left lifting property with respect to acyclic Serre fibrations, forms a cofibrantly generated model category.

1. Show that the class of weak equivalences satisfies the 2-out-of-3 property.

2. Show that weak equivalences, fibrations and cofibrations are stable under retracts.

[1] Recall that an object X in a category $\mathcal{C}$ is said to be compact if the functor $Hom(X, -)$ commutes with filtered colimits.

3. Let $\mathcal{C}$ be a category and S be a class of maps. Show that $\mathrm{LLP}(\mathrm{RLP}(\mathrm{LLP}(S))) = \mathrm{LLP}(S)$ and that $\mathrm{RLP}(\mathrm{LLP}(\mathrm{RLP}(S))) = \mathrm{RLP}(S)$.

   Let I' denote the collection of all boundary inclusions $\{\partial_n : S^{n-1} \hookrightarrow D^n\}_{n\geq 0}$ and J denote the collection of all maps $\{j_n : D^n \hookrightarrow D^n \times [0,1], x \mapsto (x,0)\}_{n\geq 0}$. Note that by definition, the class of Serre fibrations is $\mathrm{RLP}(J)$.

4. Show that $J \subseteq \mathrm{Cell}(I')$ and deduce that $\mathrm{LLP}(\mathrm{RLP}(J))) \subseteq \mathrm{LLP}(\mathrm{RLP}(I')))$.

5. Show that $\mathrm{LLP}(\mathrm{RLP}(J)) \subseteq \mathcal{W}$. Deduce that $\mathrm{Cell}(J) \subseteq \mathcal{W} \cap \mathrm{LLP}(\mathrm{RLP}(I'))$.

6. Show that every map in $\mathrm{RLP}(I')$ is a trivial Serre fibration.

7. Show that for every set A, the functor $\mathrm{Hom}(A, -) : \mathbf{Set} \to \mathbf{Set}$ commutes with α-sequential colimits for some cardinal α. Use this to prove that every topological space is small with respect to inclusions.
   Deduce that the small object argument can be applied both to the class I' and the class J.
   NB: In this case, the transfinite induction won't be indexed by ω but by a larger ordinal.

8. Show that if $f : X \to Y$ is a trivial Serre fibration, then f is in $\mathrm{RLP}(I')$. In particular, we get $\mathrm{Cof} = \mathrm{LLP}(\mathrm{RLP}(I'))$.

9. Conclude.

Exercice 3 (**Top is proper).** We endow **Top** with Quillen model category structure.

1. Prove that the category is *right proper*, that is the pullback of a weak equivalence under a fibration remains a weak equivalence.

2. Prove that **Top** is *left proper*, that is the pushout of a weak equivalence by a cofibration remains a weak equivalence.